On the formulas of meromorphic functions with periodic Herman rings

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Periodic Fatou components

Classification of periodic Fatou components (1-dim dynamics):

1. Attracting basin (Kœnigs, 1884)
2. Parabolic basin (Lean-Fatou, 1897)
3. Super-attracting basin (Böttcher, 1904)
4. Baker domain (Fatou, 1919)
5. Siegel disk (Siegel, 1942)
6. Herman ring (Arnold-Herman, 1979)

Let f be a rational map with $\deg(f) \geq 2$ or a transcendental meromorphic function.

Definition (Herman ring)

A periodic Fatou component U of f is called a **Herman ring** if U is conformally isomorphic to some annulus and if the restriction of f, or some iterate of f on U, is holomorphically conjugated to an irrational rotation of this annulus.

- Entire functions cannot have Herman rings (by maximum modulus principle).
- There are two known methods for constructing Herman rings.

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- There are two known methods for constructing Herman rings.

Arnold-Herman-Yoccoz's theorems

Theorem (Arnold, 1965)

Let $\theta \in \mathscr{D}$ be Diophantine, and $\sigma > 1$. Then $\exists \varepsilon > 0$ such that if $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a homeomorphism with rotation number $\rho(f) = \theta$, which extends to be analytic and univalent on the annulus $\{1/\sigma < |z| < \sigma\}$ and satisfies $|f(z) - e^{2\pi i\theta} z| < \varepsilon$ there, then f is conformally conjugate to $R_\theta(\zeta) = e^{2\pi i\theta}\zeta$ on $\{1/\sqrt{\sigma} < |z| < \sqrt{\sigma}\}$.

Definition (Herman numbers)

Let $\mathscr{H}$ be the set of $\theta \in \mathbb{R}$ such that every orientation-preserving analytic circle diffeomorphism of rotation number θ is analytically conjugate to a rotation.

Theorem (Herman, 1979)

$\mathscr{D} \subset \mathscr{H}$.

Theorem (Yoccoz, 2002)

There is an arithmetic characterization of $\mathscr{H}$ and $\mathscr{D} \subsetneq \mathscr{H} \subsetneq \mathscr{B}$, where $\mathscr{B}$ is Brjuno.

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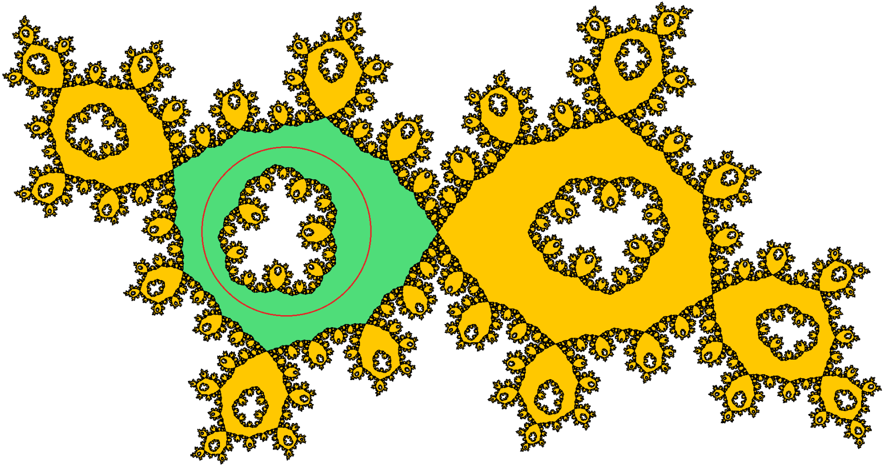
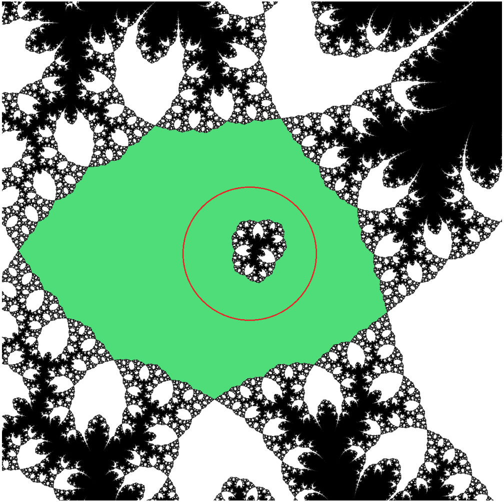
Example I



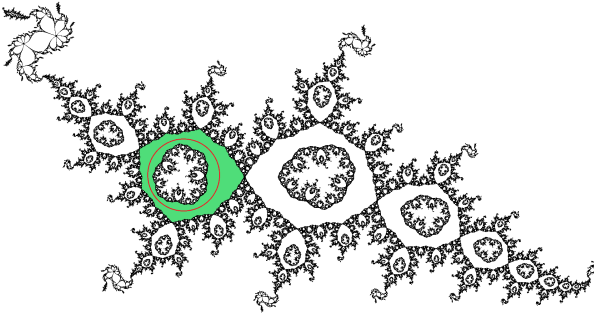
$$B(z) = e^{2\pi i t} z^2 \frac{z-4}{1-4z}, \quad \text{where } t = 0.6151732\ldots \quad \text{s.t. } \rho(B|_{\mathbb{S}^1}) = \frac{\sqrt{5}-1}{2}$$

Example II



$$A(z) = e^{2\pi i t} z e^{\frac{1}{4}(z - \frac{1}{z})}, \quad \text{where } t = 0.6145263\ldots \quad \text{s.t. } \rho(A|_{\mathbb{S}^1}) = \frac{\sqrt{5}-1}{2}$$

Example III

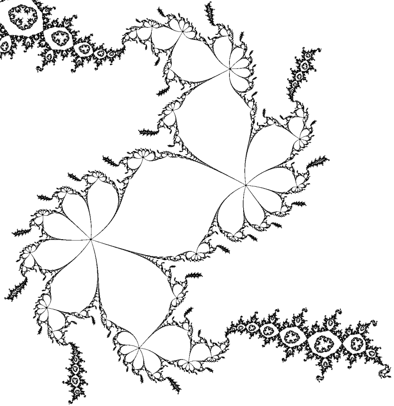


$$B_1(z) = e^{2\pi i t} z^3 \frac{z-4}{1-4z} \frac{z-a}{1-\bar{a}z}$$

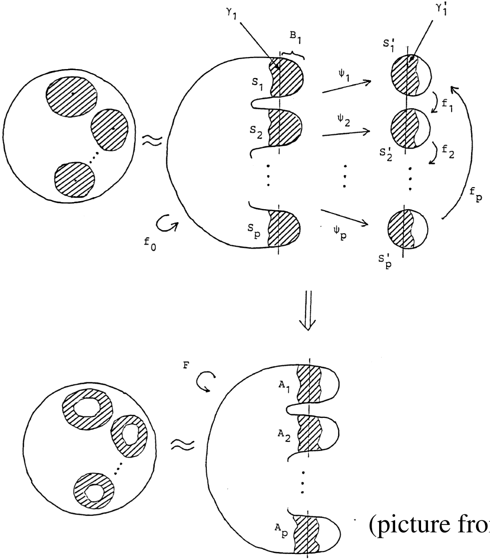
where $a \approx 17.6692736 - 11.1188197i$

$t \approx 0.6057267$

s.t. $\rho(B_1|_{\mathbb{S}^1}) = \dfrac{\sqrt{5}-1}{2}$



Shishikura's construction by qc surgery



(picture from Mitsu's paper)

Shishikura's example

The construction in §4 not only assures us the existence of Herman rings of order p, but also suggests that if the γ_1 is chosen sufficiently close to the center of S_1, then the resulting rational function F will be close to f_0 outside a neighborhood of the centers (by suitable choice of ϕ). Hence, if $f_0(z) = z^2+c_0$ has a Siegel disk of order 2 with center z_0 (such c_0 can be found by an elementary calculation), it is expected that a rational function of the form

$$F(z) = F_{a,b,c}(z) = z^2 \cdot \frac{z-a}{z-b} + c$$

has a Herman rings of order 2, for suitable $a,b \fallingdotseq z_0$ and $c \fallingdotseq c_0$.

Notice that F is close to f_0 outside a neiborhood of 0, if $a,b \fallingdotseq z_0$ and $c \fallingdotseq c_0$.

Shishikura's example

Here is an example of such parameters:

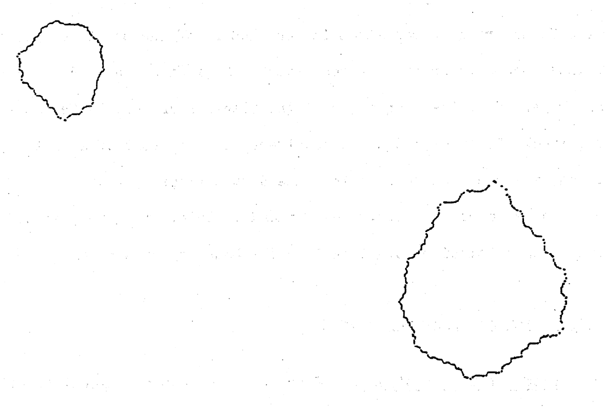
$c_0 = -0.8639244 + 0.2103677 \times i$, $z_0 = -0.0789079 - 0.2497882 \times i$,

$a = -0.0768679 - 0.2503722 \times i$, $b = -0.0809479 - 0.2492042 \times i$,

$c = -0.8648749 + 0.2103377 \times i$.

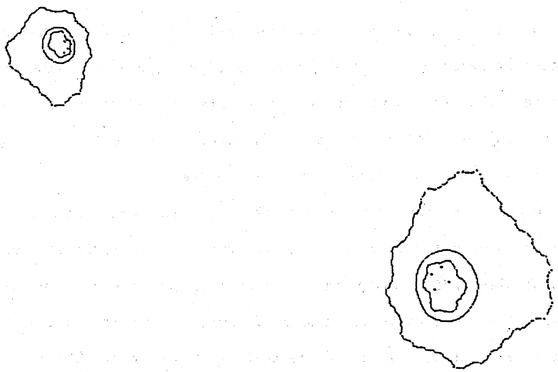
The parameters were found by numerical experiments (trial-and-error method).

Shishikura's example



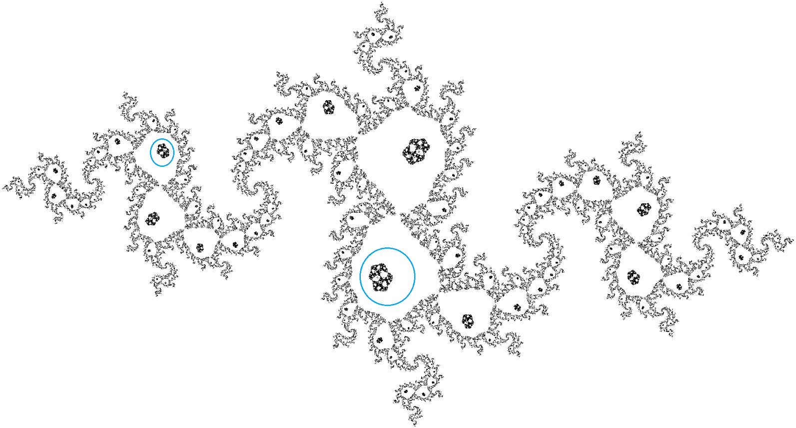
Orbit of critical point 0 of f_0.

Shishikura's example



Orbits of critical points of F.

Shishikura's example



Shishikura's question

Shishikura (1980s): Find an explicit form of a rational map which has a Herman ring of period **two** and draw its picture.

Obstruction

Theorem (Y. 2020)

A Herman ring of period $p \geq 2$ of a rational map f cannot contain any circle C with $f^{\circ p}(C) = C$.

Proof idea: Assume that $C = \mathbb{S}^1$ s.t. $f^{\circ p}(\mathbb{S}^1) = \mathbb{S}^1$. Define $\tau(z) = 1/\bar{z}$ and

$$F(z) := \begin{cases} f^{\circ p}(z) & \text{if } z \in \widehat{\mathbb{C}} \setminus \mathbb{D}, \\ \tau \circ f^{\circ p} \circ \tau^{-1}(z) & \text{if } z \in \mathbb{D}. \end{cases}$$

One obtains two cycles of Herman rings $\{A_1, A_2, \cdots, A_p\}$ and $\{\tau(A_1), \tau(A_2), \cdots, \tau(A_p)\}$ of f, where $A_1 = \tau(A_1)$ contains $\mathbb{S}^1$. This is a contradiction.

Remark: To find the formulas of rational maps having Herman rings of period $p \geq 2$, it is hard via studying analytic circle diffeomorphisms.

The formulas of rational maps

Theorem (Y. 2020)

Let θ be a Brjuno number and $p \geq 2$ an integer. There exists $a_0 \geq 3$ such that for any $a > a_0$, there are b and $u = u(a,b) \in \mathbb{C} \setminus \{0\}$ such that

$$f_{a,b}(z) = uz^2 \frac{z-a}{1-az} + b$$

has a p-cycle of Herman rings of rotation number θ and a super-attracting p-cycle containing 0. In particular, if $p = 2$, then $u = (ab-1)/(b(b-a))$.

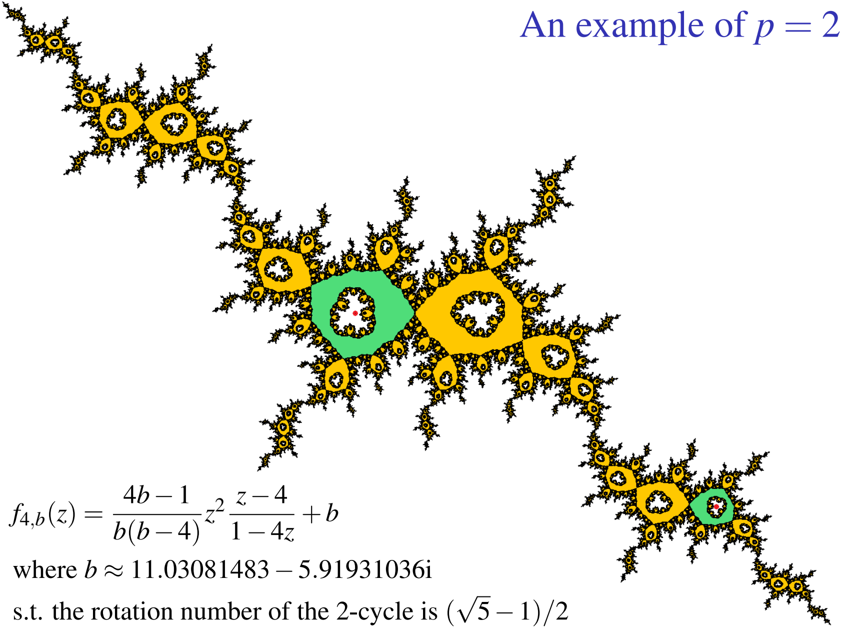
Remark: (1) It suffices to determine the parameter b essentially.

(2) The formula $f_{a,b}$ is obtained by **quasiconformal surgery**:

- Turning a p-cycle of Siegel disks of $z^2 + c$ to a p-cycle of Herman rings;
- Deformations (*changing the conformal modulus* and *twisting*) in Herman rings, and straightening the resulting cubic rational map

$$f(z) = \mu z^2 \frac{z-\alpha}{1-\omega z} + \beta \quad \text{to} \quad f_{a,b}(z) = uz^2 \frac{z-a}{1-az} + b.$$

An example of $p = 2$

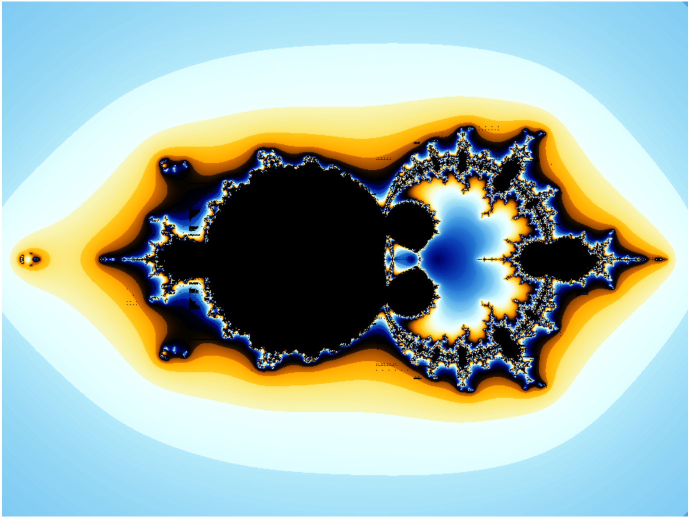


$$f_{4,b}(z) = \frac{4b-1}{b(b-4)} z^2 \frac{z-4}{1-4z} + b$$

where $b \approx 11.03081483 - 5.91931036i$

s.t. the rotation number of the 2-cycle is $(\sqrt{5}-1)/2$

How to find the parameters



The parameter plane of $f_{4,b}$ (where $p = 2$)

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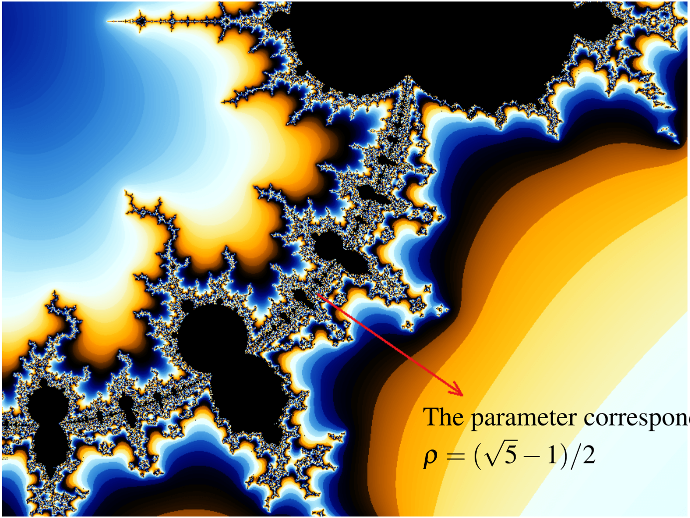

The parameter corresponding to
$\rho = (\sqrt{5}-1)/2$

The parameter plane of $f_{4,b}$ (where $p = 2$)

HR in transcendental meromorphic families

- Zheng (2000): Constructed the first example of Herman rings in transcendental meromorphic functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$.
- Domínguez and Fagella (2004): Independently, gave some examples of Herman rings of transcendental meromorphic functions in a variety of arrangements.
- Fagella and Peter (2012): Existence of Herman rings of **any given period** in transcendental meromorphic functions by qc surgery.
- Moreno Rocha (2020): Existence of Herman rings in elliptic functions.

Theorem (Y. 2020)

A Herman ring of period $p \geq 1$ of a transcendental meromorphic function f cannot contain any circle C with $f^{\circ p}(C) = C$.

Shishikura's question can be also asked for transcendental meromorphic functions:

Find an explicit form of a transcendental meromorphic function which has a Herman ring of period **one (or two)** and draw its picture.

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The pre-models

For $p = 1$, consider $G_\lambda(z) = \lambda z e^z$ (by Domínguez and Fagella, 2004). After pasting the complement of an invariant subdisk of a quadratic Siegel disk to 0, one gets

$$g_{1,a,b}(z) = b\frac{z^2}{z-a}e^z.$$

For $p = 2$, we have

Lemma (Y. 2020)

For any $\theta \in \mathscr{B}$, there exists $\kappa \in \mathbb{C} \setminus \{0\}$ such that $E(z) = \lambda z e^z + \kappa$ with $\lambda = -1/e^\kappa$ has a 2-periodic Siegel point 0 whose rotation number is θ.

Theorem (Y. 2020)

Let $\theta \in \mathscr{B}$. Then $\exists a_1 \in (0,1]$ such that $\forall 0 < a < a_1$, there exists $b \in \mathbb{C} \setminus \{0\}$ s.t.

$$m_{a,b}(z) = u\frac{z^2}{z-a}e^z + b$$

has a 2-cycle of Herman rings of rotation number θ, where $u = (a-b)/(be^b)$.

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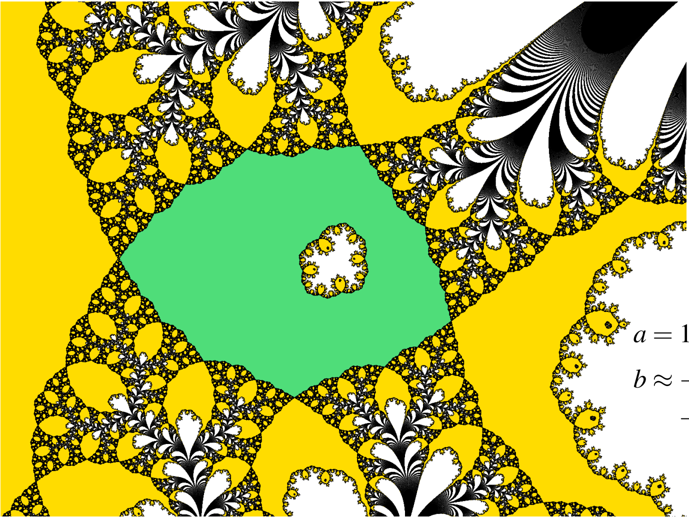
Pictures of HR in transcendental cases

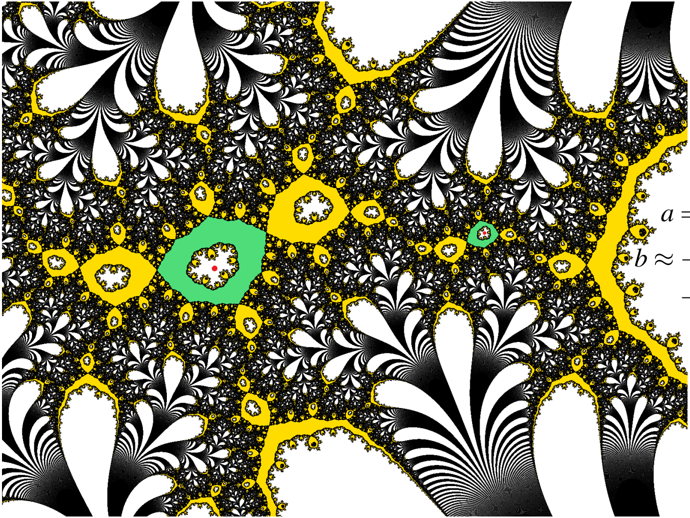


$a = 1/20$

$b \approx -0.7218804$
$-0.6929262i$

The map $g_{1,a,b}$ with a fixed Herman ring

Pictures of HR in transcendental cases



$a = 1/100$
$b \approx -1.2379676$
$-0.1653588i$

The map $m_{a,b}$ with a 2-cycle of Herman rings

Another pre-model

Proposition (Y. 2020)

For any $\theta \in \mathscr{B}$ and $p \geq 1$, there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $E_\lambda(z) = \lambda z^2 e^z$ has a p-cycle of Siegel disks with rotation number θ.

Assuming this proposition, we have

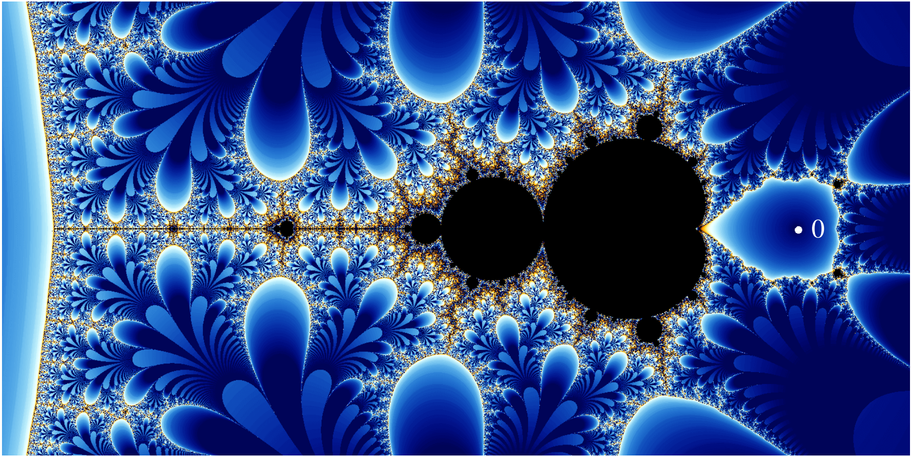
Theorem (Y. 2020)

Let $\theta \in \mathscr{B}$ and $p \geq 1$. Then there are a, b and $u = u(a,b) \in \mathbb{C} \setminus \{0\}$ s.t.

$$g_{a,b}(z) = u\,\frac{z-b}{z-a}\,z^2 e^z$$

has a p-cycle of Herman rings of rotation number θ and a super-attracting p-cycle which is different from 0.

Quadratics in a transcendental entire family



The parameter plane of $E_\lambda(z) = \lambda z^2 e^z$. Figure range: $[-21,3] \times [-6,6]$.

Mandelbrot-like set in the family

For $E_\lambda(z) = \lambda z^2 e^z$, we have

Theorem (Y. 2020)

Let

$$\Lambda := \{\lambda \in \mathbb{C} : \tfrac{2}{5} < |\lambda| < 51\} \setminus \mathbb{R}^+, \quad \textit{and}$$
$$\mathscr{D} := \{\lambda \in \mathbb{C} : \tfrac{1}{2} \leq |\lambda| \leq 50 \textit{ and } \tfrac{1}{10} \leq \arg\lambda \leq 2\pi - \tfrac{1}{10}\}.$$

Then we have a holomorphic family of quadratic-like mappings $\{(E_\lambda; U_\lambda, V_\lambda)\}_{\lambda\in\Lambda}$ *which satisfies*

1. U_λ *contains exactly one critical point* -2*, and* $E_\lambda(-2)+2$ *turns around zero once as* λ *turns around* $\partial\mathscr{D}$ *once; and*
2. $E_\lambda(-2) \in V_\lambda \setminus U_\lambda$ *for every* $\lambda \in \partial\mathscr{D}$.

In particular, there exists a ***Mandelbrot-like set*** *in the interior of* $\mathscr{D}$ *such that the main cardioid hyperbolic component has period one.*

A transcendental family containing SD

For the exponential maps:

$$E_\kappa(z) = e^z + \kappa.$$

Theorem (Rempe-Schleicher, 2009)

For any $\theta \in \mathscr{B}$ and $p \geq 1$, there exists $\kappa \in \mathbb{C}$ such that E_κ contains a p-cycle of Siegel disks with rotation number θ.

From the family E_κ, by using a similar surgery, we obtain:

Theorem (Y., 2021)

Let θ be a Brjuno number and $p \geq 1$ an integer. Then there are a, b and $u = u(a,b) \in \mathbb{C} \setminus \{0\}$ such that

$$h_{a,b}(z) = u\,\frac{z-b}{z-a}\,e^z$$

*has a p-cycle of Herman rings (can be **unbounded**) of rotation number θ and a super-attracting p-cycle.*

It is hard to draw a picture ...

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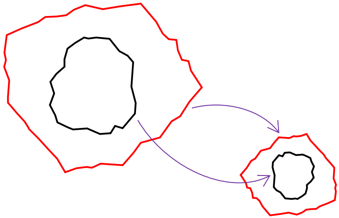
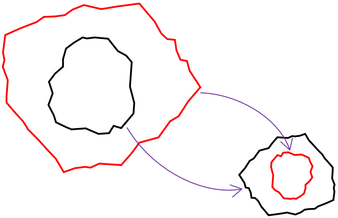
More examples of Herman rings

- **Nested** Herman rings: A 2-cycle Herman rings $\{A_1, A_2\}$ is called *nested* if $A_1 \subset \mathrm{int}(A_2)$ or $A_2 \subset \mathrm{int}(A_1)$.



Non-nested Herman rings



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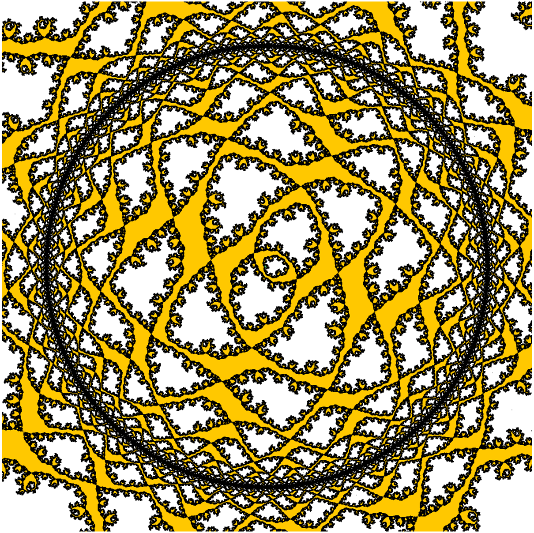
Theorem (Y. 2020; Based on Shishikura 1987)

For any $\theta \in \mathscr{B}$, there exists $(r,t) \in (0,1)^2$ such that $h_{r,t}$ has a 2-cycle of ***nested*** *Herman rings of rotation number θ, where*

$$h_{r,t}(z) = e^{2\pi \text{i}t} \left(\frac{z - \frac{1}{r}}{1 - \frac{z}{r}} \right)^3 \frac{z - r^2 e^{-2\pi \text{i}t}}{1 - r^2 e^{2\pi \text{i}t} z}.$$

Nested Herman rings



HR of $h_{r,t}$, where
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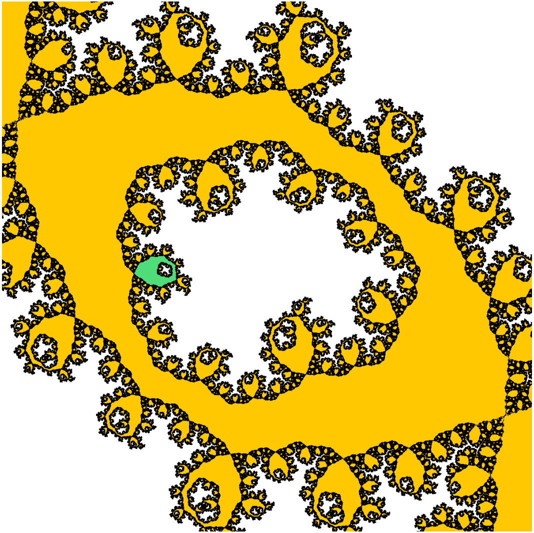
Nested Herman rings



Zoom of the HR of $h_{r,t}$, where
$r = 1/40$
$t \approx 0.34172383$

- Find the formulas and draw pictures of more complicated configurations of p-cycle of Herman rings. For example, 3-cycle of nested Herman rings.
- Study the degenerate Herman rings, i.e, what happens as the modulus of the Herman ring tends to 0 (or to ∞)?
- Does there exist a rational map with exactly 3 critical values having a Herman ring? (The period is at least 4, according to Hu-Xiao, 2019)

THANK YOU FOR YOUR ATTENTION !

Some questions on Herman rings

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