The high type quadratic Siegel disks are Jordan domains

YANG Fei

Nanjing University

joint with Mitsuhiro Shishikura

TOPICS IN COMPLEX DYNAMICS 2019
FROM COMBINATORICS TO TRANSCENDENTAL DYNAMICS

Barcelona University, Barcelona

March 25, 2019

Siegel disk and continued fractions

Let $0 < \alpha < 1$ be irrational, f non-linear holo., $f(0) = 0$ and $f'(0) = e^{2\pi i\alpha}$.
The *maximal* region in which f is conjugate to $R_\alpha(z) = e^{2\pi i\alpha}z$ is a simply connected domain Δ_f called the **Siegel disk** of f centered at 0.

Siegel disk and continued fractions

Let $0<\alpha<1$ be irrational, f non-linear holo., $f(0)=0$ and $f'(0)=e^{2\pi i\alpha}$.
The *maximal* region in which f is conjugate to $R_\alpha(z)=e^{2\pi i\alpha}z$ is a simply connected domain Δ_f called the **Siegel disk** of f centered at 0.

Let

$$\alpha=[0;a_1,a_2,\cdots,a_n,\cdots]=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots}}}$$

be the **continued fraction expansion** of α. Then

$$\frac{p_n}{q_n}=[0;a_1,a_2,\cdots,a_n]=\cfrac{1}{a_1+\cfrac{1}{\ddots+\cfrac{1}{a_n}}}$$

converges to α exponentially fast.

Siegel-Brjuno-Yoccoz

Diophantine condition of order $\leq \kappa$:

$$\mathscr{D}(\kappa) := \left\{ \alpha \in (0,1) : \ \exists \varepsilon > 0 \text{ s.t. } \left| \alpha - \frac{p}{q} \right| > \frac{\varepsilon}{q^\kappa} \text{ for every rational } \frac{p}{q} \right\}.$$

Theorem (Siegel, 1942)

The holomorphic germ f has a Siegel disk at 0 if $\alpha \in \mathscr{D}(\kappa)$ for some $\kappa \geq 2$.

- $\cap_{\kappa>2}\mathscr{D}(\kappa)$ has full measure.
- $\mathscr{D}(2)$ has measure 0 and $\alpha \in \mathscr{D}(2)$ is of **bounded type**, i.e. $\sup_n\{a_n\} < \infty$.
- $\alpha \in \mathscr{D} = \cup_{\kappa\geq 2}\mathscr{D}(\kappa) \Leftrightarrow$ $\sup_n\{\frac{\log q_{n+1}}{\log q_n}\} < \infty$.

Siegel-Brjuno-Yoccoz

Diophantine condition of order $\leq \kappa$:

$$\mathscr{D}(\kappa) := \left\{ \alpha \in (0,1) : \ \exists \varepsilon > 0 \text{ s.t. } \left| \alpha - \frac{p}{q} \right| > \frac{\varepsilon}{q^\kappa} \text{ for every rational } \frac{p}{q} \right\}.$$

Theorem (Siegel, 1942)

The holomorphic germ f has a Siegel disk at 0 if $\alpha \in \mathscr{D}(\kappa)$ for some $\kappa \geq 2$.

- $\cap_{\kappa>2}\mathscr{D}(\kappa)$ has full measure.
- $\mathscr{D}(2)$ has measure 0 and $\alpha \in \mathscr{D}(2)$ is of **bounded type**, i.e. $\sup_n\{a_n\} < \infty$.
- $\alpha \in \mathscr{D} = \cup_{\kappa \geq 2}\mathscr{D}(\kappa) \Leftrightarrow \sup_n\{\frac{\log q_{n+1}}{\log q_n}\} < \infty$.

Theorem (Brjuno, 1965)

The holomorphic germ f has a Siegel disk at 0 if α belongs to

$$\mathscr{B} = \left\{ \alpha \in (0,1) \setminus \mathbb{Q} : \sum_n \frac{\log q_{n+1}}{q_n} < \infty \right\}.$$

Remark: $\mathscr{D} \subsetneq \mathscr{B}$.

Conjecture (Douady, 1986)

If a non-linear holomorphic function (entire or rational) has a Siegel disk, then the rotation number is necessarily in $\mathscr{B}$.

Theorem (Brjuno, 1965)

The holomorphic germ f has a Siegel disk at 0 if α belongs to

$$\mathscr{B} = \left\{ \alpha \in (0,1) \setminus \mathbb{Q} : \sum_n \frac{\log q_{n+1}}{q_n} < \infty \right\}.$$

Remark: $\mathscr{D} \subsetneq \mathscr{B}$.

Siegel-Brjuno-Yoccoz

Conjecture (Douady, 1986)

If a non-linear holomorphic function (entire or rational) has a Siegel disk, then the rotation number is necessarily in $\mathscr{B}$.

Theorem (Yoccoz, 1988)

If $\alpha \notin \mathscr{B}$, then $P_\alpha(z) = e^{2\pi i\alpha}z + z^2$ has no Siegel disk at the origin.

Remark: Douady's conjecture is still open even for cubic polynomials.

Some progresses have been made by Pérez-Marco, Geyer, Okuyama, Manlove, Cheraghi ...

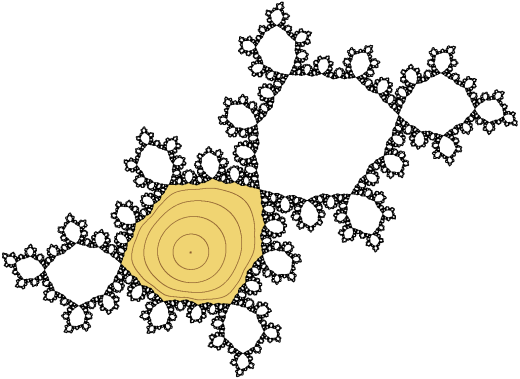
Theorem (Brjuno, 1965)

The holomorphic germ f has a Siegel disk at 0 if α belongs to

$$\mathscr{B} = \left\{ \alpha \in (0,1) \setminus \mathbb{Q} : \sum_n \frac{\log q_{n+1}}{q_n} < \infty \right\}.$$

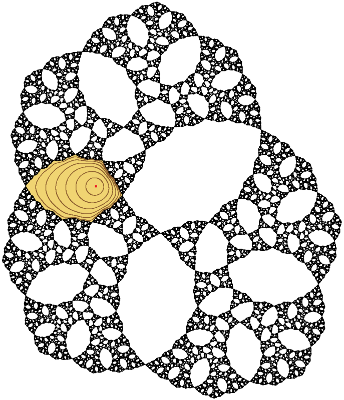
Remark: $\mathscr{D} \subsetneq \mathscr{B}$.

Siegel disks



The Siegel disk of $f(z) = e^{2\pi i \alpha} z + z^2$, where

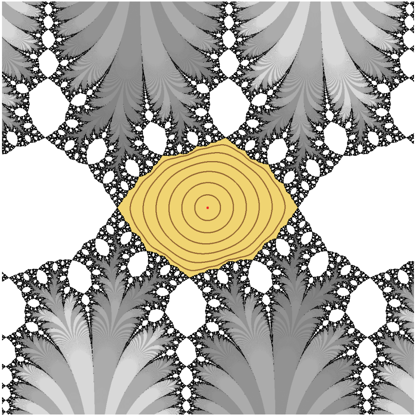
$$\alpha = \frac{\sqrt{5}-1}{2} = [0; 1, 1, 1, \cdots]$$



The Siegel disk of $f(z) = \dfrac{e^{\pi i(\sqrt{5}-1)} z}{(1-z)^2}$

Siegel disks

The Siegel disk of $f(z) = e^{\pi i(\sqrt{5}-1)} z e^z$



The Siegel disk of $f(z) = e^{\pi i(\sqrt{5}-1)/2} \sin(z)$

Douady-Sullivan's conjecture

Conjecture (Douady-Sullivan, 1986)

The Siegel disk of a rational map (deg ≥ 2) is always a Jordan domain.

Douady-Sullivan's conjecture

Conjecture (Douady-Sullivan, 1986)

The Siegel disk of a rational map ($\deg \geq 2$) is always a Jordan domain.

When $\alpha \in \mathscr{D}(2)$ is of **bounded type**:

Theorem (Zhang, 2011)

The bounded type Siegel disk of a rational map ($\deg \geq 2$) *is a quasi-disk.*

- (Douady-Ghys-Herman-Świątek, 1987) quadratic poly
- (Zakeri, 1999) cubic poly
- (Shishikura, 2001) all poly
- (Yampolsky-Zakeri, 2001) some quadratic rational map

Douady-Sullivan's conjecture

Conjecture (Douady-Sullivan, 1986)

The Siegel disk of a rational map (deg ≥ 2) is always a Jordan domain.

When $\alpha \in \mathscr{D}(2)$ is of **bounded type**:

Theorem (Zhang, 2011)

The bounded type Siegel disk of a rational map (deg ≥ 2) is a quasi-disk.

- (Douady-Ghys-Herman-Świątek, 1987) quadratic poly
- (Zakeri, 1999) cubic poly
- (Shishikura, 2001) all poly
- (Yampolsky-Zakeri, 2001) some quadratic rational map

Theorem (Zakeri, 2010)

The bounded type Siegel disk of a non-linear $f(z) = P(z)e^{Q(z)}$ is a quasi-disk, where P, Q are polys., $f(0) = 0, f'(0) = \lambda = e^{2\pi i\alpha}$.

- (Geyer, 2001) $f(z) = \lambda z e^z$
- (Keen-Zhang, 2009) $f(z) = (\lambda z + az^2)e^z$

Douady-Sullivan's conjecture

Conjecture (Douady-Sullivan, 1986)

The Siegel disk of a rational map (deg ≥ 2) is always a Jordan domain.

When $\alpha \in \mathscr{D}(2)$ is of **bounded type**:

Theorem (Zhang, 2011)

The bounded type Siegel disk of a rational map (deg ≥ 2) *is a quasi-disk.*

- (Douady-Ghys-Herman-Świątek, 1987) quadratic poly
- (Zakeri, 1999) cubic poly
- (Shishikura, 2001) all poly
- (Yampolsky-Zakeri, 2001) some quadratic rational map
- (Zhang, 2005) $f(z) = \lambda \sin(z)$
- (Y., 2013) $f(z) = \lambda \sin(z) + a \sin^3(z)$
- (Chéritat, 2006) some "simple" entire functions
- (Chéritat-Epstein, 2018) some holo. maps with at most 3 singular values.

Douady-Sullivan's conjecture

Conjecture (Douady-Sullivan, 1986)

The Siegel disk of a rational map (deg ≥ 2) is always a Jordan domain.

When $\alpha \in \mathscr{PZ}$ is of **Petersen-Zakeri type**:

$$\log a_n = O(\sqrt{n}) \text{ as } n \to \infty,$$

where $\mathscr{D}(2) \subsetneq \mathscr{PZ} \subset \cap_{\kappa>2}\mathscr{D}(\kappa)$, and $\mathscr{PZ}$ has full measure in $(0,1)$:

Theorem (Petersen-Zakeri, 2004)

For all $\alpha \in \mathscr{PZ}$, the Siegel disk of $P_\alpha(z) = e^{2\pi i\alpha}z + z^2$ is a Jordan domain.

- (Zhang, 2014) all polynomials
- (Zhang, 2016) $f(z) = e^{2\pi i\alpha}\sin(z)$

Douady-Sullivan's conjecture

Conjecture (Douady-Sullivan, 1986)

The Siegel disk of a rational map (deg ≥ 2) is always a Jordan domain.

When $\alpha \in \mathscr{PZ}$ is of **Petersen-Zakeri type**:

$$\log a_n = O(\sqrt{n}) \text{ as } n \to \infty,$$

where $\mathscr{D}(2) \subsetneq \mathscr{PZ} \subset \cap_{\kappa>2}\mathscr{D}(\kappa)$, and $\mathscr{PZ}$ has full measure in $(0,1)$:

Theorem (Petersen-Zakeri, 2004)

For all $\alpha \in \mathscr{PZ}$, the Siegel disk of $P_\alpha(z) = e^{2\pi i\alpha}z + z^2$ is a Jordan domain.

- (Zhang, 2014) all polynomials
- (Zhang, 2016) $f(z) = e^{2\pi i\alpha}\sin(z)$

Theorem (Avila-Buff-Chéritat, 2004)

$\exists\,\alpha$ s.t. the boundary of the Siegel disk of P_α is smooth.

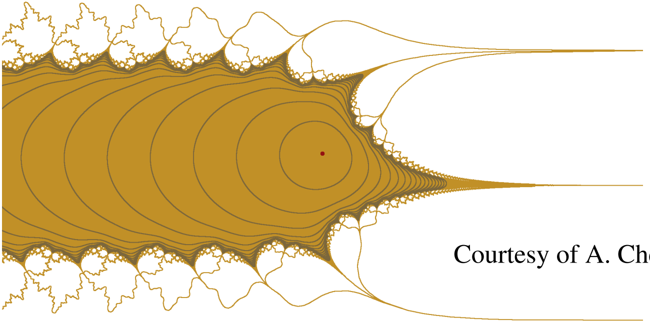
Theorem (Buff-Chéritat, 2007)

$\exists\,\alpha$ s.t. the boundary of the Siegel disk of P_α is C^r but not C^{r+1}.

Some related work has been done by Pérez-Marco, Rogers, Shen, ...

Counter-examples

Siegel disk of $f(z) = \lambda e^{z-\lambda}$, where $\lambda = e^{\pi i(\sqrt{5}-1)}$:



Courtesy of A. Chéritat

Theorem (Chéritat, 2011)

There is a holomorphic germ f such that the corresponding Siegel disk Δ_f is compactly contained in $Dom(f)$ but $\partial\Delta_f$ is a ***pseudo-circle****, which is not locally connected.*

Conjecture (Herman, 1986?)

The boundary of the Siegel disk (non-linear, entire or rational) contains at least one singular value if and only if the rotation number $\alpha \in \mathscr{H}$.

Herman's condition:

$$\mathscr{H} := \left\{ \alpha \in (0,1) \setminus \mathbb{Q} \;\middle|\; \begin{array}{l} \text{every orientation-preserving analytic circle diffeo.} \\ \text{of rotation number } \alpha \text{ is anal. conj. to } z \mapsto e^{2\pi i \alpha} z \end{array} \right\}.$$

- (Herman-Yoccoz, 1984): $\mathscr{D} \subsetneq \mathscr{H} \subsetneq \mathscr{B}$;
- (Yoccoz, 2002): Arithmetic characterization of $\mathscr{H}$:

$$\mathscr{H} = \{\alpha \in \mathscr{B} : \forall m \geq 0, \exists n > m \text{ s.t. } r_{\alpha_{n-1}} \circ \cdots \circ r_{\alpha_m}(0) \geq \mathscr{B}(\alpha_n)\},$$

where $\alpha_k = [0; a_{k+1}, a_{k+2}, \cdots]$, $\mathscr{B}(\alpha_n)$ is the Brjuno sum of α_n and

$$r_\alpha(x) = \begin{cases} \frac{1}{\alpha}\left(x - \log \frac{1}{\alpha} + 1\right) & \text{if } \quad x \geq \log \frac{1}{\alpha}, \\ e^x & \text{if } \quad x < \log \frac{1}{\alpha}. \end{cases}$$

Conjecture (Herman, 1986?)

The boundary of the Siegel disk (non-linear, entire or rational) contains at least one singular value if and only if the rotation number $\alpha \in \mathscr{H}$.

Herman's conjecture (the 'if' part) holds in the following cases:

- (Ghys, 1984): $\Delta_f \Subset Dom(f)$ and $\partial\Delta_f$ is a Jordan curve.
- (Herman, 1985): $f(z) = z^d + c$ and $f(z) = e^{az}$, where $d \geq 2$ and $a \in \mathbb{C} \setminus \{0\}$.
- (Rogers, 1998): f polynomial, then $\partial\Delta_f$ contains a critical point or $\partial\Delta_f$ is indecomposable continuum.
- (Graczyk-Świątek, 2003): $\Delta_f \Subset Dom(f)$ and α is of bounded type.
- (Chéritat-Roesch, 2016): The poly. with **two** critical values.
- (Benini-Fagella, 2018): A special class of transcendental entire functions with **two** singular values.

Some related work has also been done by Rippon, Rempe, Buff-Fagella, ...

Buff-Chéritat-Rempe (2009) proved the 'only if' part for a family of toy models.

Theorem (Shishikura-Y., 2018)

*Let α be an irrational number of sufficiently **high type**, and assume that $P_\alpha(z) = e^{2\pi i\alpha} z + z^2$ has a Siegel disk Δ_α. Then $\partial\Delta_\alpha$ is a Jordan curve, and $-e^{2\pi i\alpha}/2 \in \partial\Delta_\alpha$ if and only if $\alpha \in \mathscr{H}$.*

High type: if α belongs to

$$\mathrm{HT}_N := \{\alpha = [0; a_1, a_2, \cdots] \in (0,1) \setminus \mathbb{Q} \mid a_n \geq N \text{ for all } n \geq 1\}$$

for some large N.

HT_N has non-empty intersection with the usual types of irrational numbers: bounded type, Petersen-Zakeri type, Herman type, Brjuno type ...

Theorem (Shishikura-Y., 2018)

*Let α be an irrational number of sufficiently **high type**, and assume that $P_\alpha(z) = e^{2\pi i\alpha}z + z^2$ has a Siegel disk Δ_α. Then $\partial\Delta_\alpha$ is a Jordan curve, and $-e^{2\pi i\alpha}/2 \in \partial\Delta_\alpha$ if and only if $\alpha \in \mathscr{H}$.*

High type: if α belongs to

$$\mathrm{HT}_N := \{\alpha = [0; a_1, a_2, \cdots] \in (0,1) \setminus \mathbb{Q} \mid a_n \geq N \text{ for all } n \geq 1\}$$

for some large N.

HT_N has non-empty intersection with the usual types of irrational numbers: bounded type, Petersen-Zakeri type, Herman type, Brjuno type ...

Cheraghi (2017), independently, gave another proof of the Main result. He studied the topology of the post-critical set of all maps in the Inou-Shishikura's class IS_α (in particular, of P_α) and for all $\alpha \in \mathrm{HT}_N$.

Main result

Theorem (Shishikura-Y., 2018)

*Let α be an irrational number of sufficiently **high type**, and assume that $P_\alpha(z) = e^{2\pi i\alpha}z + z^2$ has a Siegel disk Δ_α. Then $\partial\Delta_\alpha$ is a Jordan curve, and $-e^{2\pi i\alpha}/2 \in \partial\Delta_\alpha$ if and only if $\alpha \in \mathscr{H}$.*

High type: if α belongs to

$$\mathrm{HT}_N := \{\alpha = [0; a_1, a_2, \cdots] \in (0,1) \setminus \mathbb{Q} \mid a_n \geq N \text{ for all } n \geq 1\}$$

for some large N.

HT_N has non-empty intersection with the usual types of irrational numbers: bounded type, Petersen-Zakeri type, Herman type, Brjuno type ...

Cheraghi (2017), independently, gave another proof of the Main result. He studied the topology of the post-critical set of all maps in the Inou-Shishikura's class IS_α (in particular, of P_α) and for all $\alpha \in \mathrm{HT}_N$.

Avila-Lyubich (2015): $\partial\Delta_f$ is a quasi-circle if $f \in IS_\alpha \cup \{P_\alpha\}$ with $\alpha \in \mathrm{HT}_N \cap \mathscr{D}(2)$.

Inou-Shishikura's invariant class

Our proof is also valid for all the maps in Inou-Shishikura's class IS_0:

$$IS_0 \supsetneq \left\{ f : Dom(f) \to \mathbb{C} \;\middle|\; \begin{array}{l} 0 \in Dom(f) \text{ open} \subset \mathbb{C}, f \text{ is holo. in } Dom(f), \\ f(0)=0, f'(0)=1, f : Dom(f) \setminus \{0\} \to \mathbb{C}^* \text{ is a} \\ \textbf{branched covering} \text{ with } \textbf{a unique critical value} \\ cv_f, \text{ all critical points are of } \textbf{local degree 2} \end{array} \right\}.$$

The following maps (their variations or renormalization) are contained in IS_α:

- $P_\alpha(z) = e^{2\pi i\alpha} z + z^2$;
- $g_\alpha(z) = e^{2\pi i\alpha} \dfrac{z}{(1-z)^2}$;
- $P_{n,\alpha}(z) = e^{2\pi i\alpha} z \left(1 + \dfrac{z}{n}\right)^n, n \geq 2$;
- $E_\alpha(z) = e^{2\pi i\alpha} z e^z$,
- $S_\alpha(z) = e^{\pi i\alpha} \sin(z)$.

Inou-Shishikura's invariant class

Our proof is also valid for all the maps in Inou-Shishikura's class IS_0:

$$IS_0 \supsetneq \left\{ f : Dom(f) \to \mathbb{C} \;\middle|\; \begin{array}{l} 0 \in Dom(f) \text{ open} \subset \mathbb{C}, f \text{ is holo. in } Dom(f), \\ f(0)=0,\, f'(0)=1, f : Dom(f)\setminus\{0\} \to \mathbb{C}^* \text{ is a} \\ \textbf{branched covering} \text{ with } \textbf{a unique critical value} \\ cv_f, \text{ all critical points are of } \textbf{local degree 2} \end{array} \right\}.$$

The following maps (their variations or renormalization) are contained in IS_α:

- $P_\alpha(z) = e^{2\pi i\alpha} z + z^2$;
- $g_\alpha(z) = e^{2\pi i\alpha} \dfrac{z}{(1-z)^2}$;
- $P_{n,\alpha}(z) = e^{2\pi i\alpha} z \left(1 + \dfrac{z}{n}\right)^n, n \geq 2$;
- $E_\alpha(z) = e^{2\pi i\alpha} z e^z$,
- $S_\alpha(z) = e^{\pi i\alpha} \sin(z)$.

Theorem (Inou-Shishikura, 2008)

$\exists \varepsilon_0 > 0$, s.t. if $0 < \alpha < \varepsilon_0$ then the near-parabolic renorm.

$$\mathscr{R} : IS_\alpha \cup \{P_\alpha, g_\alpha\} \to IS_{1/\alpha}$$

is well-defined. Moreover,

- *$\mathscr{R}$ can be iterated infinitely many times if $\alpha \in \mathrm{HT}_N$ for $N > 1/\varepsilon_0$.*
- *The operator $\mathscr{R}$ is hyperbolic.*

Idea of the proof I

For $f_0 := f \in IS_\alpha \cup \{P_\alpha, g_\alpha\}$ with $\alpha_0 := \alpha = [0; a_1, a_2, \cdots] \in \mathrm{HT}_N$, define $f_n = \mathscr{R}^{\circ n} f_0$.

Then $f_n \in IS_{\alpha_n}$ for all $n \geq 1$, where $\alpha_n = [0; a_{n+1}, a_{n+2}, \cdots]$.

Idea of the proof I

For $f_0 := f \in IS_\alpha \cup \{P_\alpha, g_\alpha\}$ with $\alpha_0 := \alpha = [0; a_1, a_2, \cdots] \in \mathrm{HT}_N$, define $f_n = \mathscr{R}^{\circ n} f_0$.
Then $f_n \in IS_{\alpha_n}$ for all $n \geq 1$, where $\alpha_n = [0; a_{n+1}, a_{n+2}, \cdots]$.

For the **first** part (Douady-Sullivan's conjecture), the main steps are:

1. For each $n \in \mathbb{N}$, construct a continuous curve $\gamma_n^0 : [0,1] \to \mathbb{C}$ in the Fatou coordinate plane of f_n, s.t. $\Phi_n^{-1}(\gamma_n^0)$ is a continuous closed curve in Δ_n;
2. Obtain a sequence of continuous curves $\{\gamma_0^n : [0,1] \to \mathbb{C}\}_{n \in \mathbb{N}}$ in the Fatou coordinate plane of f_0 by **renormalization tower**, s.t. $\{\Phi_0^{-1}(\gamma_0^n)\}_{n \in \mathbb{N}}$ is a sequence of continuous **closed curves** in Δ_0;
3. Prove that $\{\gamma_0^n : [0,1] \to \mathbb{C}\}_{n \in \mathbb{N}}$ converges uniformly to a continuous curve $\gamma^\infty : [0,1] \to \mathbb{C}$ and show that $\Phi_0^{-1}(\gamma^\infty)$ is exactly $\partial \Delta_0$.

Idea of the proof I

For $f_0 := f \in IS_\alpha \cup \{P_\alpha, g_\alpha\}$ with $\alpha_0 := \alpha = [0; a_1, a_2, \cdots] \in \mathrm{HT}_N$, define $f_n = \mathscr{R}^{\circ n} f_0$.

Then $f_n \in IS_{\alpha_n}$ for all $n \geq 1$, where $\alpha_n = [0; a_{n+1}, a_{n+2}, \cdots]$.

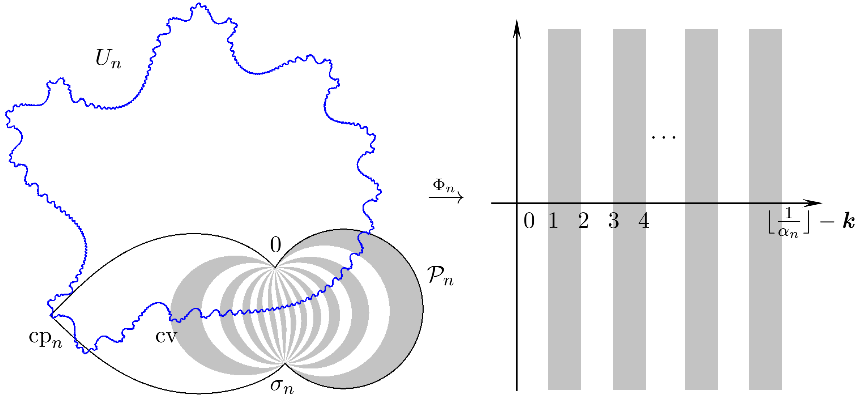
For the **first** part (Douady-Sullivan's conjecture), the main steps are:

1. For each $n \in \mathbb{N}$, construct a continuous curve $\gamma_n^0 : [0,1] \to \mathbb{C}$ in the Fatou coordinate plane of f_n, s.t. $\Phi_n^{-1}(\gamma_n^0)$ is a continuous closed curve in Δ_n;
2. Obtain a sequence of continuous curves $\{\gamma_0^n : [0,1] \to \mathbb{C}\}_{n\in\mathbb{N}}$ in the Fatou coordinate plane of f_0 by **renormalization tower**, s.t. $\{\Phi_0^{-1}(\gamma_0^n)\}_{n\in\mathbb{N}}$ is a sequence of continuous **closed curves** in Δ_0;
3. Prove that $\{\gamma_0^n : [0,1] \to \mathbb{C}\}_{n\in\mathbb{N}}$ converges uniformly to a continuous curve $\gamma^\infty : [0,1] \to \mathbb{C}$ and show that $\Phi_0^{-1}(\gamma^\infty)$ is exactly $\partial\Delta_0$.

Key point: The convergence of the curves $\{\gamma_0^n : [0,1] \to \mathbb{C}\}_{n\in\mathbb{N}}$ is based on the **contraction** of renormalization operator (consider the inverse).

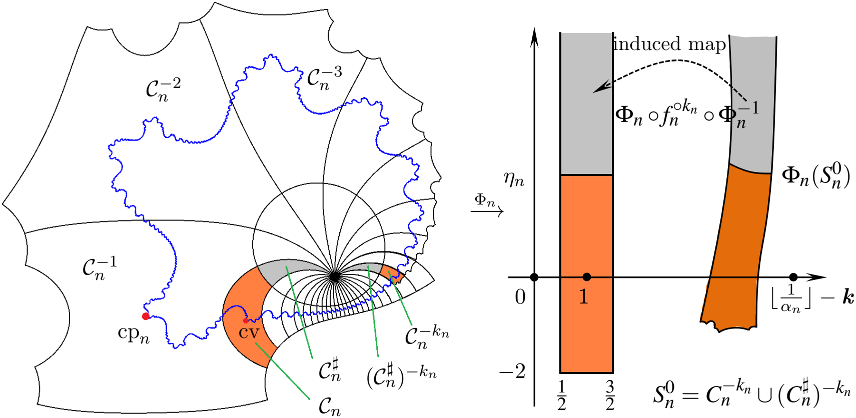
Note: The convergence may not be exponentially fast!

Fatou coordinates and near-parabolic renorm.

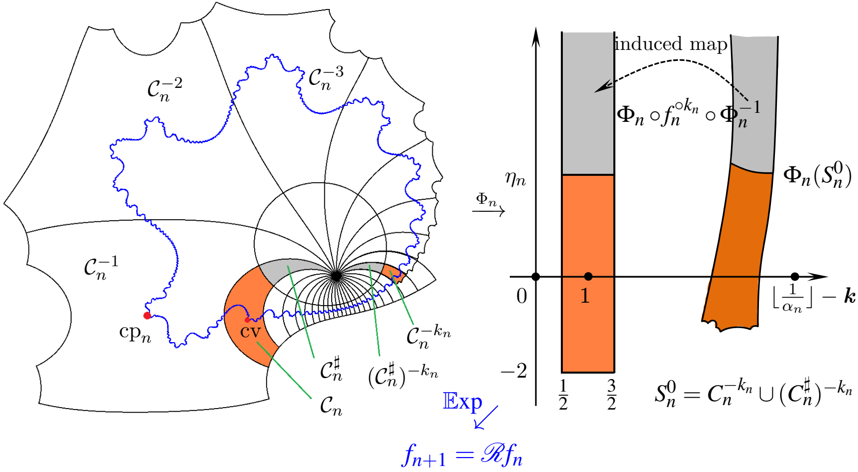


For each f_n, the perturbed petal $\mathscr{P}_n$ and Fatou coordinate Φ_n satisfy $\Phi_n(\mathrm{cv}) = 1$, $\Phi_n(\mathscr{P}_n) = \{\zeta \in \mathbb{C} : 0 < \mathrm{Re}\,\Phi_n(z) < \lfloor \frac{1}{\alpha_n} \rfloor - \boldsymbol{k}\}$ and $\Phi_n(f_n(z)) = \Phi_n(z) + 1$.

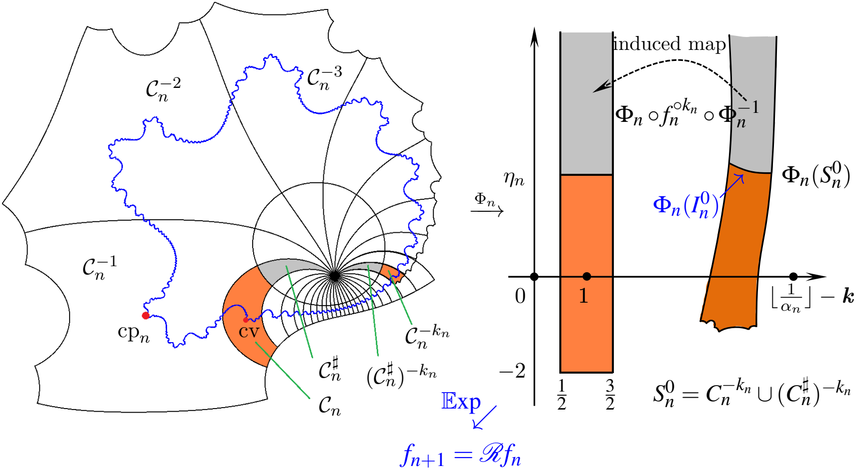
Fatou coordinates and near-parabolic renorm.



Fatou coordinates and near-parabolic renorm.



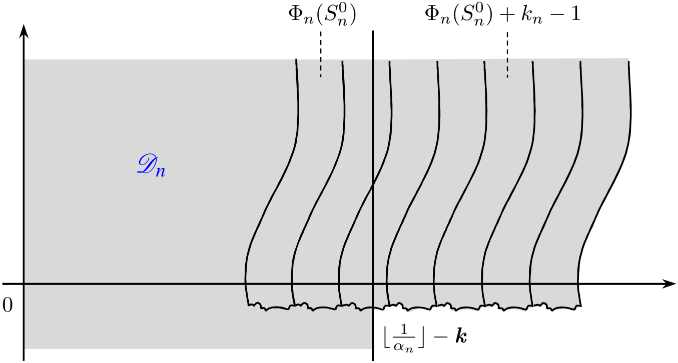
Fatou coordinates and near-parabolic renorm.

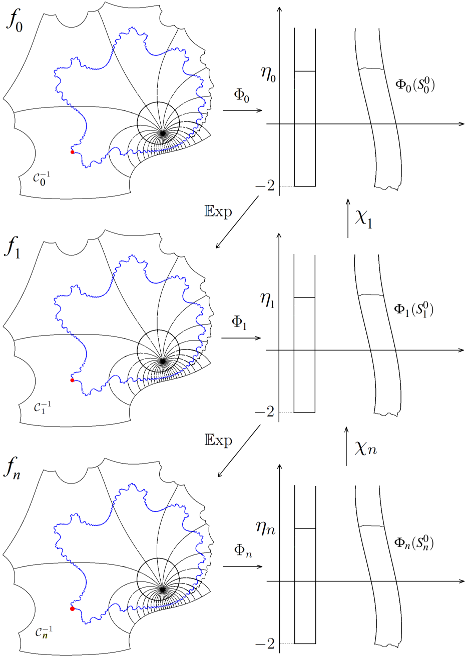


Renormalization tower

The domain of definition of Φ_n^{-1} (of level $n \in \mathbb{N}$) can be extended to:

$$\mathscr{D}_n = \Phi_n(\mathscr{P}_n) \bigcup_{j=0}^{k_n+\boldsymbol{k}'} (\Phi_n(S_n^0)+j)$$





Renormalization tower

The domain of definition of Φ_n^{-1} (of level $n \in \mathbb{N}$) can be extended to:

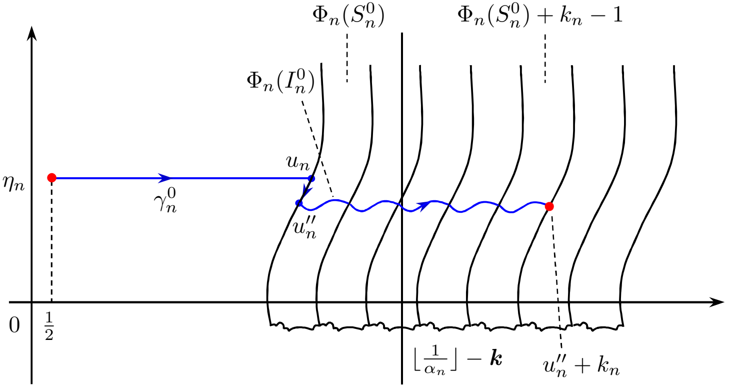
$$\mathscr{D}_n = \Phi_n(\mathscr{P}_n) \bigcup_{j=0}^{k_n + \boldsymbol{k}'} (\Phi_n(S_n^0) + j)$$

$\exists$ anti-holo. map $\chi_n : \mathscr{D}_n \to \mathscr{D}_{n-1}$ s.t.

$$\begin{array}{ccc} & & \mathscr{D}_{n-1} \\ & \overset{\mathbb{E}\mathrm{xp}}{\swarrow} & \uparrow \chi_n \\ \mathbb{C} \setminus \{0\} & \xleftarrow{\Phi_n^{-1}} & \mathscr{D}_n \end{array}$$

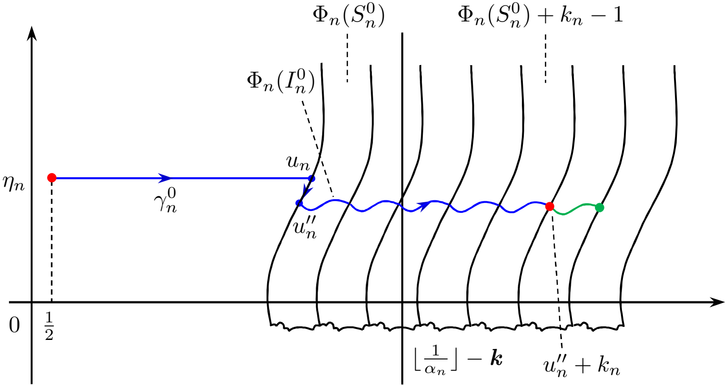
and $\chi_n(1) = \boldsymbol{k}'' \leq C$.

$\chi_n = \chi_{n,0} : \mathscr{D}_n \to \mathscr{D}_{n-1}$ is uniformly contractive w.r.t. **hyperbolic metrics**.



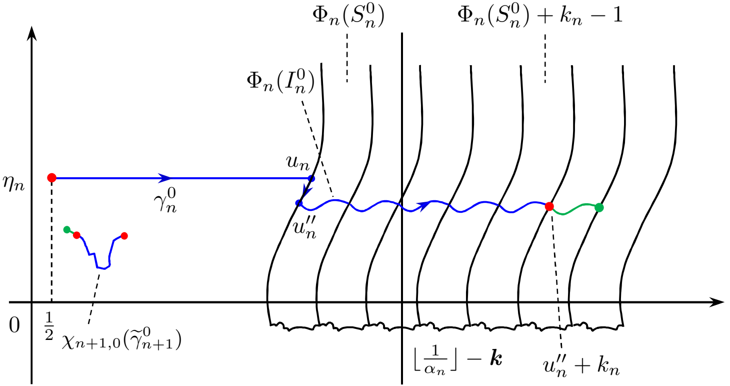
The height $\eta_n = \frac{\mathscr{B}(\alpha_{n+1})}{2\pi} + \frac{M}{\alpha_n}$ is chosen s.t. $\Phi_n^{-1}(\gamma_n^0)$ is a closed curve in Δ_n.

The construction of the curves



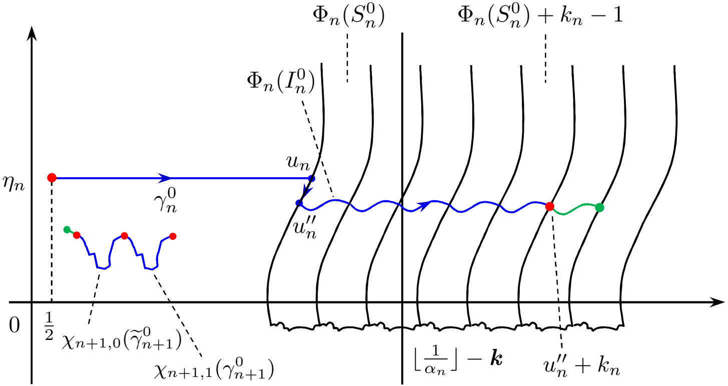
The height $\eta_n = \frac{\mathscr{B}(\alpha_{n+1})}{2\pi} + \frac{M}{\alpha_n}$ is chosen s.t. $\Phi_n^{-1}(\gamma_n^0)$ is a closed curve in Δ_n.

The construction of the curves



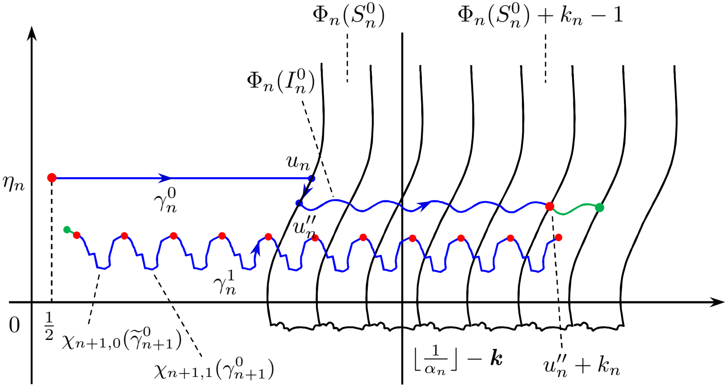
The height $\eta_n = \frac{\mathscr{B}(\alpha_{n+1})}{2\pi} + \frac{M}{\alpha_n}$ is chosen s.t. $\Phi_n^{-1}(\gamma_n^0)$ is a closed curve in Δ_n.

The construction of the curves



The height $\eta_n = \frac{\mathscr{B}(\alpha_{n+1})}{2\pi} + \frac{M}{\alpha_n}$ is chosen s.t. $\Phi_n^{-1}(\gamma_n^0)$ is a closed curve in Δ_n.

The construction of the curves



The height $\eta_n=\frac{\mathscr{B}(\alpha_{n+1})}{2\pi}+\frac{M}{\alpha_n}$ is chosen s.t. $\Phi_n^{-1}(\gamma_n^0)$ is a closed curve in Δ_n.

The sequence of curves is convergent

Proposition

There exists a constant $K > 0$ such that for all $n \in \mathbb{N}$, we have

$$\sum_{i=0}^{n} \sup_{t\in[0,1]} |\gamma_0^i(t) - \gamma_0^{i+1}(t)| \leq K.$$

In particular, $(\gamma_0^n(t) : [0,1] \to \mathbb{C})_{n\in\mathbb{N}}$ converges uniformly as $n \to \infty$.

Key of the proof:

Study the contraction factors between the adjacent renornormalization levels.

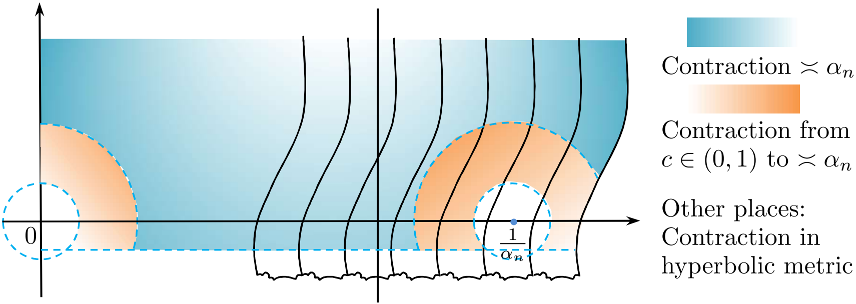
The sequence of curves is convergent

Proposition

There exist positive constants C_0, C_1 and C_2 such that for all $n \geq 1$,

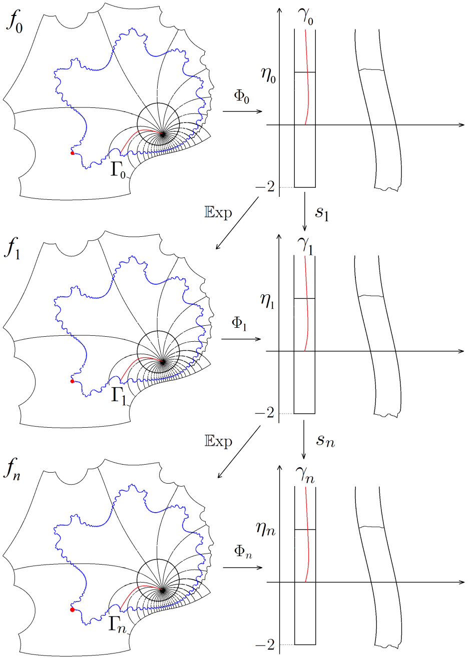
1. (Cheraghi, 2013) If $\zeta \in \mathscr{D}_n$ with $\operatorname{Im}\zeta \geq 1/(4\alpha_n)$, then
$$|\chi_n'(\zeta) - \alpha_n| \leq C_0 \alpha_n e^{-2\pi\alpha_n \operatorname{Im}\zeta}.$$
2. (Shishikura-Y., 2018) If $\zeta \in \mathscr{D}_n$ with $\operatorname{Im}\zeta \in [-2, 1/(4\alpha_n)]$ and $\rho := \min\{|\zeta|, |\zeta - 1/\alpha_n|\} \geq C_1$, then
$$|\chi_n'(\zeta)| \leq \frac{\alpha_n}{1 - e^{-2\pi\alpha_n(\rho - C_2\log(2+\rho))}} \left(1 + \frac{C_0}{\rho}\right),$$
where C_1 and C_2 are chosen such that $\rho - C_2 \log(2+\rho) \geq 2$ if $\rho \geq C_1$.

The sequence of curves is convergent



Remark: The convergence of $\{\gamma_0^n\}_{n\in\mathbb{N}}$ is exponentially fast if

- α_0 is of bounded type; or
- $\mathscr{B}(\alpha_{n+1}) \geq C/\alpha_n$ for some $C > 0$ (for example $a_{n+1} = e^{a_n}$).



Idea of the proof II

For the **second** part (Herman's conjecture), the main steps are:

1. For each $n \in \mathbb{N}$, construct a canonical simple arc

$$\gamma_n : [0, 1) \to \mathbb{C}$$

in $\mathscr{D}_n$ with $\gamma_n(0) = 1$, s.t.

$$\Gamma_n := \Phi_n^{-1}(\gamma_n)$$

is a simple arc in $Dom(f_n)$ **connecting cv and** 0, and

$$s_n(\gamma_{n-1}) = \gamma_n,$$

where $s_n := \Phi_n \circ \mathbb{E}\text{xp}$.

Idea of the proof II

2. Define a class of irrational numbers $\widetilde{\mathscr{H}}_N$ in $\mathscr{B}_N = \mathscr{B} \cap \mathrm{HT}_N$:

$$\widetilde{\mathscr{H}}_N = \left\{ \alpha \in \mathscr{B}_N \,\middle|\, \begin{array}{l} \forall \zeta \in \gamma_0 \setminus \{1\}, \exists n \geq 1, \text{ s.t.} \\ \mathrm{Im}\, s_n \circ \cdots \circ s_1(\zeta) \geq \widetilde{\mathscr{B}}(\alpha_n) \end{array} \right\},$$

where

$$\widetilde{\mathscr{B}}(\alpha_n) = \frac{\mathscr{B}(\alpha_{n+1})}{2\pi} + M.$$

3. Prove that $\mathrm{cv} \in \partial \Delta_0$ if and only if $\alpha \in \widetilde{\mathscr{H}}_N$.

Idea of the proof II

Lemma

$\exists$ constants $D_0, D_1 > 0$ s.t. for all $n \geq 1$,

1. If $\zeta \in \gamma_{n-1}$ with $\operatorname{Im}\zeta \geq \frac{1}{2\pi}\log\frac{1}{\alpha_n} + D_0$, then
$$\left|\operatorname{Im} s_n(\zeta) - \frac{1}{\alpha_n}\left(\operatorname{Im}\zeta - \frac{1}{2\pi}\log\frac{1}{\alpha_n}\right)\right| \leq \frac{D_1}{\alpha_n}.$$
2. If $\zeta \in \gamma_{n-1}$ with $\operatorname{Im}\zeta < \frac{1}{2\pi}\log\frac{1}{\alpha_n} + D_0$, then
$$\left|\log(3 + \operatorname{Im} s_n(\zeta)) - 2\pi \operatorname{Im}\zeta\right| \leq D_1.$$

Recall: Arithmetic characterization of $\mathscr{H}$ (Yoccoz, 2002):

$$\mathscr{H} = \left\{\alpha \in \mathscr{B} \;\middle|\; \begin{array}{l} \forall m \geq 0, \exists n > m \text{ s.t.} \\ r_{n-1} \circ \cdots \circ r_m(0) \geq \mathscr{B}(\alpha_n) \end{array}\right\},$$

where

$$r_n(x) = \begin{cases} \frac{1}{\alpha_n}\left(x - \log\frac{1}{\alpha_n} + 1\right), & \text{if } x \geq \log\frac{1}{\alpha_n}, \\ e^x, & \text{if } x < \log\frac{1}{\alpha_n}. \end{cases}$$

2. Define a class of irrational numbers $\widetilde{\mathscr{H}}_N$ in $\mathscr{B}_N = \mathscr{B} \cap \mathrm{HT}_N$:
$$\widetilde{\mathscr{H}}_N = \left\{\alpha \in \mathscr{B}_N \;\middle|\; \begin{array}{l} \forall \zeta \in \gamma_0 \setminus \{1\}, \exists n \geq 1, \text{ s.t.} \\ \operatorname{Im} s_n \circ \cdots \circ s_1(\zeta) \geq \widetilde{\mathscr{B}}(\alpha_n) \end{array}\right\},$$
where
$$\widetilde{\mathscr{B}}(\alpha_n) = \frac{\mathscr{B}(\alpha_{n+1})}{2\pi} + M.$$
3. Prove that $\mathrm{cv} \in \partial\Delta_0$ if and only if $\alpha \in \widetilde{\mathscr{H}}_N$.
4. Prove that $\widetilde{\mathscr{H}}_N = \mathscr{H} \cap \mathrm{HT}_N$.

Idea of the proof II

Lemma

$\exists$ constants $D_0, D_1 > 0$ s.t. for all $n \geq 1$,

1. If $\zeta \in \gamma_{n-1}$ with $\operatorname{Im}\zeta \geq \frac{1}{2\pi}\log\frac{1}{\alpha_n} + D_0$, then
$$\left|\operatorname{Im} s_n(\zeta) - \frac{1}{\alpha_n}\left(\operatorname{Im}\zeta - \frac{1}{2\pi}\log\frac{1}{\alpha_n}\right)\right| \leq \frac{D_1}{\alpha_n}.$$
2. If $\zeta \in \gamma_{n-1}$ with $\operatorname{Im}\zeta < \frac{1}{2\pi}\log\frac{1}{\alpha_n} + D_0$, then
$$\left|\log(3 + \operatorname{Im} s_n(\zeta)) - 2\pi\operatorname{Im}\zeta\right| \leq D_1.$$

Recall: Arithmetic characterization of $\mathscr{H}$ (Yoccoz, 2002):

$$\mathscr{H} = \left\{\alpha \in \mathscr{B} \;\middle|\; \begin{array}{l}\forall m \geq 0, \exists n > m \text{ s.t.} \\ r_{n-1}\circ\cdots\circ r_m(0) \geq \mathscr{B}(\alpha_n)\end{array}\right\},$$

where

$$r_n(x) = \begin{cases} \frac{1}{\alpha_n}\left(x - \log\frac{1}{\alpha_n} + 1\right), & \text{if } x \geq \log\frac{1}{\alpha_n}, \\ e^x, & \text{if } x < \log\frac{1}{\alpha_n}. \end{cases}$$

2. Define a class of irrational numbers $\widetilde{\mathscr{H}_N}$ in $\mathscr{B}_N = \mathscr{B} \cap \mathrm{HT}_N$:
$$\widetilde{\mathscr{H}_N} = \left\{\alpha \in \mathscr{B}_N \;\middle|\; \begin{array}{l}\forall \zeta \in \gamma_0 \setminus \{1\}, \exists n \geq 1, \text{ s.t.} \\ \operatorname{Im} s_n\circ\cdots\circ s_1(\zeta) \geq \widetilde{\mathscr{B}}(\alpha_n)\end{array}\right\},$$
where
$$\widetilde{\mathscr{B}}(\alpha_n) = \frac{\mathscr{B}(\alpha_{n+1})}{2\pi} + M.$$
3. Prove that $\mathrm{cv} \in \partial\Delta_0$ if and only if $\alpha \in \widetilde{\mathscr{H}_N}$.
4. Prove that $\widetilde{\mathscr{H}_N} = \mathscr{H} \cap \mathrm{HT}_N$.

THANK YOU FOR YOUR ATTENTION !