Dimension paradox of irrationally indifferent attractors

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$$f(z) = \lambda z + a_2 z^2 + \cdots, \text{ where } \lambda \in \mathbb{C} \setminus \{0\}.$$

$f : U \to \mathbb{C}$ defines a dynamical system: $z_{n+1} = f(z_n)$.

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3. $\lambda = e^{2\pi i \alpha}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$: **irrationally indifferent**

Irrationally indifferent

Near $0, f(z) = e^{2\pi i\alpha} z + O(z^2)$ is close to the aperiodic rotation $R_\alpha : z \mapsto e^{2\pi i\alpha} z$ $(\alpha \in \mathbb{R} \setminus \mathbb{Q})$.

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Theorem (Cremer, 1928)

*Any **rational map** f ($\deg(f) = d \geq 2$) **cannot** be locally linearized at 0 if*

$$\limsup_{q\to\infty} \sqrt[d^q]{1/|\lambda^q - 1|} = \infty.$$

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Diophantine condition of order $\leq \kappa$:

$$\mathscr{D}(\kappa) := \left\{ \alpha \in (0,1) : \; \exists \varepsilon > 0 \text{ s.t. } \left| \alpha - \frac{p}{q} \right| > \frac{\varepsilon}{q^\kappa} \text{ for every rational } \frac{p}{q} \right\}.$$

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1. $\cap_{\kappa>2} \mathscr{D}(\kappa)$ has full measure.
2. $\mathscr{D}(2)$ has measure 0 and $\alpha \in \mathscr{D}(2)$ is called **bounded type** (**constant type**). This is equivalent to the continued fractional expansion

$$\alpha = [0; a_1, a_2, \cdots, a_n, \cdots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

   satisfies $\sup_n \{a_n\} < \infty$.
3. Let $p_n/q_n = [0; a_1, \cdots, a_n]$. Then $\alpha \in \mathscr{D} = \cup_{\kappa \geq 2} \mathscr{D}(\kappa) \Leftrightarrow \sup_n \{ \frac{\log q_{n+1}}{\log q_n} \} < \infty$.

Brjuno type

Theorem (Brjuno, 1965)

*Any **holomorphic germ** f **can** be locally linearized at* 0 *if* α *belongs to*

$$\mathscr{B} = \left\{ \alpha \in (0,1) \setminus \mathbb{Q} : \sum_n \frac{\log q_{n+1}}{q_n} < \infty \right\}.$$

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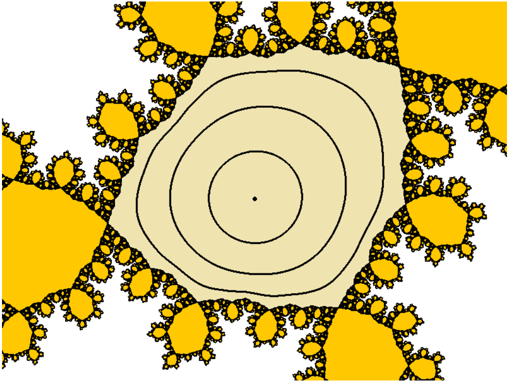
Conjecture (Douady, 1986): If a non-linear holomorphic function (entire or rational) is locally linearizable, then the rotation number is necessarily in $\mathscr{B}$.

Remark: This conjecture is still open for cubic polynomials.

Siegel disk and Cremer point

The **maximal** open subset of U containing 0 in which f is conjugated to its linear part $R_\alpha : z \mapsto e^{2\pi i\alpha} z$ is the **Siegel disk** Δ_f centered at 0.



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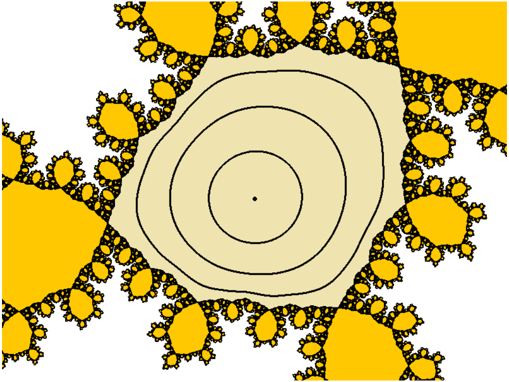

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If f not linearizable at 0, then 0 is called a **Cremer point** of f.

Aim and motivation

A rational map either acts ergodically on the sphere, or its **post-critical set** $P(f)$ (i.e., the closure of the critical orbits) behaves as a measure-theoretic **attractor**:

$$\lim_{n\to\infty} d_{\widehat{\mathbb{C}}}(f^{\circ n}(z), P(f)) = 0, \text{ for almost all } z \in J(f).$$

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Aim: To understand the local dynamics near an irrationally indifferent fixed point:

- Near the closure of the Siegel disk;
- Near the Cremer point;

and to study the properties of the irrationally indifferent **attractors** (i.e., the post-critical set associated to the Siegel disk or Cremer point).

Recall

$$P_\alpha(z) = e^{2\pi i\alpha}z + z^2 \text{ with } \alpha \in \mathbb{R}\setminus\mathbb{Q}.$$

Topology of the boundaries of Siegel disks:

- (Douady-Herman, Zakeri, Shishikura, Zhang, 1980s-2011): For rational f with $\deg(f) \geq 2$, $\partial\Delta_f$ is a Jordan curve if α is of **bounded type**.
- (Petersen-Zakeri, 2004): for **almost all** α, $\partial\Delta_{P_\alpha}$ is a Jordan curve.
- (Zhang, 2014): for all non-linear poly. f and **almost all** α, $\partial\Delta_f$ is a Jordan curve.
- (Geyer, Chéritat, Keen-Zhang, Zakeri, Zhang, 2001-2016) The boundaries of the Siegel disks of some **transcendental entire functions** are Jordan domains.

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Remark: The topology of $\partial\Delta_{P_\alpha}$ has not been completely understood.

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Geometry of the boundaries of Siegel disks:

- (McMullen, 1998): $\dim_H(\partial\Delta_{P_\alpha}) \leq \dim_H(J_{P_\alpha}) < 2$ if α is of **bounded type**.
- (Graczyk-Jones, 2002): if $\partial\Delta_f$ quasicircle and $\text{cp} \in \partial\Delta_f$, then $\dim_H(\partial\Delta_f) > 1$.
- (Avila-Buff-Chéritat, 2004): $\exists\, \alpha$ s.t. $\partial\Delta_{P_\alpha}$ is smooth, hence $\dim_H(\partial\Delta_{P_\alpha}) = 1$.
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Problem: What are the topology and geometry of the attractors of the **Cremer** case?

Hedgehogs

Let f be a non-linear holomorphic system with the form

$$f(z) = e^{2\pi i\alpha} z + \mathscr{O}(z^2), \text{ where } \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Pérez-Marco (1997) proved that if f and f^{-1} are defined and univalent in a **neighborhood** of the closure of a Jordan domain $U \subset \mathbb{C}$ containing 0, then there exists a compact, full and connected set $K = K_{f,U}$ contained in $\overline{U}$ such that $0 \in K$, $K \cap \partial U \neq \emptyset$ and $f(K) = f^{-1}(K) = K$.

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If ∂U is C^1-smooth, Pérez-Marco (1994, 1996) proved that

- K is in unique;
- the non-linearizable hedgehogs (i.e. 0 is a Cremer point) have no interior and they are **not locally connected** at any point different from the fixed point.

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Milnor (2006): “As far as I know, no useful picture of the Julia set near such a point has ever been produced, either by computer or by theory.”

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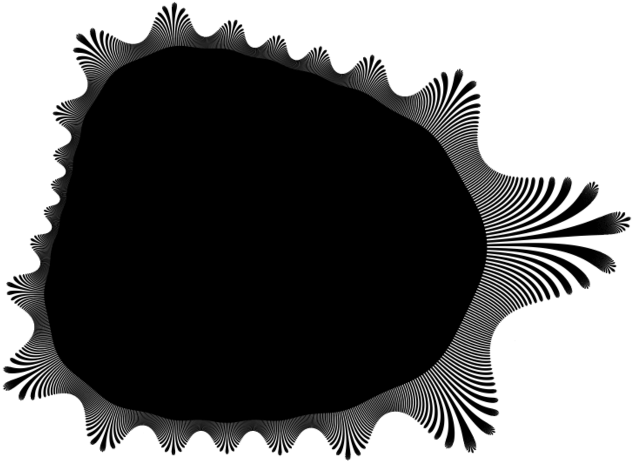
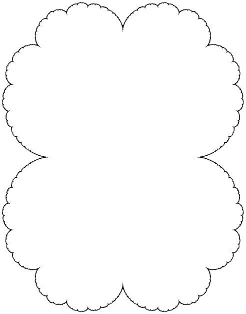
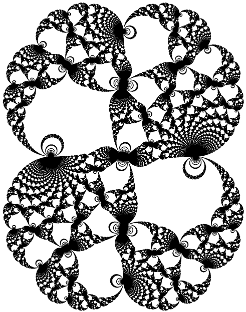


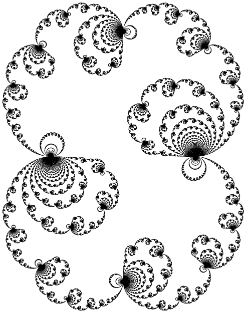
A hedgehog of a **toy model** (not a holomorphic map) by Chéritat

Parabolic bifurcation

The irrationally indifferent and parabolic cases are related by dynamical bifurcations. It was known that the Julia set **does not depend continuously** at the parabolic parameters. One of the important phenomenon is **parabolic bifurcation**.

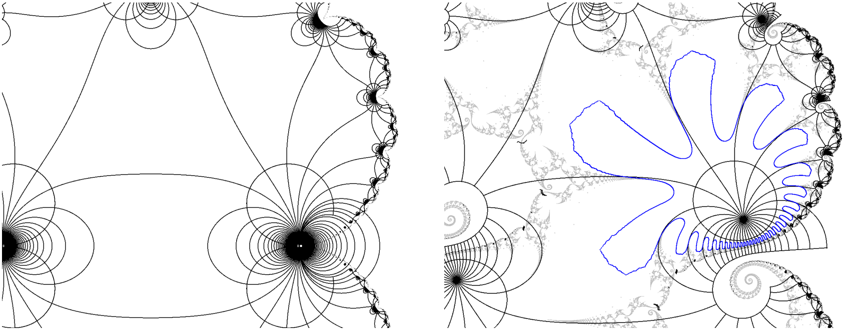






Parabolic bifurcation

Although the Julia set **does not depend continuously** at the parabolic parameters, it turns out that the (perturbed) Fatou coordinate does (restricted on some truncated chessboard).



Developments

The main tools to analyze such bifurcation are **Fatou coordinates** and **horn maps**, which were developed by:

1. Douady-Hubbard (1984-85): landing of external rays at the M-set (Orsay notes), the straightening of polynomial-like maps;
2. Lavaurs (1989): the non-local connectivity of the connectedness locus of cubic polynomials (Ph.d thesis);
3. Douady (1994): the discontinuity of the Julia sets;
4. Shishikura (1998): the Hausdorff dim of ∂M (an invariant class);
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6. Inou-Shishikura (2006): near-parabolic renormalization (a **new invariant class**).

Remark: **Near-parabolic renormalization** $\mathscr{R}$ (acted on Inou-Shishikura's class) is a very powerful tool to study the quadratic maps with irrationally indifferent fixed points. In particular, one can use this tool to study the properties of Cremer attractors.

Inou-Shishikura's class

Let $P(z) = z(1+z)^2$. Then P has a parabolic fixed point at 0 and **two simple** critical points -1 and $\mathrm{cp}_P = -\frac{1}{3}$ with $P(-1) = 0$ and $\mathrm{cv} = P(\mathrm{cp}_P) = -\frac{4}{27}$.

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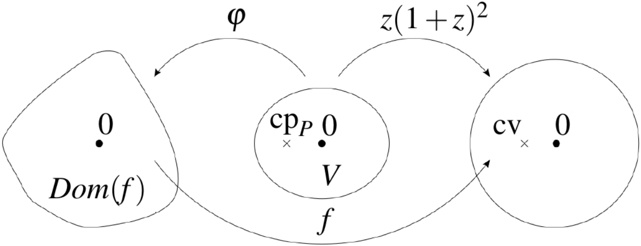
$$IS_0 := \left\{ f = P \circ \varphi^{-1} : \varphi(V) \to \mathbb{C} \;\middle|\; \begin{array}{l} \varphi : V \to \mathbb{C} \text{ is univalent,} \\ \varphi(0) = 0,\ \varphi'(0) = 1 \end{array} \right\}.$$



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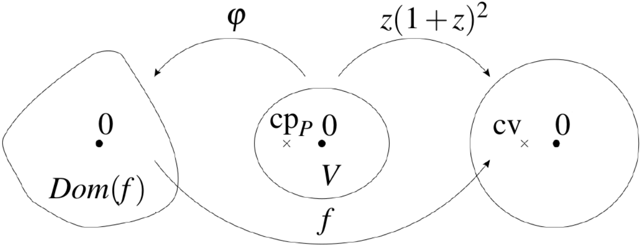

$$IS_\alpha = \{ f(z) = f_0(e^{2\pi i\alpha} z) : e^{-2\pi i\alpha} \cdot Dom(f_0) \to \mathbb{C} \mid f_0 \in IS_0 \}.$$

Let N be a positive integer. Denote

$$\mathrm{HT}_N := \{\alpha = [0; a_1, a_2, \cdots] \in (0,1) \setminus \mathbb{Q} \mid a_n \geq N \text{ for all } n \geq 1\}.$$

Theorem (Inou-Shishikura, 2006)

There are two Jordan domains V and V' satisfying $V \Subset V'$ and an integer $N \geq 2$ such that for all $f \in IS_\alpha \cup \{P_\alpha\}$ with $\alpha \in (0, 1/N]$, then $\mathscr{R}f$ is well-defined so that

$$\mathscr{R}f = P \circ \psi^{-1} \in IS_{1/\alpha}.$$

Moreover, ψ extends to a univalent function from V' to $\mathbb{C}$. In particular, if $\alpha \in \mathrm{HT}_N$, then $\mathscr{R}$ can be iterated infinitely many times.

Infinite renormalization

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$$\mathscr{R}f = P \circ \psi^{-1} \in IS_{1/\alpha}.$$

Moreover, ψ extends to a univalent function from V' to $\mathbb{C}$. In particular, if $\alpha \in \mathrm{HT}_N$, then $\mathscr{R}$ can be iterated infinitely many times.

Problem: How large is N?

Infinite renormalization

Let N be a positive integer. Denote

$$\mathrm{HT}_N := \{\alpha = [0; a_1, a_2, \cdots] \in (0,1) \setminus \mathbb{Q} \mid a_n \geq N \text{ for all } n \geq 1\}.$$

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Problem: How large is N? 100?

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Problem: How large is N? 100? Maybe even 10^6 is a good result.

Post-critical sets and Herman's condition

Theorem (Inou-Shishikura, 2006)

$\forall \alpha \in \mathrm{HT}_N$ *and all* $f \in IS_\alpha \cup \{P_\alpha\}$, $\mathrm{Dom}(f)$ *contains the post-critical set*

$$\Lambda_f = \overline{\bigcup_{k\in\mathbb{N}} f^{\circ k}(\mathrm{cp}_f)}.$$

Remark: $\Lambda_f \cup \Delta_f$ is the **maximal hedgehog** of f centered at 0, where Δ_f is the Siegel disk (if any) of f centered at 0.

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Herman's condition:

$$\mathscr{H} := \left\{ \alpha \in (0,1)\setminus\mathbb{Q} \;\middle|\; \begin{array}{l} \text{every orientation-preserving analytic circle diffeo.} \\ \text{of rotation number } \alpha \text{ is anal. conj. to } z \mapsto e^{2\pi i\alpha} z \end{array} \right\}.$$

Herman-Yoccoz (1984): $\mathscr{D} \subsetneq \mathscr{H} \subsetneq \mathscr{B}$; and
Yoccoz (2002): Arithmetic characterization of $\mathscr{H}$.

Theorem (Cheraghi, 2017)

$\forall \alpha \in \mathrm{HT}_N$ *and all* $f \in IS_\alpha \cup \{P_\alpha\}$*, one of the following statements hold:*

- *If* $\alpha \in \mathscr{H}$*, then* $\mathrm{cp}_f \in \partial\Delta_f$ *and* $\Lambda_f = \partial\Delta_f$ *is a* ***Jordan curve****;*
- *If* $\alpha \in \mathscr{B} \setminus \mathscr{H}$*, then* $\mathrm{cp}_f \notin \partial\Delta_f$ *and* Λ_f *is a* ***one-side hairy circle****;*
- *If* $\alpha \notin \mathscr{B}$*, then* 0 *is Cremer and* Λ_f *is a* ***Cantor bouquet****.*

Topology of attractors

Theorem (Cheraghi, 2017)

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Independently,

Theorem (Shishikura-Y., 2016)

$\forall\alpha \in \mathrm{HT}_N$ *and all* $f \in IS_\alpha \cup \{P_\alpha\}$, *then*

- *If* $\alpha \in \mathscr{B}$ $(\Leftrightarrow \Delta_f \neq \emptyset)$, *then* $\partial\Delta_f$ *is a* ***Jordan curve****;*
- $\mathrm{cp}_f \in \partial\Delta_f$ *if and only if* $\alpha \in \mathscr{H}$.

Remark: Ghys, Herman, Pérez-Marco, Geyer, Chéritat-Roesch, Benini-Fagella studied the sufficiency of the Herman's condition.

Cantor bouquet

Definition

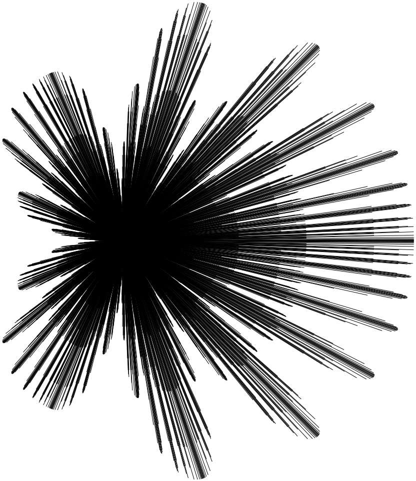
A **Cantor bouquet** is a compact subset of the plane which is homeomorphic to a set of the form

$$\{re^{2\pi i\theta} \in \mathbb{C} : 0 \leq r \leq R(\theta)\},$$

where $R : \mathbb{R}/\mathbb{Z} \to [0, \infty)$ satisfies

1. $R^{-1}(0)$ is dense in $\mathbb{R}/\mathbb{Z}$;
2. $(\mathbb{R}/\mathbb{Z}) \setminus R^{-1}(0)$ is dense in $\mathbb{R}/\mathbb{Z}$;
3. for each $\theta_0 \in \mathbb{R}/\mathbb{Z}$,

$$\limsup_{\theta \to \theta_0^+} R(\theta) = R(\theta_0) = \limsup_{\theta \to \theta_0^-} R(\theta).$$



One-sided hairy circle

Definition

A **one-sided hairy circle** is a compact subset of the plane which is homeomorphic to a set of the form

$$\{re^{2\pi i\theta} \in \mathbb{C} : 1 \le r \le R(\theta)\},$$

where $R : \mathbb{R}/\mathbb{Z} \to [1, \infty)$ satisfies

1. $R^{-1}(1)$ is dense in $\mathbb{R}/\mathbb{Z}$;
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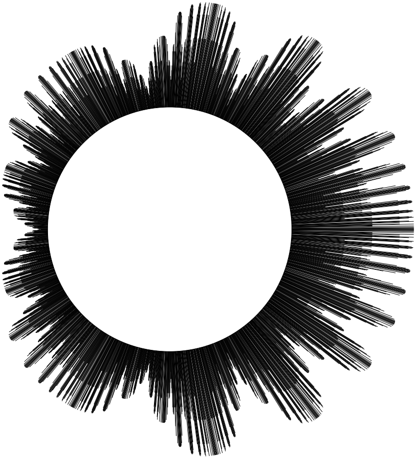

Main results

Cheraghi (2013, 2019): $\text{Area}(\Lambda_f) = 0$ for all $f \in IS_\alpha \cup \{P_\alpha\}$ with $\alpha \in \text{HT}_N$.

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There exists $N > 0$ such that for all $f \in IS_\alpha \cup \{P_\alpha\}$ with $\alpha \in \mathrm{HT}_N \setminus \mathscr{H}$, then

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Corollary

For all $\alpha \in \text{HT}_N \setminus \mathscr{H}$, $J(P_\alpha)$ has Hausdorff dimension two.

We use $\mathscr{E}_f$ to denote the set of *one-sided* endpoints (which are not contained in $\overline{\Delta}_f$) of the components of $\Lambda_f \setminus \overline{\Delta}_f$.

Theorem (Dimension paradox)

*There exist two **infinite** sets $J \subset \mathscr{B} \setminus \mathscr{H}$ and $S \subset (\mathbb{R} \setminus \mathbb{Q}) \setminus \mathscr{B}$ such that for all $\alpha \in (J \cup S) \cap \mathrm{HT}_N$ and all $f \in IS_\alpha \cup \{P_\alpha\}$, we have*

$$\dim_H\left((\Lambda_f \setminus \overline{\Delta}_f) \setminus \mathscr{E}_f\right) = 1 \quad \textit{and} \quad \dim_H(\mathscr{E}_f) = 2.$$

Remark: J contains $\alpha = [0; a_1, a_2, \cdots]$, where

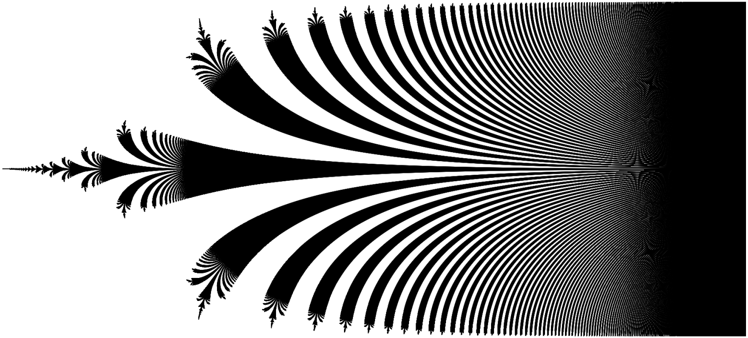
$$a_1 = N \text{ and } a_{n+1} = \lfloor e^{2^n a_n} \rfloor \text{ for } n \geq 1$$

and S contains $\alpha = [0; a_1, a_2, \cdots]$, where

$$a_1 = N \text{ and } a_{n+1} = \lfloor e^{e^{a_n}} \rfloor \text{ for } n \geq 1.$$

Results on exponential maps

Devaney-Krych (1984): $J(\lambda e^z)$ is a Cantor bouquet, for $0 < \lambda < 1/e$.
McMullen (1987): $\dim_H(J(\lambda e^z)) = 2$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.



Silva (1988): hairs are C^∞.
Karpińska (1999): $\dim_H(\textit{end points}) = 2$ and $\dim_H(\textit{hairs without end points}) = 1$.

McMullen's criterion

Proposition (McMullen, 1987)

Let $\{\mathscr{K}_n\}_{n=0}^{\infty}$ be a sequence satisfying the **nesting conditions**. Let $\delta_{n+1} > 0$ such that for all $1 \leq i \leq l_n$ and $K_{n,i} \in \mathscr{K}_n$, we have

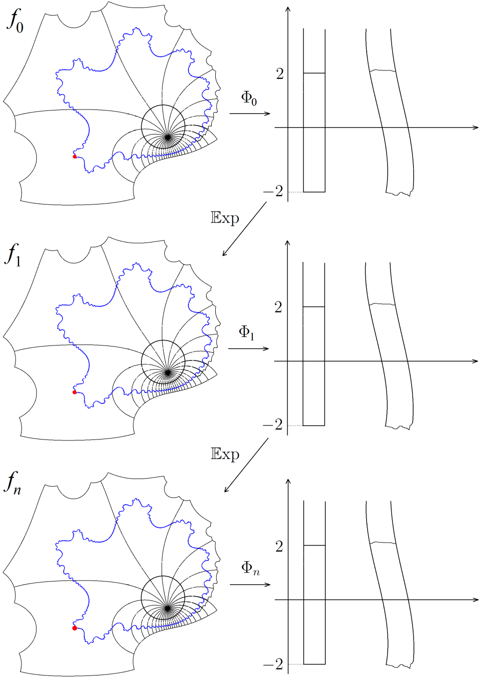
$$\operatorname{density}(\mathscr{K}_{n+1}, K_{n,i}) := \operatorname{density}\Big(\bigcup_{j=1}^{l_{n+1}} K_{n+1,j}, K_{n,i}\Big) \geq \delta_{n+1}.$$

Suppose that for each $K_{n,i} \in \mathscr{K}_n$ with $n \geq 1$,

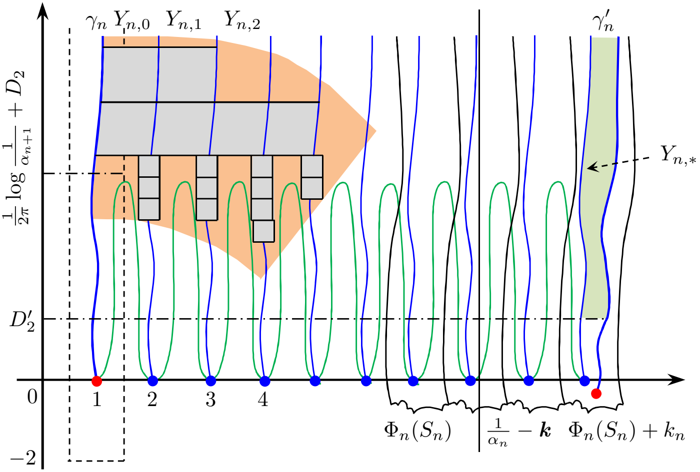
$$\operatorname{diam} K_{n,i} \leq d_n < 1, \text{ where } d_n \to 0 \text{ as } n \to \infty.$$

Then the Hausdorff dimension of $\bigcap_n \mathscr{K}_n$ satisfies

$$\dim_H\Big(\bigcap_{n\in\mathbb{N}} \mathscr{K}_n\Big) \geq 2 - \limsup_{n\to\infty} \frac{\sum_{k=1}^{n+1} |\log \delta_k|}{|\log d_n|}.$$


f_0
Φ_0
2
-2
$\mathbb{E}$xp
f_1
Φ_1
2
-2
$\mathbb{E}$xp
f_n
Φ_n
2
-2

Find sets satisfying the nesting conditions



THANK YOU FOR YOUR ATTENTION !