Parabolic renormalization and applications

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Complex dynamical systems

$f : X \to X$

$$x \mapsto f(x) \mapsto f(f(x)) \mapsto f(f(f(x))) \mapsto \cdots \mapsto f^{\circ n}(x) \mapsto \cdots$$

Complex dynamical systems: X complex manifold, f holomorphic.

- Complex 1-dim:

$$X = \widehat{\mathbb{C}},\ f \text{ rational map (e.g. } z^2 + c,\ z^n + \lambda / z^n);$$
$$X = \mathbb{C},\ f \text{ transcendental entire function (e.g. } \lambda e^z, \lambda \sin z).$$

- Complex high dim: $X = \mathbb{C}^n$ with $n \geqslant 2$ (e.g. Henon map):

$$f : \mathbb{C}^2 \to \mathbb{C}^2 : \begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z^2 + c - aw \\ z \end{pmatrix},\ a \neq 0.$$

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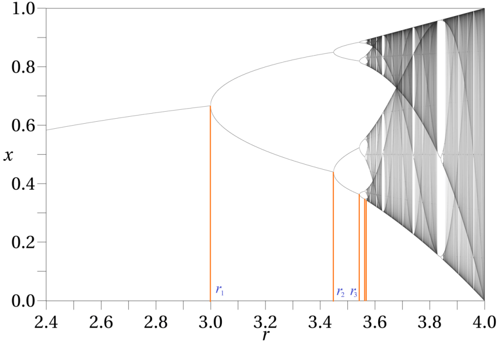
Partially motivated by: **Newton's method** for root-finding: $f(z) = z - \dfrac{g(z)}{g'(z)}$.

Model of population variation: **Logistic map**

$$x_{n+1} = rx_n(1 - x_n), \quad \text{where } r \in [0,4] \text{ and } x_n \in [0,1].$$



Period-doubling bifurcations:

The limit

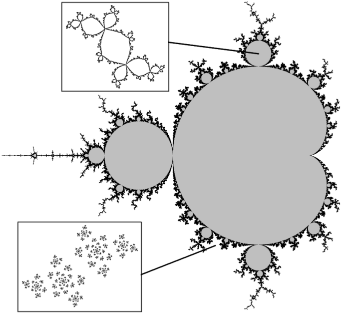
$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

exists, and equals to

4.6692016091029906...

Universality of the **Feigenbaum constant**: δ is the same for $x_{n+1} = r\sin(x_n)$, ...

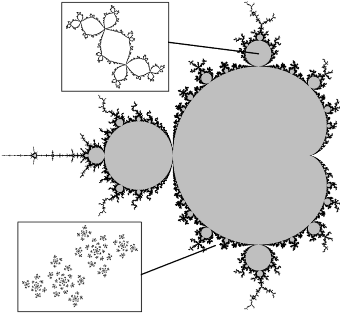
- Lanford (Bull. AMS 1982; Computer-assisted)
- Lyubich (Ann. Math. 1999; non-numerical proof by complex dynamics)



Objects in 1-dim

- **Dynamical planes** (Julia and Fatou sets / components, ...)
- **Parameter spaces** (Mandelbrot / Multibrot set, bifurcation loci, hyperbolic components, capture domains, ...)

Dimension, measure, connectivity, local connectivity, ergodicity, rigidity, hyperbolic density ...



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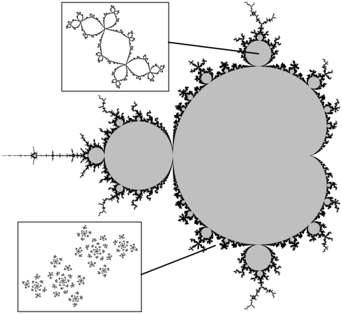
Douady (1980s): You first plow in the dynamical planes and then harvest in the parameter space.

Main tools (far from exhaustive):

- *Classical*: Montel, distortion theorems, modulus, hyperbolic metric;
- *QC*: Surgery, holomorphic motion, Thurston theorem, Teichmüller theory;
- *Dynamics*: renormalization theories (various: polynomial-like, sector, parabolic and near-parabolic, pacman);
- Puzzles, orbifold, several complex variables, arithmetics ...



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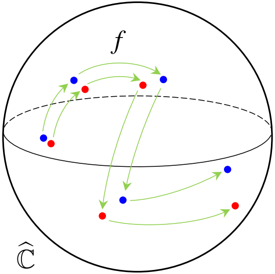
Fatou and Julia



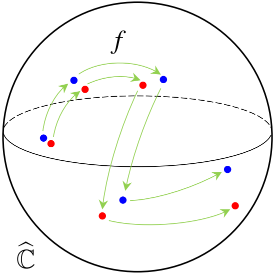
Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be rational.

The **Fatou set** (or stable set) of f:

$F(f) := \{z \in \widehat{\mathbb{C}} : \{f^{\circ n}\}_{n\in\mathbb{N}}$ is equicontinuous at $z\}$.

The **Julia set** (or chaotic set) $J(f) := \widehat{\mathbb{C}} \setminus F(f)$.

Fatou and Julia



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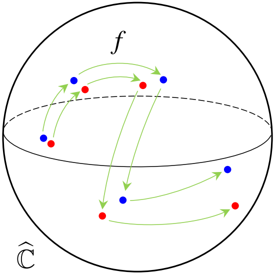
The following maps satisfy $J(f) = \widehat{\mathbb{C}}$:

$$f(z) = \frac{(z^2+1)^2}{4z(z^2-1)}, \quad f(z) = 1 - \frac{2}{z^2}.$$

If $J(f) \neq \widehat{\mathbb{C}}$, then they have no interior. How complex can they be?

Fatou and Julia



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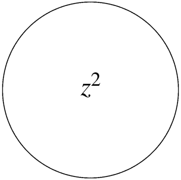
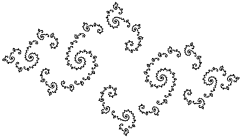
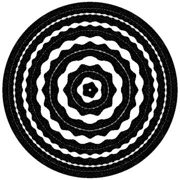




$z^2 - 0.78 + 0.23i$



$z^2 + 10^{-5}/z^3$

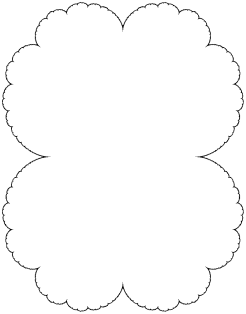
Sullivan (1983):
If f is *hyperbolic*,
then $\dim_H J(f) < 2$.

Pf. Distortion theorem.

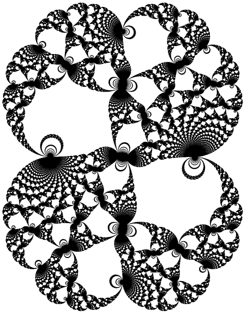
Also true for *parabolic* maps.

Parabolic bifurcations

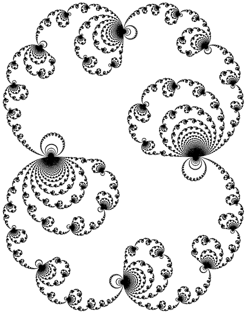
Douady (1994, see also Qiu-Yin 1995) and Wu (1999) proved that $J(f)$ **does not move continuously** at parabolic parameters. One of the important phenomenon is the **parabolic bifurcation**.



$z^2 + \frac{1}{4}$

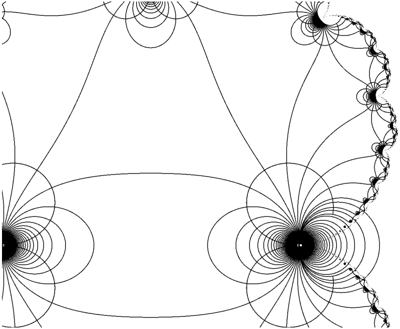


$z^2 + \frac{1}{4} + \varepsilon_1$

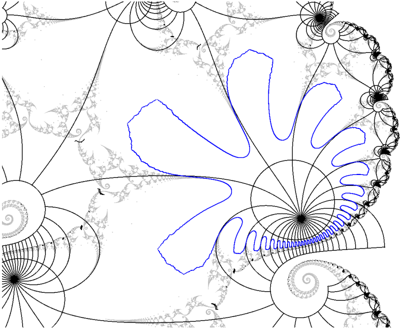


$z^2 + \frac{1}{4} + \varepsilon_2$

Although $J(f)$ **does not move continuously** at parabolic parameters, it turns out that the (perturbed) *Fatou coordinate* does (restricting on some truncated chessboard).



$z^2+\frac{1}{4}$

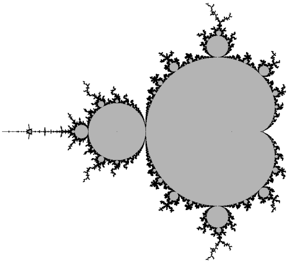


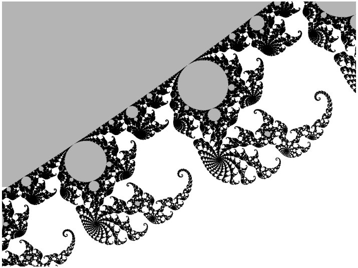
$z^2+\frac{1}{4}+\varepsilon$

Hausdorff dim of J and M

Denote $p_c(z) = z^2 + c$, where $c \in \mathbb{C}$. The **Mandelbrot set** is

$$\mathrm{M} := \{c \in \mathbb{C} : \lim_{n\to\infty} p_c^{\circ n}(0) \neq \infty\}.$$



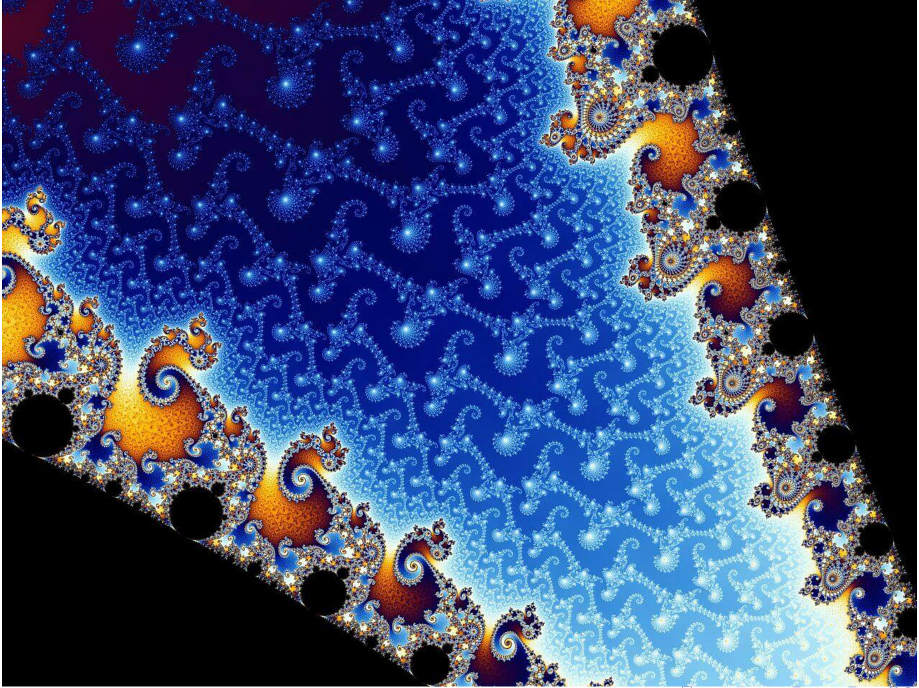


Theorem (Shishikura, Ann. Math., 1998)

- $\dim_H(J(p_c)) = 2$ *for generic* $c \in \partial\mathrm{M}$;
- $\dim_H(\partial\mathrm{M}) = 2$.

Ingredients in the proof:

- Parabolic perturbation and *twice* near-parabolic renormalization;
- Holomorphic motion.



Area of Julia sets

For polynomials,

Zero area of Julia sets

$\Rightarrow$ (NILF) No invariant line field Conjecture

$\Rightarrow$ (HD) Hyperbolic density Conjecture

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Zero area: Very fruitful results, especially for quadratic polynomials:

- (Douady-Hubbard, Lyubich 1980s):
  Geometrically finite;
- (Lyubich, Shishikura 1991):
  At most finitely (polynomial-like)renorm and without irrationally indifferent periodic points (*Siegel* or *Cremer*);
- (Petersen 1996, McMullen 1998, Yampolsky 1999, Petersen-Zakeri 2004):
  Siegel disks with almost all rotation numbers;
- (Yarrington 1995, Avila-Lyubich 2008, A. Dudko-Sutherland 2020):
  Some ∞- renorm.

Area of Julia sets

Conjecture (Douady, 1990s)

There exist quadratic polynomials having Julia sets of **positive area**.

Theorem (Buff-Chéritat, Ann. Math., 2012)

There exist quadratic polynomials having Julia sets of positive area, and moreover, they either have a Cremer point, a Siegel point, or are ∞-renormalizable.

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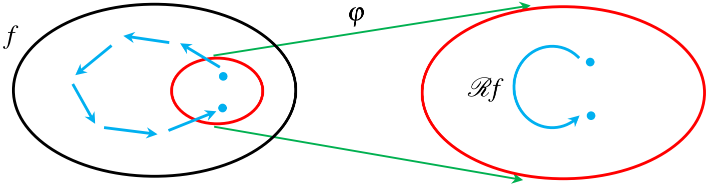
Theorem (Avila-Lyubich, Ann. Math., 2022)

There exist ∞-renormalizable quadratic polynomials having locally connected *Julia sets of positive area.*

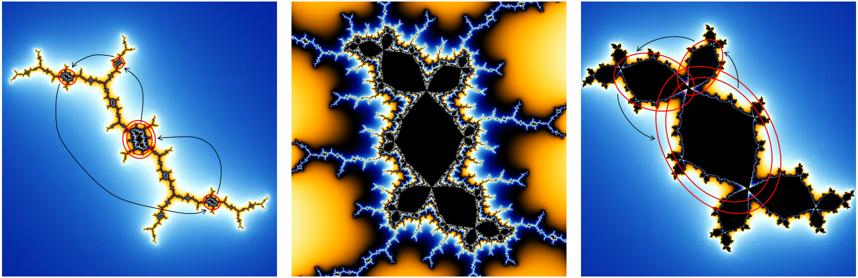
A key ingredient in both proofs:

Inou-Shishikura's invariant class under **parabolic / near-parabolic renormalization** to control the post-critical sets of perturbations of polynomials.

(Polynomial-like) renormalization



Renormalization $\mathscr{R}f$ = first return map of f after rescaling
$= \varphi \circ f^{\circ k} \circ \varphi^{-1}$ (if return time $\equiv k$)

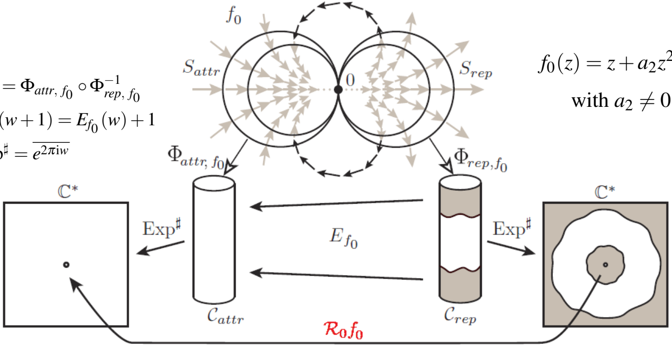


Parabolic and near-parabolic renormalizations

$E_{f_0} = \Phi_{attr,\, f_0} \circ \Phi^{-1}_{rep,\, f_0}$

$E_{f_0}(w+1) = E_{f_0}(w) + 1$

$\mathrm{Exp}^\sharp = \overline{e^{2\pi i w}}$



$f_0(z) = z + a_2 z^2 + \cdots$

with $a_2 \neq 0$

$\mathscr{R}_0 f_0(0) = 0$

$(\mathscr{R}_0 f_0)'(0) = 1$

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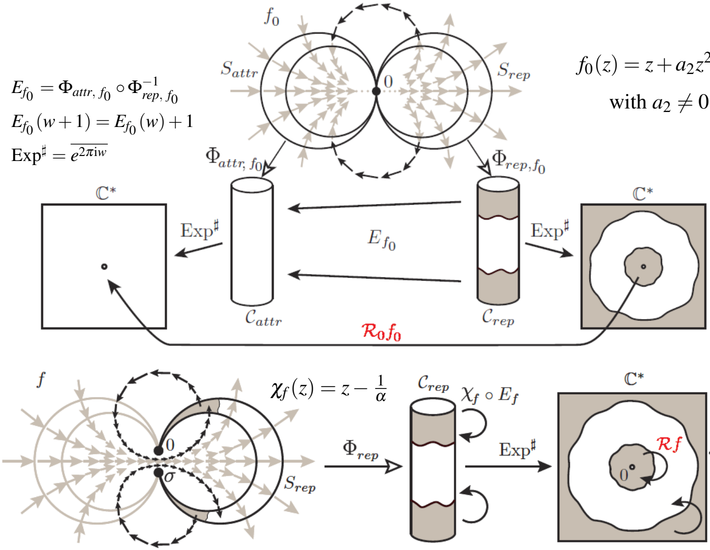

$f(z) = e^{2\pi i\alpha} z + O(z^2)$ with small α, $|\arg(\alpha)| < \frac{\pi}{4}$

Two invariant classes

To prove $\dim_H(\partial \mathrm{M}) = 2$, Shishikura (1998) introduced a class $\mathscr{F}_0$ of holomorphic parabolic maps, s.t.

$$\mathscr{R}_0(\mathscr{F}_0) \subset \mathscr{F}_0,$$

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For a near-parabolic map $f = e^{2\pi i\alpha} h$ with $h \in \mathscr{F}_0$, the **near-parabolic renorm** can be expressed as a skew product:

$$\mathscr{R} : (\alpha, h) \mapsto (\tfrac{1}{\alpha}, \mathscr{R}_\alpha h),$$

where $\mathscr{R}_\alpha$ is the **renorm in the fiber direction** and $(\mathscr{R}_\alpha h)(z) = z + O(z^2)$. However,

$$\mathscr{R}_\alpha(\mathscr{F}_0) \not\subset \mathscr{F}_0 \quad \text{for any small } \alpha \neq 0.$$

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Inou and Shishikura (2008) introduced a class $\mathscr{F}_1$ of holomorphic parabolic maps, s.t.

$$\mathscr{R}_\alpha(\mathscr{F}_1) \subset \mathscr{F}_1 \text{ for all sufficiently small } \alpha.$$

Then $\mathscr{R}$ can be iterated infinitely many times on $f = e^{2\pi i \alpha} h$, where $h \in \mathscr{F}_1$ and α is of sufficiently **high type** (the continued fraction coefficients are all large enough).

MLC conjecture

Conjecture (Douady-Hubbard, 1980s)

The Mandelbrot set is locally connected.

For quadratic polynomials $p_c(z) = z^2 + c$,

$$\text{MLC} \Longrightarrow \text{NILF conjecture} \Longleftrightarrow \text{HD conjecture.}$$

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MLC holds in the following cases:

- At most finitely renormalizable (Yoccoz 1990s);
- Some infinitely renormalizable (Lyubich 1997, Jiang 2003, Kahn 2006, Kahn-Lyubich 2008-2009, Levin 2011, Cheraghi-Shishikura 2015, Dudko-Lyubich 2023-2024);
- On $\mathbb{R}$ (Dudko-Kahn-Lyubich 2023, announced).

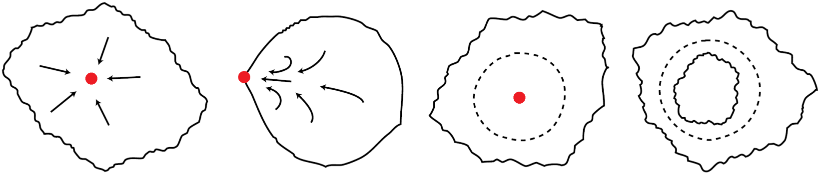
Cheraghi and Shishikura's proof is based on the near-parabolic renormalization.

Sullivan's eventually periodic theorem

Theorem (Sullivan, Ann. Math., 1985)

The Fatou components of all rational maps are eventually periodic.

Classification of periodic Fatou components of rational maps:



(super-)attracting basin 1884, 1904 | parabolic basin 1897 | Siegel disk 1942 | Herman ring 1979

Regularity of boundaries of Fatou components

Theorem (Roesch-Yin, Sci. China Math., 2022)

All bounded attracting and parabolic components of polynomials are Jordan domains.

Puzzle techniques: Kozlovski-Shen-van Strien nest + Kahn-Lyubich modulus lemma.

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Conjecture (Douady-Sullivan, 1986)

The Siegel disk of any rational map (deg $\geqslant 2$) is a Jordan domain.

Theorem (Petersen-Zakeri, Ann. Math., 2004)

For almost all α, the Siegel disk of $P_\alpha(z) = e^{2\pi i\alpha}z + z^2$ is a Jordan domain.

Theorem (Zhang, Invent. Math., 2011)

All bounded type Siegel disks of rational maps are quasi-disks.

Tools: Blaschke models + surgery + (relative Schwarz lemma).

High type Siegel disks

Theorem (Shishikura-Y., JEMS, 2024)

*Let α be of sufficiently **high type**, and assume that $P_\alpha(z) = e^{2\pi i\alpha} z + z^2$ has a Siegel disk Δ_α. Then*

- *$\partial\Delta_\alpha$ is a Jordan curve; and*
- *$\partial\Delta_\alpha$ contains a critical point if and only if α is of Herman type.*

High type:

$$\mathrm{HT}_N := \{\alpha = [0; a_1, a_2, \cdots] \in (0,1) \setminus \mathbb{Q} \mid a_n \geqslant N \text{ for all } n \geqslant 1\}$$

for some large N.

Main tool: **Near-parabolic renormalization** (based on Inou-Shishikura's class $\mathscr{F}_1$).

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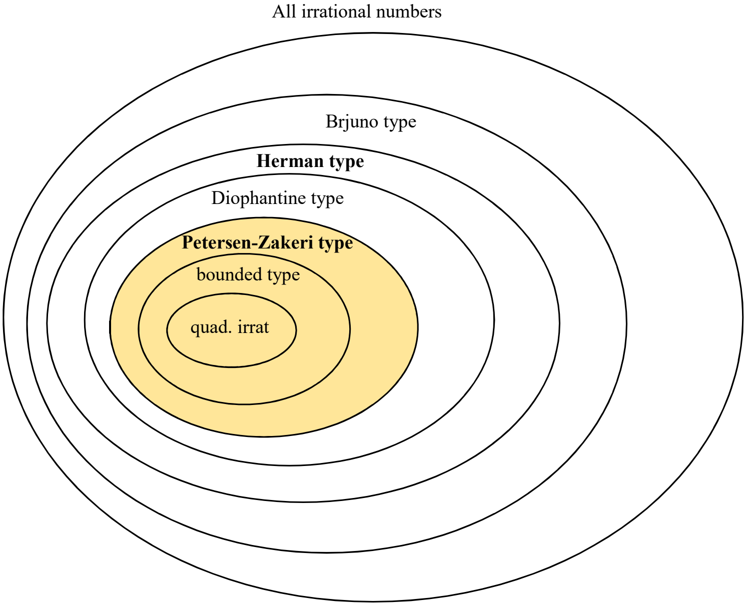
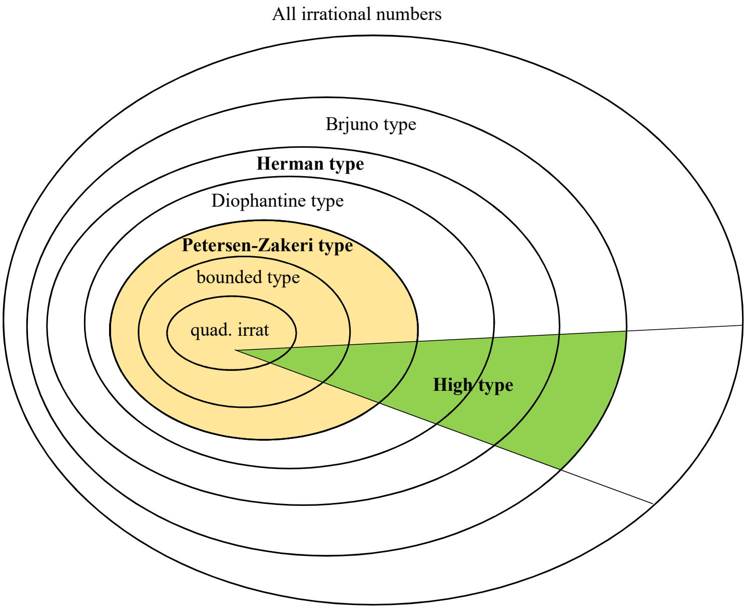
Cheraghi (2022, arXiv), independently, gave another proof.


All irrational numbers
Brjuno type
Herman type
Diophantine type
Petersen-Zakeri type
bounded type
quad. irrat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Cantor Julia sets

Theorem (Qiu-Yin, Sci. China Math., 2009; Kozlovski-van Strien, PLMS, 2009)

The Julia set of a polynomial is a Cantor set ***if and only if*** *each critical component of the filled-in Julia set is aperiodic.*

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Question (folk)

Does there exist a Cantor Julia set having positive area?

Based on the parabolic perturbation and Shishikura's result:

Theorem (Y., IMRN, 2021)

There exist Cantor Julia sets having Hausdorff dimension two.

More Julia sets of positive area

Theorem (Y., 2024)

For any meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ in Shishikura's class $\mathscr{F}_0$, there exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $f_\alpha = e^{2\pi i\alpha} f$ has a Cremer point (resp. Siegel disk) and a Julia set of positive area.

In particular, the following maps belong to $\mathscr{F}_0$:

1. ***Polynomials**: $z \mapsto z(1+z)^n$ for $n \geqslant 1$, and $z \mapsto z(1+z)^2(1+\frac{1}{2}z)$;*
2. ***Rationals**: $z \mapsto z/(1-z)^n$ for $n \geqslant 2$, and $z \mapsto z(1-z)^3/\left(1-\frac{8}{9}z\right)$.*

Remark:

- Every $f_\alpha = e^{2\pi i\alpha} f$ with $f : \mathbb{C} \to \widehat{\mathbb{C}}$ in $\mathscr{F}_0$ is **not** polynomial-like renorm;
- Polynomial case: Qiao-Qu (2020) for $n = 2$, X. Zhang (2022) for $n \geqslant 21$.

The renormalization in the proof **depends only on near-parabolic**, but not on sector and others.

Other harvests and further developments

1. Cheraghi (CMP 2013, ASENS 2019): Zero area of post-critical set;
2. Cheraghi-Chéritat (Invent. Math. 2015): Marmi-Moussa-Yoccoz conjecture;
3. Cheraghi-DeZotti-Y. (arXiv 2020): H-dim of post-critical set and dim paradox;
4. Y. (Adv. Math. 2024): Smooth degenerate Herman rings;

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Extend to higher local degrees:

- (Chéritat 2022): Invariant class for unicritical maps (No numerical calculations);
- (Y. 2024) Invariant class for cubic unicritical maps (following Inou-Shishikura).

Developments:

- Extend high type to more rotation numbers:
  [Kapiamba 2022], [D. Dudko-Lyubich 2022], [Qu 2024];
- Extend unicritical to multi-critical?

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Questions

(1) (Milnor 1992) Does there exist a Cremer Julia set of area zero?

(2) (Avila-Lyubich 2015) Does there exist $c \in \mathbb{R}$ s.t. $\text{area}(J(z^2+c)) > 0$?

Thank you for your attention!