Siegel disks

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Linearization problem

Consider the non-linear holomorphic germ

$$f(z) = \lambda z + a_2 z^2 + \cdots, \text{ where } \lambda \in \mathbb{C} \setminus \{0\}.$$

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3. $\lambda = e^{2\pi i \alpha}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$: **irrational indifferent**

Irrational indifferent

Near 0, $f(z) = e^{2\pi i\alpha} z + O(z^2)$ is close to the aperiodic rotation $R_\alpha : z \mapsto e^{2\pi i\alpha} z$ ($\alpha \in \mathbb{R} \setminus \mathbb{Q}$).

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   Any **rational map** f ($\deg(f) = d \geq 2$) **cannot** be locally linearized at 0 if

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Irrational numbers

Diophantine condition of order $\leq \kappa$:

$$\mathscr{D}(\kappa) := \left\{ \alpha \in (0,1) : \ \exists \varepsilon > 0 \text{ s.t. } \left| \alpha - \frac{p}{q} \right| > \frac{\varepsilon}{q^{\kappa}} \text{ for every rational } \frac{p}{q} \right\}.$$

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1. $\cap_{\kappa>2} \mathscr{D}(\kappa)$ has full measure.
2. $\mathscr{D}(2)$ has measure 0 and $\alpha \in \mathscr{D}(2)$ is called **bounded type** (constant type). This is equivalent to the continued fractional expansion

$$\alpha = [0; a_1, a_2, \cdots, a_n, \cdots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

   satisfies $\sup\{a_n\} < \infty$.
3. Let $p_n/q_n = [0; a_1, \cdots, a_n]$. Then $\alpha \in \mathscr{D} = \cup_{\kappa \geq 2} \mathscr{D}(\kappa) \Leftrightarrow \sup_n \{ \frac{\log q_{n+1}}{\log q_n} \} < \infty$.

Definition

$0 \in U$, an open subset of $\mathbb{C}$

$f(z) = e^{2\pi i\alpha} z + O(z^2) : U \subset \mathbb{C} \to \mathbb{C}$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

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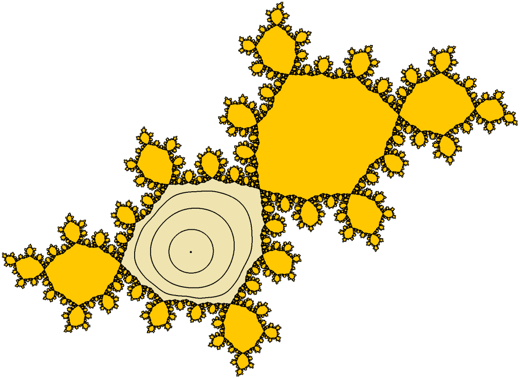
Note that, apart from its center, a Siegel disk **cannot contain any periodic point, critical point**. On the other hand it can contain an iterated image of a critical point.

Siegel and his disks

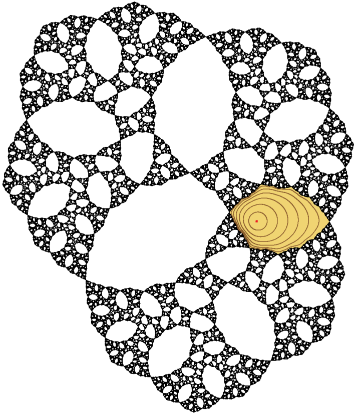
Carl Ludwig Siegel (1896-1981)



The Siegel disk of $f(z) = e^{2\pi i\alpha}z + z^2$, where

$$\alpha = \frac{\sqrt{5}-1}{2} = [0; 1, 1, 1, \cdots]$$

Siegel disks



The Siegel disk of $f(z) = \dfrac{e^{\pi i(\sqrt{5}-1)} z}{(1+z)^2}$

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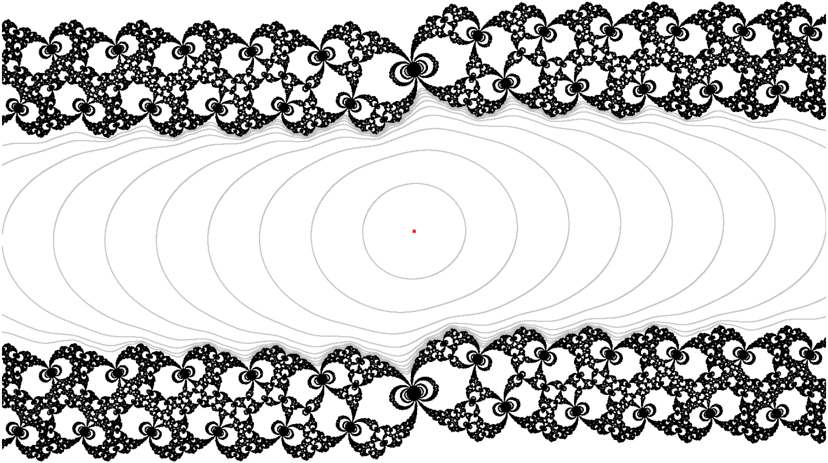
Siegel disks



The Siegel disk of $f(z) = e^{\pi i(\sqrt{5}-1)} \tan(z)$, rotated by 90°

Brjuno type

Recall: $p_n/q_n = [0; a_1, \cdots, a_n]$ and $\alpha \in \mathscr{D} \Leftrightarrow \sup_n\{\frac{\log q_{n+1}}{\log q_n}\} < \infty$.

Brjuno (Dokl. Akad. Nauk USSR 1965):

Any **holomorphic germ** f **can** be locally linearized at 0 if α belongs to

$$\mathscr{B} = \left\{\alpha \in (0,1) \setminus \mathbb{Q} : \sum_n \frac{\log q_{n+1}}{q_n} < \infty\right\}.$$

Remark: $\mathscr{D} \subsetneq \mathscr{B}$.

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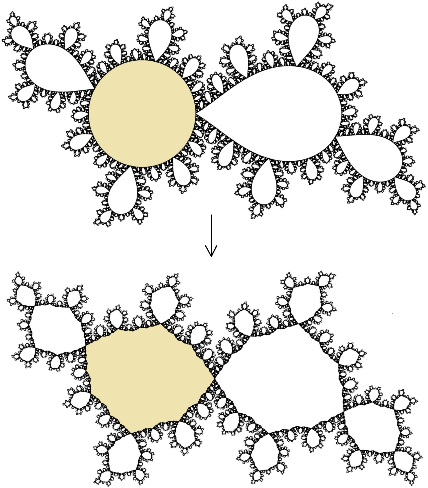
Yoccoz (C. R. Acad. Sci. Paris 1988):

If $\alpha \notin \mathscr{B}$, then $f(z) = e^{2\pi i\alpha} z + z^2$ is not locally linearizable at the origin. Moreover, every neighborhood of 0 contains infinitely many periodic orbits.

Regularity of the boundary



Blaschke model for quadratic Siegel disk

Herman-Świątek (1986, 1998):

Let $f : \mathbb{T} \to \mathbb{T}$ be a real-analytic critical circle homeomorphism of rotation number α. Then f is quasisymmetrically conjugate to $R_\alpha(z) = e^{2\pi i\alpha} z$ if and only if α is of bounded type.

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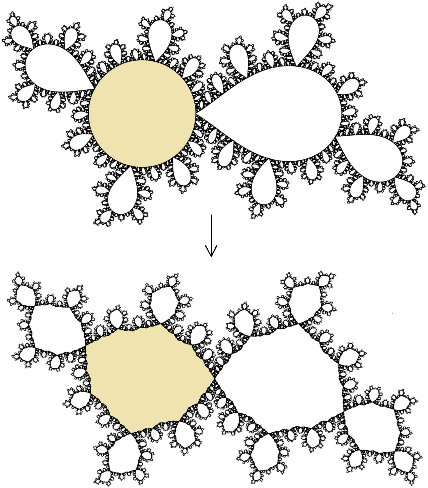

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Douady-Ghys-Herman-Shishikura (1986):

The boundary of the Siegel disk of $f(z) = e^{2\pi i \alpha} z + z^2$ is a quasicircle if α is of bounded type.

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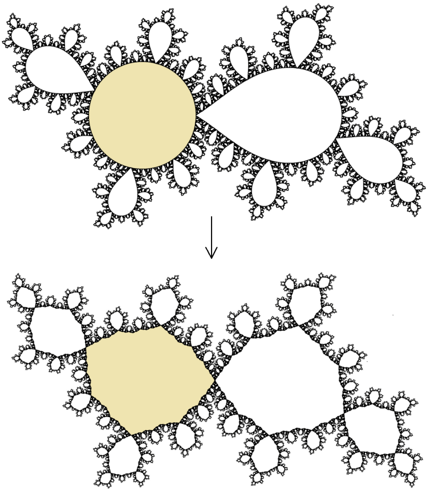

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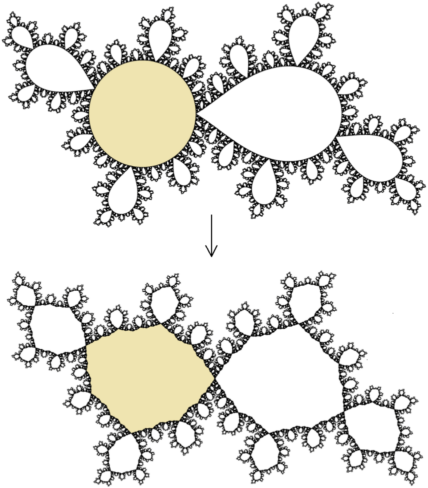

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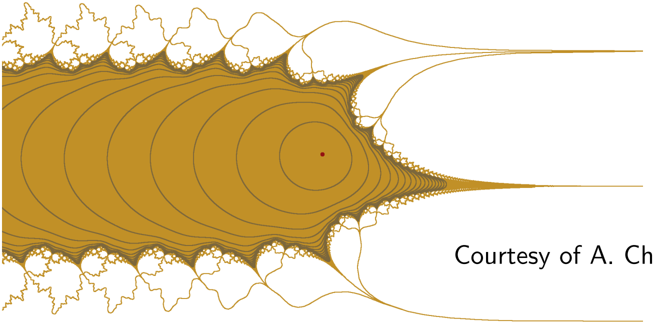
Conjecture 2 (Douady-Sullivan, 1986):
The Siegel disk of a **rational map** (deg ≥ 2) is always a Jordan domain.

Counter-examples

Siegel disk of $f(z) = \lambda e^{z-\lambda}$, where $\lambda = e^{\pi i(\sqrt{5}-1)}$:



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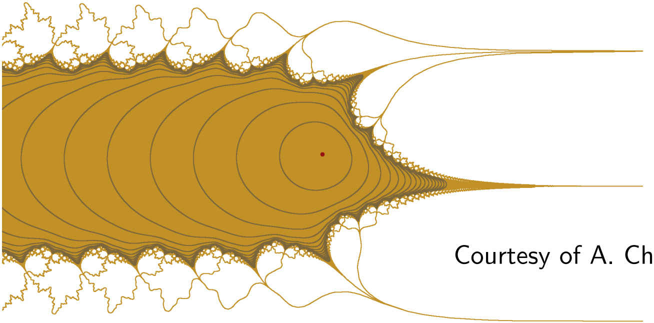

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Chéritat (Math. Ann. 2011):

There exists a holomorphic germ f such that the corresponding Siegel disk Δ_f is compactly contained in $Dom(f)$ but $\partial\Delta_f$ is a **pseudo-circle**, which is not locally connected.

Herman type

Herman's condition:

$$\mathscr{H} := \left\{ \alpha \in (0,1) \setminus \mathbb{Q} \;\middle|\; \begin{array}{l} \text{every orientation-preserving analytic circle diffeo.} \\ \text{of rotation number } \alpha \text{ is anal. conj. to } z \mapsto e^{2\pi i\alpha} z \end{array} \right\}.$$

1. Herman-Yoccoz (1984): $\mathscr{D} \subsetneq \mathscr{H} \subsetneq \mathscr{B}$; and
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   Yoccoz (2002): Arithmetic characterization of $\mathscr{H}$.
2. Ghys (1984): If $\alpha \in \mathscr{H}$, $\Delta_f \Subset Dom(f)$ and $\partial\Delta_f$ is a Jordan curve, then $\partial\Delta_f$ contains a critical point.
3. Herman (1985): If $\alpha \in \mathscr{H}$, $\Delta_f \Subset Dom(f)$ and and $f|_{\partial\Delta_f}$ is injective, then $\partial\Delta_f$ contains a critical point. In particular, $f|_{\partial\Delta_f}$ is injective if $f(z) = z^d + c$ or e^{az}, where $d \geq 2$ and $a \in \mathbb{C} \setminus \{0\}$.
4. Herman (1986): $\exists \alpha \in \mathscr{B}$ s.t. $\partial\Delta_f$ contains no crit. pt, where f is a quad. poly.

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Conjecture 3 (Herman, 1986?): The boundary of the Siegel disk contains a singular value if and only if the rotation number $\alpha \in \mathscr{H}$.

Three conjectures

Three conjectures (for non-linear map):

1. (Douady, 1986) If a **holomorphic function** has a Siegel disk, then the rotation number is necessarily of Brjuno type.
2. (Douady-Sullivan, 1986) The boundary of the Siegel disk of a **rational map** (deg ≥ 2) is always a Jordan curve.
3. (Herman, 1986?) The boundary of the Siegel disk of a **holomorphic function** contain a singular value if and only if the rotation number is of Herman type.

Douady's conjecture

Siegel disk exists $\Rightarrow \alpha \in \mathscr{B}$

Douady's conjecture holds in the following cases:

1. Yoccoz (C. R. Acad. Sci. Paris 1988): $f(z) = z^2 + c$.

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4. Okuyama (J. Math. Soc. Japan 2001): 'Piecewise 1-subhyperbolic' polynomials.
5. Okuyama (Kodai Math. J. 2005): '1-hyperbolic structurally finite' entire function $f = \int P e^Q$, where P, Q are polynomials.

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6. Geyer (arXiv, 2015): Julia-saturated poly., including $f(z) = z^d + c$, $d \geq 2$.
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The proofs are based on a perturbation lemma, which is essentially established by Yoccoz (1988) and improved by Pérez-Marco (1993).

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$\partial\Delta$ is a Jordan curve

Douady-Sullivan's conjecture holds in the following cases:

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   1. Douady-Herman- $\cdots$ (Séminaire Bourbaki 1986): quadratic polynomials.

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   5. G. Zhang (Invent. Math. 2011): all rational maps.

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2. $\alpha \in \mathscr{D}(2)$ is of bounded type (transcendental entire functions):
   1. Geyer (Tran. AMS, 2001): $f(z) = \lambda z e^z$.
   2. Keen-G. Zhang (Ergod. Th. Dynam. Sys. 2009): $f(z) = \lambda(z + az^2)e^z$.
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1. $\alpha \in \mathscr{D}(2)$ is of bounded type (rational maps):
   1. Douady-Herman- $\cdots$ (Séminaire Bourbaki 1986): quadratic polynomials.
   2. Zakeri (Comm. Math. Phys. 1999): cubic polynomials.
   3. Geyer (Tran. AMS, 2001): $f(z) = \lambda z(1 + z/d)^d$.
   4. Shishikura (unpublished 2001): all polynomials.
   5. G. Zhang (Invent. Math. 2011): all rational maps.

2. $\alpha \in \mathscr{D}(2)$ is of bounded type (transcendental entire functions):
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   2. Keen-G. Zhang (Ergod. Th. Dynam. Sys. 2009): $f(z) = \lambda(z + az^2)e^z$.
   3. Zakeri (Duke Math. J 2010): $f(z) = Pe^Q$, where P, Q are polynomials.
   4. G. Zhang (Illinois J. Math. 2005): $f(z) = \lambda \sin(z)$.
   5. Y. (Acta Math. Sin. (Engl. Ser.) 2013): $f(z) = \lambda \sin(z) + a \sin^3(z)$

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4. α satisfies L. Shen's condition $\mathscr{S}$:

$$\log^2 a_n \leq n \cdot \log n \cdot \log\log n \cdots \underbrace{\log\log \cdots \log}_{k} n.$$

where k is a fixed integer.

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Most of the proofs above are based on studying Blaschke models.

Herman's conjecture

$\partial\Delta$ contains a singular value $\Leftrightarrow$ $\alpha \in \mathscr{H}$

Herman's conjecture (the 'if' part) holds in the following cases:

1. Ghys (C. R. Acad. Sci. Paris 1984): $\Delta_f \Subset Dom(f)$ and $\partial\Delta_f$ is a Jordan curve.
2. Herman (CMP 1985): $f(z) = z^d + c$ and $f(z) = e^{az}$, where $d \geq 2$ and $a \in \mathbb{C} \setminus \{0\}$.

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Chéritat-Roesch's proof is based on: A refinement of Herman's proof, two theorems of Mañé, and the separation theorem of Goldberg-Milnor-Poirier-Kiwi.

Topology and geometry

1. Regularity of the boundary of the Siegel disks:
   1. Rogers (ETDS 1992): Tame or wild (periodic pt on $\partial\Delta \Rightarrow$ wild).
   2. Avila-Buff-Chéritat (Acta Math. 2004): Smooth boundary for $f(z) = e^{2\pi i\alpha} z + z^2$.
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3. Self-similarity, scaling ratios and triangles in SD:
   1. McMullen (Acta Math. 1998): Self-similarity, $f(z)=e^{2\pi i\alpha}z+z^2$, quad. irrat. α.
   2. Buff-Henriksen (Math. Res. Let. 1999): A lower bound for the ratio of self-similarity.
   3. Gaidashev (CMP 2015): Lower and upper bounds for the ratio of self-similarity.

Topology and geometry

4. Local connectivity of the Julia set containing a Siegel disk:

1. Petersen (Acta Math. 1996) and Yampolsky (ETDS 1999): $f(z) = e^{2\pi i\alpha} z + z^2$ for bounded type α.
2. Yampolsky-Zakeri (JAMS 2001): Quadratic rational map containing two bounded type Siegel disks.
3. Petersen-Zakeri (Ann. Math. 2004): $f(z) = e^{2\pi i\alpha} z + z^2$ for $\alpha \in \mathscr{P}\mathscr{Z}$.
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5. Conformal radius $r(\alpha)$ and the error function Υ:

$$\Upsilon(\alpha) = \log r(\alpha) + Y(\alpha), \text{ where } Y(\alpha) = \sum_{n\geq 0} \alpha_1 \cdots \alpha_n \log \frac{1}{\alpha_{n+1}}.$$

1. Yoccoz (1988): $\Upsilon > C$.
2. Buff-Chéritat (Invent. Math. 2004): $\Upsilon < 16$.
3. Buff-Chéritat (Ann. Math. 2006): Υ is continuous.
4. Marmi-Moussa-Yoccoz conjecture (CMP 1997): Υ is $1/2$-Hölder continuous.

Topology and geometry

6. Small cycles:

   1. Yoccoz (1988): $\alpha \notin \mathscr{B}$, f quad. poly. $\Rightarrow$ small cycles.
   2. Pérez-Marco (Thesis, 1990): Cremer $+ \sum \frac{\log\log(q_{n+1})}{q_n} < \infty \Rightarrow$ small cycles.

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Theorem (Fatou, 1920s):

Let f be a rational map or a transcendental entire or meromorphic function, and suppose that f has a Siegel disk. Then its boundary is contained in the postsingular set of f.

Improved by

1. Mañé (Bol. Soc. Bras. Mat. 1993) for rational maps: $\partial\Delta \subset \overline{O^+(c)}$, where $c \in \omega(c)$.
2. Rempe-van Strien (Tran. AMS 2011) for exponential maps: $\partial\Delta \subset \overline{O^+(s)}$, where $s \in \omega(s)$.

Near-para. renorm. and Inou-Shishikura's class

In 2006, Inou and Shishikura introduced a class of holomorphic maps IS_0 which is invariant under the **parabolic renormalization** operator $\mathscr{R}_0$. Accordingly, the perturbed class IS_α ($\alpha \neq 0$ is very small) can be acted under the **near-parabolic renormalization** operator $\mathscr{R}$.

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The operator $\mathscr{R}$ can be iterated infinitely times on IS_α if

$$\alpha \in \mathrm{HT}_N := \{\alpha = [0; a_1, a_2, \cdots] \in (0,1) \setminus \mathbb{Q} \mid a_n \geq N \text{ for all } n \geq 1\}$$

for some large N.

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Near parabolic renormalization is a very **powerful tool** to study the dynamics of irrational indifferent fixed points.

Remarkable applications

Some remarkable applications of Inou-Shishikura's class (for high type):

1. Buff-Chéritat (Ann. Math. 2012): Quadratic Julia sets with positive area.
2. Cheraghi (CMP 2013): Zero area of post-critical set for Brjuno.
3. Avila-Cheraghi (to appear JEMS 2014): Statistical properties (uniquely ergodic on the post-critical set) and small cycles.
4. Cheraghi-Chéritat (Invent. Math. 2015): Marmi-Moussa-Yoccoz conjecture.
5. Avila-Lyubich (arXiv 2015): Quadratic Feigenbaum Julia sets with positive area.
6. Cheraghi-Shishikura (arXiv 2015): MLC at infinitely satellite renormalization pts.
7. Cheraghi (to appear Ann. Sci. École Norm. Sup. 2016): Douady's conjecture and zero area of post-critical set for non-Brjuno.
8. Shishikura-Y. (arXiv 2016): Douady-Sullivan's conjecture and Herman's conjecture.
9. Chéritat-Y. (preprint 2016): Self-similarity of bounded type Siegel disks.
10. Cheraghi (arXiv 2017): Topology of the invariant hedgehogs.

Maps in Inou-Shishikura's class

Inou-Shishikura (2006):

$$IS_0 \supsetneq \left\{ f : Dom(f) \to \mathbb{C} \;\middle|\; \begin{array}{l} 0 \in Dom(f) \text{ open} \subset \mathbb{C}, f \text{ is holo. in } Dom(f), \\ f(0)=0,\ f'(0)=1, f : Dom(f) \setminus \{0\} \to \mathbb{C}^* \text{ is a} \\ \textbf{branched covering} \text{ with } \textbf{a unique critical value} \\ cv_f,\ \text{all critical points are of } \textbf{local degree 2} \end{array} \right\}.$$

The following maps (their variations or renormalization) are contained in IS_α:

1. $f_\alpha(z) = e^{2\pi i \alpha} z + z^2$;
2. $g_\alpha(z) = e^{2\pi i \alpha} \dfrac{z}{(1-z)^2}$;
3. $P_{n,\alpha}(z) = e^{2\pi i \alpha} z \left(1 + \dfrac{z}{n}\right)^n$, where $n \geq 2$;
4. $E_\alpha(z) = e^{2\pi i \alpha} z e^z$,
5. $S_\alpha(z) = e^{\pi i \alpha} \sin(z)$.

Main result

Theorem (Shishikura-Y. 2016):

Let $\alpha \in HT_N$ and assume that $P_\alpha(z) = e^{2\pi i\alpha} z + z^2$ has a Siegel disk Δ_α. Then $\partial\Delta_\alpha$ is a Jordan curve, and $-e^{2\pi i\alpha}/2 \in \partial\Delta_\alpha$ if and only if α is of Herman type.

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Key ingredients in the proof: Compare the contraction factors between the renormalization towers and Yoccoz's characterization on the Brjuno numbers and Herman numbers.

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For $\alpha \in (0,1)$ and $x \in \mathbb{R}$, define

$$r_\alpha(x) = \begin{cases} \dfrac{1}{\alpha}\left(x - \log\dfrac{1}{\alpha} + 1\right) & \text{if} \quad x \geq \log\dfrac{1}{\alpha}, \\ e^x & \text{if} \quad x < \log\dfrac{1}{\alpha}. \end{cases}$$

Theorem (Yoccoz's characterization on $\mathscr{H}$, 2002):

$$\mathscr{H} = \{\alpha \in \mathscr{B} : \forall\, m \geq 0, \exists\, n > m \text{ s.t. } r_{\alpha_{n-1}} \circ \cdots \circ r_{\alpha_m}(0) \geq \mathscr{B}(\alpha_n)\}.$$

Related topics

Some related topics include:

1. Herman rings (Shishikura, Milnor, Fagella, Henriksen, Nayak $\cdots$)
2. Cremer and hedgehogs (Pérez-Marco, Kiwi, Blokh, Oversteegen, Childers, Biswas $\cdots$)
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Two possible directions: Extend the near-parabolic renormalization theory to

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THANK YOU FOR YOUR ATTENTION !