Local connectivity of Julia sets for rational maps with Siegel disks

YANG Fei

Nanjing University

Joint with Shuyi Wang, Gaofei Zhang and Yanhua Zhang

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Complex dynamical systems

$f : X \to X$

$$x \mapsto f(x) \mapsto f(f(x)) \mapsto f(f(f(x))) \mapsto \cdots \mapsto f^{\circ n}(x) \mapsto \cdots$$

Complex dynamical systems: X complex manifold, f holomorphic.

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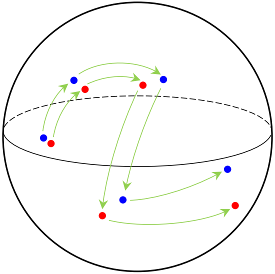

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map.

The **Fatou set** (or stable set) of f:

$$F(f) := \{z \in \widehat{\mathbb{C}} : \ \{f^{\circ n}\}_{n \in \mathbb{N}} \text{ is equicontinuous at } z\}.$$

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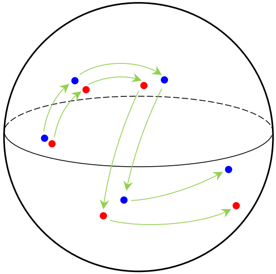

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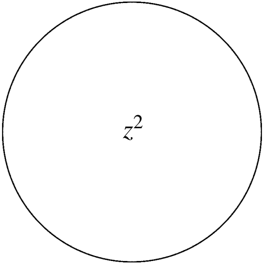
The **Julia set** (or chaotic set) $J(f) := \widehat{\mathbb{C}} \setminus F(f)$.

Each connected component of $F(f)$ (resp. $J(f)$) is called a **Fatou** (resp. **Julia**) **component**.

Julia / Fatou sets



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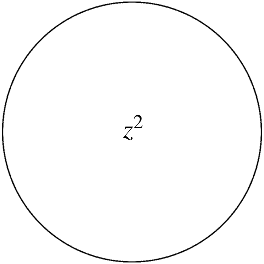
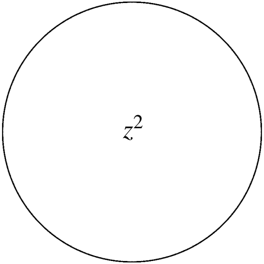
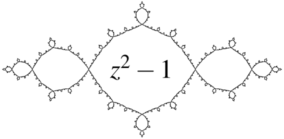


$z^2 - 1$?

Julia / Fatou sets





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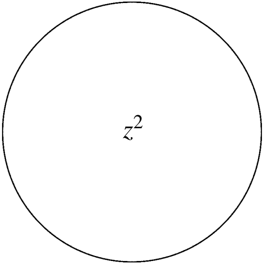
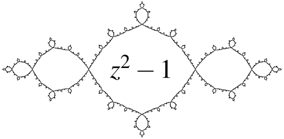
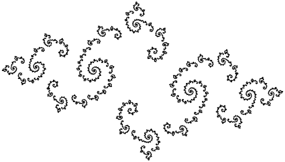


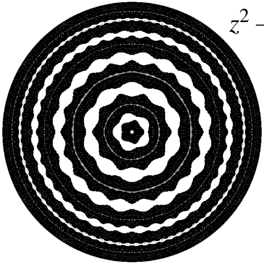
z^2



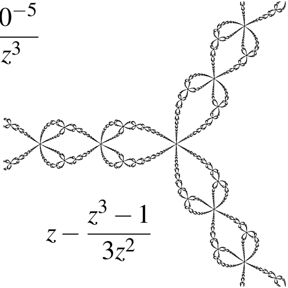
$z^2 - 1$



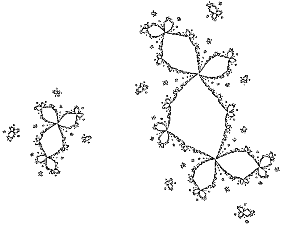
$z^2 - 0.78 + 0.23i$



$z^2 + \frac{10^{-5}}{z^3}$



$z - \frac{z^3 - 1}{3z^2}$



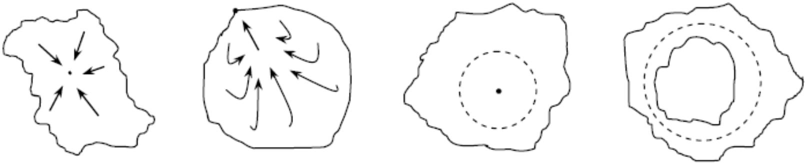
$z^3 - 0.48z + 0.7 + 0.5i$

Sullivan's eventually periodic theorem

Theorem (Sullivan, Ann. Math., 1985)

The Fatou components of all rational maps are eventually periodic.

Classification of periodic Fatou components of rational maps:

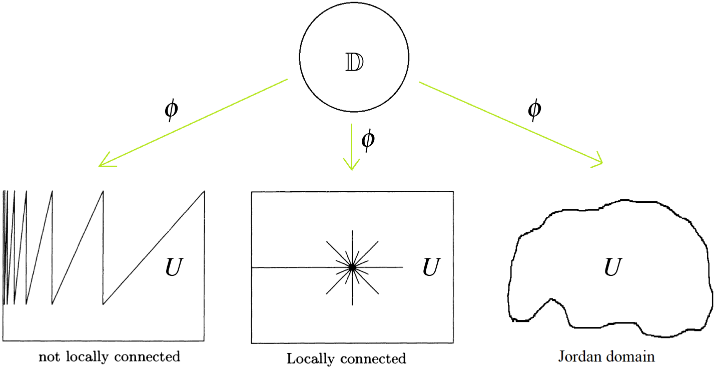


(super-)attracting basin 1884, 1904 | parabolic basin 1897 | Siegel disk 1942 | Herman ring 1979

Study Julia sets from Fatou sets

Carathéodory (1913):

The biholomorphic map $\phi : \mathbb{D} \to U \subset \widehat{\mathbb{C}}$ can be extended to a continuous map $\overline{\phi} : \overline{\mathbb{D}} \to \overline{U} \Longleftrightarrow \partial U$ is **L**ocally **C**onnected.
In particular, $\overline{\phi}$ is a homeomorphism $\Longleftrightarrow \partial U$ is a Jordan curve.



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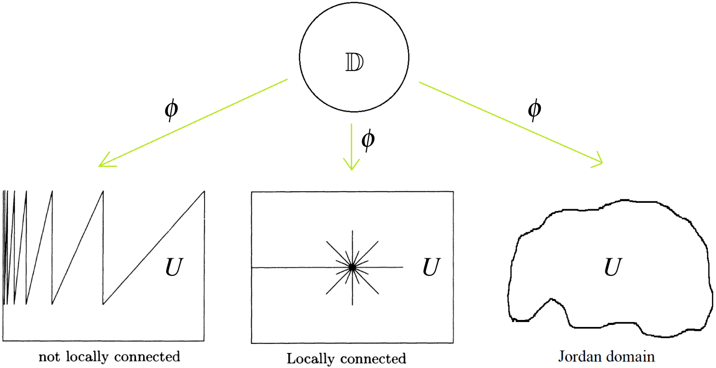
The biholomorphic map $\phi : \mathbb{D} \to U \subset \widehat{\mathbb{C}}$ can be extended to a continuous map $\overline{\phi} : \overline{\mathbb{D}} \to \overline{U} \iff \partial U$ is **Locally Connected**.
In particular, $\overline{\phi}$ is a homeomorphism $\iff \partial U$ is a Jordan curve.



If LC holds: Dynamics from Fatou components to the Julia set continuously.

Local connectivity of Julia sets

The **connected** Julia sets are **locally connected** in the following cases:

- Hyperbolic, Subhyperbolic (Douady-Hubbard, 1980s)
- Geometrically finite (Tan-Yin, 1996)
- Semi-hyperbolic (Carleson-Jones-Yoccoz, 1994; Yin, 1999)
- Collet-Eckmann (Graczyk-Smirnov, 1998)
- Real polynomials (Levin-van Strien, 1998; Clark-van Strien-Trejo, 2017)
- At most finitely many renormalizable polynomials (Yoccoz, 1980s; Kozlovski-van Strien, 2009)
- Some infinitely renormalizable quadratics (Lyubich, Kahn, Levin, ..., 1997-now)
- Newton maps (Roesch, 2008; Drach-Schleicher, 2022; Wang-Yin-Zeng, 2022)
- McMullen maps (Qiu-Wang-Yin, 2012)
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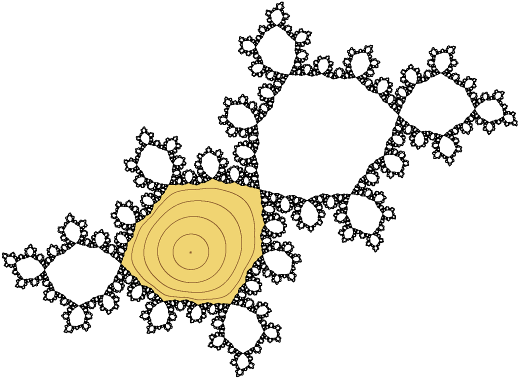
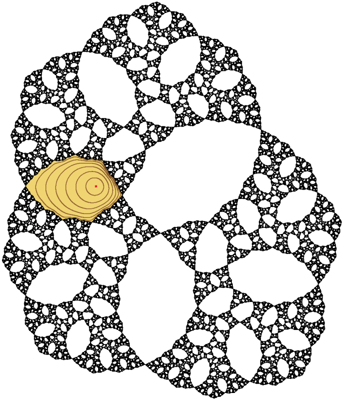
Expanding metrics: Hyperbolic, orbifold, ...
Combinatorial tools: Puzzles developed by Yoccoz, Branner-Hubbard, Lyubich, Kozlovski-Shen-van Strien, Roesch, ...
Analytic tools: Grötzsch inequality on modulus, Kahn-Lyubich covering lemma ...

Siegel disks



The Siegel disk of $f(z) = e^{2\pi i\alpha} z + z^2$, where
$$\alpha = \frac{\sqrt{5}-1}{2} = [0; 1, 1, 1, \cdots]$$



The Siegel disk of $f(z) = \dfrac{e^{2\pi i\alpha} z}{(1-z)^2}$

LC of Julia sets: with Siegel disks

Theorem (Petersen, Acta Math., 1996)

For any bounded type α, the Julia set of $P(z) = e^{2\pi i\alpha} z + z^2$ is locally connected.

$\alpha = [0; a_1, a_2, \cdots, a_n, \cdots]$ is of **bounded type** if $\sup_n \{a_n\} < \infty$.

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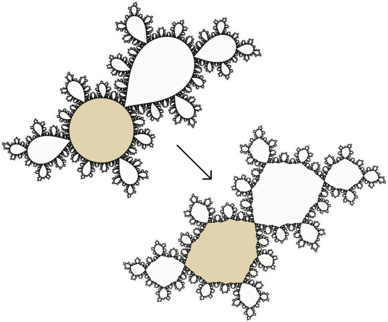
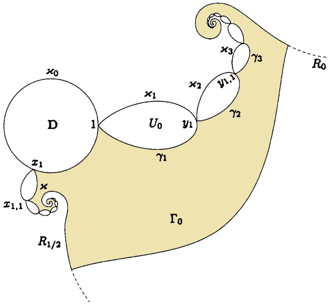


Blaschke model $f_\alpha(z) = e^{2\pi i t(\alpha)} z^2 \dfrac{z-3}{1-3z}$



Petersen's puzzles

Main result

Main Theorem (Wang-Y.-Zhang-Zhang, arXiv, 2022)

Suppose f is a ***rational*** *map with Siegel disks such that $J(f)$ is connected, and moreover, the forward orbit of every critical point of f satisfies one of the following:*

1. *It is finite; or*
2. *It lies in an attracting basin; or*
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It provides an alternative proof of Petersen's result without using puzzles.

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Immediate consequences: The Julia sets of following maps are locally connected:

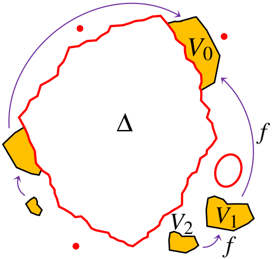
- Cubic polynomials in Zakeri's curves;
- Cubic Newton maps with a bounded type Siegel disk;
- McMullen maps $z \mapsto z^m + \lambda/z^n$ ($m \geqslant 2$, $n \geqslant 1$) with a bounded type Siegel disk.

Main Lemma

The *postcritical set* of f is $\mathscr{P}(f) := \overline{\bigcup_{n\geqslant 1} f^n(\mathrm{Crit}(f))}$.

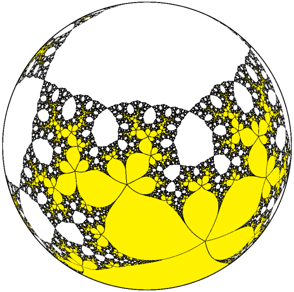
Main Lemma (WYZZ, 2022)

Let f be $\deg \geqslant 2$ rational having a fixed bounded type Siegel disk Δ with $\mathrm{dist}_{\widehat{\mathbb{C}}}(\mathscr{P}(f)\setminus\partial\Delta, \partial\Delta) > 0$. Then $\forall\, \varepsilon > 0$, and $\forall$ Jordan domain $V_0 \subset \widehat{\mathbb{C}} \setminus \overline{\Delta}$ with $\emptyset \neq \overline{V}_0 \cap \mathscr{P}(f) \subset \partial\Delta$, $\exists N = N(\varepsilon, V_0, f) \geq 1$, s.t. $\mathrm{diam}_{\widehat{\mathbb{C}}}(V_n) < \varepsilon$ for all $n \geqslant N$, where V_n is any connected component of $f^{-n}(V_0)$.



Parallel to **classical Shrinking Lemma** (Tan-Yin 1996, Lyubich-Minsky 1997): $\overline{V}_0 \cap \mathscr{P}(f) = \emptyset$.

Further developments



Work in progress (Fu-Y., based on the Main Lemma):

- LC of the Julia set of Siegel rational maps with **parabolic** points;
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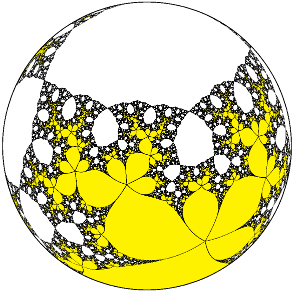

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Potential topics and challenges:

- Extending the bounded type rotation numbers to unbounded type;
- Allowing the critical orbits to have larger degrees of freedom.

Thank you for your attention!