Sierpiński carpet Julia sets of holomorphic maps

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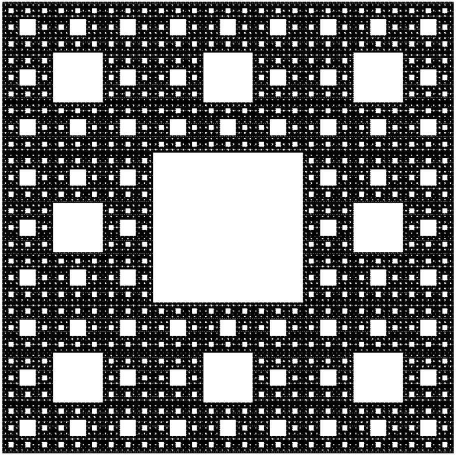
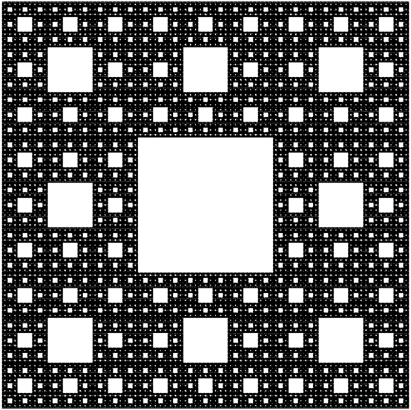
Sierpiński carpets



The middle ninths square Sierpiński carpet S_3

Sierpiński carpets

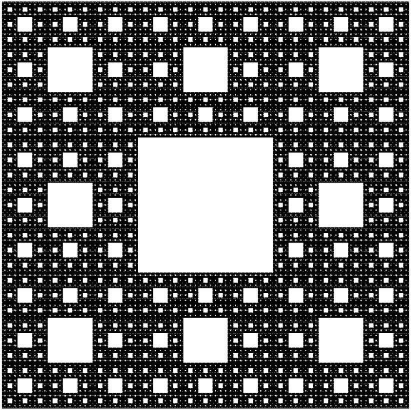


carpet S_3

According to Whyburn (1958), if a subset $S \subset \widehat{\mathbb{C}}$ satisfies

- S is compact;
- S is connected;
- S is locally connected;
- S has empty interior; and
- The complementary components of S are bounded by pairwise disjoint simple closed curves;

then S is homeomorphic to S_3, and S is also called a **Sierpiński carpet** (**carpet** in short).



carpet S_3

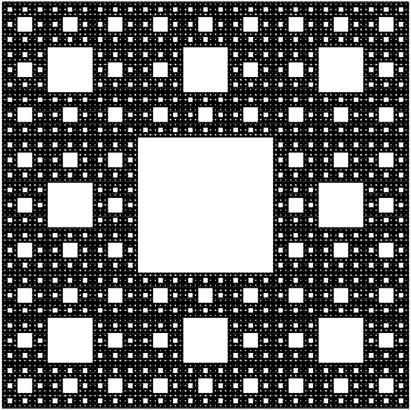
Equivalently, S can be written as

$$S = \widehat{\mathbb{C}} \setminus \bigcup_{i \in \mathbb{N}} D_i,$$

where

- Each D_i is a Jordan disk;
- $\mathrm{diam}(D_i) \to 0$ as $i \to \infty$;
- The **peripheral circles** $\{C_i = \partial D_i\}_{i \in \mathbb{N}}$ satisfy $C_i \cap C_j = \emptyset$ for all $i \neq j$; and
- S has empty interior.

Sierpiński carpets



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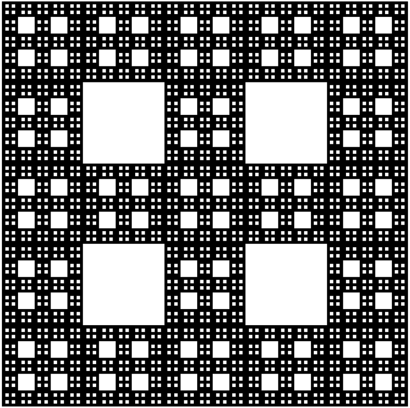
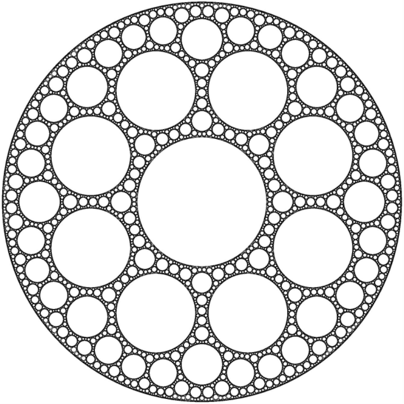


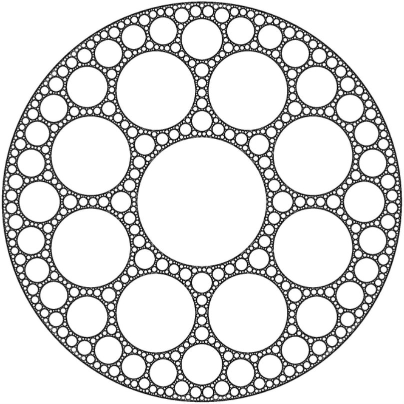
carpet $F_{5,1}$

Sierpiński carpets



a **round** carpet

Sierpiński carpets



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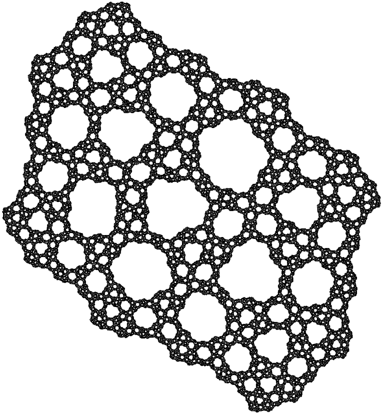

a carpet **Julia set**

Interesting subjects

All Sierpiński carpets are the same in the sense of **homeomorphism**.

People are interested in their geometric properties:

- Quasisymmetric equivalent;
- Quasisymmetric rigidity;
- Area and Hausdorff dimension.

Dynamical classification:

- Topological conjugacy on the Julia sets.

Quasisymmetric geometries

A homeomorphism $f : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called **quasisymmetric** if there is a distortion control function $\eta : [0, \infty) \to [0, \infty)$ which is also a homeomorphism such that

$$\frac{d_Y(f(a), f(b))}{d_Y(f(a), f(c))} \leq \eta\left(\frac{d_X(a, b)}{d_X(a, c)}\right)$$

for every triple of distinct points $a, b, c \in X$, and in this case (X, d_X), (Y, d_Y) are called **quasisymetrically equivalent**.

Note: $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is quasisymmetric $\Longleftrightarrow$ It is quasiconformal.

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Two basic questions in quasiconformal geometry are

1. Determine whether two given homeomorphic metric spaces are quasisymmetrically equivalent;
2. Study the quasisymmetric rigidity (q.s. group) of a given metric space.

Quasisymmetrics of Sierpiński carpets

The set of quasisymmetric equivalence classes of (round) Sierpiński carpets in $\widehat{\mathbb{C}}$ is uncountable (Bonk-Kleiner-Merenkov, 2009).

The investigations of quasisymmetric equivalence between the given Sierpiński carpet and a round carpet (**q.s. uniformization**) were partly motivated by a problem in geometric group theory:

Kapovich-Kleiner conjecture (2000): if G is a Gromov hyperbolic group such that $\partial_\infty G$ is a Sierpiński carpet, then $\partial_\infty G$ is quasisymmetrically equivalent to a round Sierpiński carpet in $\widehat{\mathbb{C}}$.

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Theorem (Bonk-Merenkov, Ann. Math., 2013)

(1) *Every quasisymmetric self-map of the standard carpet S_3 is a Euclidean isometry.*
(2) *The carpets S_p and S_q are quasisymmetrically equivalent if and only if $p = q$.*

Zeng-Su (2015) generalized this result to a special class of carpets $F_{n,p}$, and they proved that any S_m and $F_{n,p}$ are not quasisymmetrically equivalent.

Main tool: **carpet modulus**, which is an quasiconformal invariant.

Quasisymmetrics of Sierpiński carpets

Theorem (Bonk, Invent. Math., 2011)

Suppose that the peripheral circles $\{C_i\}_{i\in\mathbb{N}}$ of a carpet S are ***uniformly relatively separated*** *and are* ***uniform quasicircles****. Then there exists a quasiconformal map $\phi:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ such that $\phi(C_i)$ is a round circle for all $i\in\mathbb{N}$.*

By uniformly relatively separated, it means that $\exists c>0$ s.t. the *relative distance* $\Delta(C_i,C_j)$ of any two peripheral circles satisfies

$$\Delta(C_i,C_j):=\frac{\mathrm{dist}_\sigma(C_i,C_j)}{\min\{\mathrm{diam}_\sigma(C_i),\mathrm{diam}_\sigma(C_j)\}}\geq c.$$

Remark: Any S_m or $F_{n,p}$ is quasisymmetrically equivalent to a round carpet.

Quasisymmetrics of Sierpiński carpet Julia sets

Theorem (Bonk-Lyubich-Merenkov, Adv. Math., 2016)

Let f, g be two ***post-critically finite*** *rational maps whose Julia sets $J(f)$ and $J(g)$ are Sierpiński carpets. Then*

- *The peripheral circles $\mathscr{C}_f$ of $J(f)$ (resp. $J(g)$) are uniform quasicircles and uniformly relatively separated. In particular, $J(f)$ (resp. $J(g)$) is q.s. equivalent to a round carpet.*
- *Every q.s. homeo. from $J(f)$ onto $J(g)$ is the restriction of a Möbius map.*
- *The quasisymmetric group $QS(J(f))$ (resp. $J(g)$) is finite.*

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Qiu-Y.-Zeng (2019) generalized this result to **semi-hyperbolic** rational maps.

Quasisymmetrics of Sierpiński carpet Julia sets

Yin (1999): A rational map f is called **semi-hyperbolic** if the critical points in $J(f)$ are non-recurrent and f has no parabolic periodic points.

Theorem (Qiu-Y.-Zeng, Fund. Math., 2019)

Let f be a semi-hyperbolic rational map whose Julia set $J(f)$ is a Sierpiński carpet. Then

- $\mathscr{C}_f$ *are uniform quasicircles.*
- $\mathscr{C}_f$ *are uniformly relatively separated if and only if* $P_f \cap \mathscr{C}_f = \emptyset$, *where* P_f *is the post-critical set of* f.
- *If* $P_f \cap \mathscr{C}_f = \emptyset$, *then the quasisymmetric group* $QS(J(f))$ *is finite.*
- *Suppose that* $P_f \cap \mathscr{C}_f = \emptyset$ *and* $J(g)$ *is also a Sierpiński carpet,* $\sharp(F(f) \cap P_f) < \infty$, $\sharp(F(g) \cap P_g) < \infty$. *Then any quasisymmetric homeomorphism between* $J(f)$ *and* $J(g)$ *is the restriction of a Möbius transformation.*

New ingredients

New ingredients in the proof:

Lemma (The modulus controls the relative distance)

Let $A \subset \widehat{\mathbb{C}}$ be an annulus with two boundary components γ_1 and γ_2. If the modulus of A satisfies $\text{mod}(A) \geq m > 0$, then there exists a constant $C(m) > 0$ depending only on m such that the relative distance of γ_1 and γ_2 satisfies $\Delta(\gamma_1, \gamma_2) \geq C(m) > 0$.

Tool in the proof: Teichmüller's module theorem.

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Tool in the proof: Teichmüller's module theorem.

Lemma (Post-critically infinite and semi-hyperbolic)

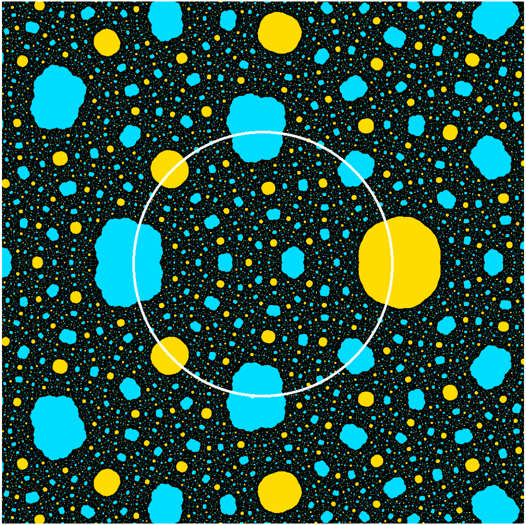
There exists a real quadratic polynomial $P_c(z) = z^2 + c$ such that:

- P_c is post-critically infinite in $J(P_c)$; and
- 0 is non-recurrent and $\omega(0) \cap \beta(P_c) = \emptyset$.

Kawahira-Kisaka (2018) also constructed such examples.

Milnor and Tan Lei's example

(1993)



The parameters

$a \approx -0.138115091$ and
$b \approx 0.303108805$

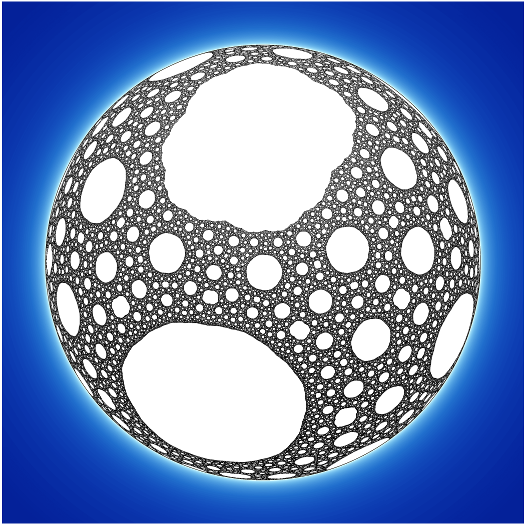
are chosen such that

$$f(z) = a\left(z + \frac{1}{z}\right) + b$$

has two super-attracting periodic points with periods 3 and 4.

Pilgrim's example

(1994)



The parameter

$$c \approx -0.69562$$

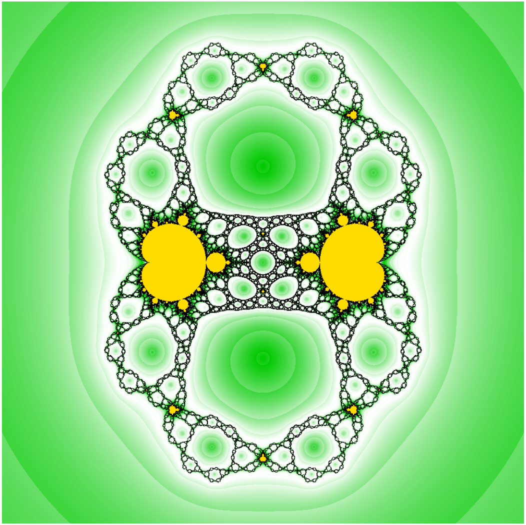
is chosen such that the post-critical set of

$$f_c(z) = c\frac{(z-1)^2(z+2)}{3z-2}$$

has exactly 4 points.

Devaney-Look-Uminsky's Sierpiński holes

(2005)



For $n \geq 2$ and $m \geq 1$, suppose that one free critical point ω_0 of

$$f_\lambda(z) = z^n + \frac{\lambda}{z^m}$$

is attracted by the immediate super-attracting basin B_λ of ∞. If

$$f_\lambda^{\circ k}(\omega_0) \in T_\lambda \neq B_\lambda$$

for some $k \geq 2$, then $J(f_\lambda)$ is a Sierpiński carpet.

Devaney-Fagella-Garijo-Jarque's result (2014)

For each $p \geq 1$, define

$$V_p = \left\{ f \in \mathrm{Rat}_2 \,\middle|\, \begin{array}{l} f \text{ has a super-attracting} \\ \text{periodic pt with period } p \end{array} \right\}.$$

In particular, the non-escaping locus of V_1 is the Mandelbrot set.

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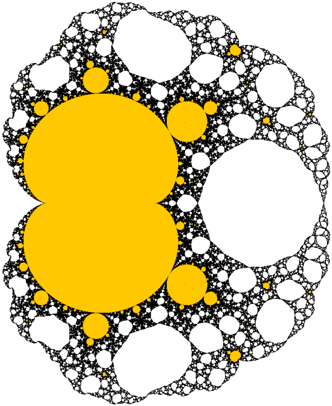

Non-escaping locus in V_2

For each $p \geq 1$, define

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According to Aspenberg-Yampolsky (2009) and Devaney-Fagella-Garijo -Jarque (2014), V_p contains Sierpiński holes if and only if $p \geq 3$.

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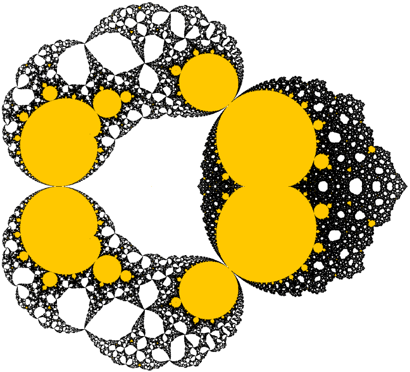

Non-escaping locus in V_3

For each $p \geq 1$, define

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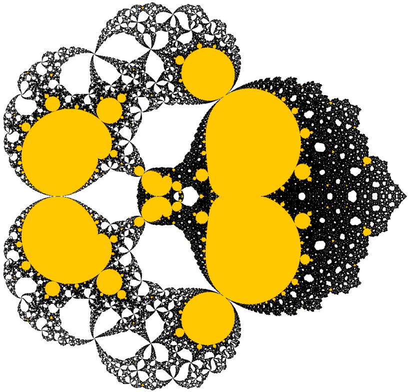

Non-escaping locus in V_4

For each $p \geq 1$, define

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According to Aspenberg-Yampolsky (2009) and Devaney-Fagella-Garijo -Jarque (2014), V_p contains Sierpiński holes if and only if $p \geq 3$.

Carpet Julia sets in other families

- Steinmetz (2008), the Morosawa-Pilgrim family:

$$f_t(z) = t\left(1 + \frac{(4/27)z^3}{1-z}\right), \quad \text{where } t \in \mathbb{C}^*.$$

- Look (2010), models for imaginary 3-circle inversions:

$$G_\rho(z) = \frac{-\rho^2 z^2}{z^3 - 1}, \quad \text{where } \rho \in \mathbb{C}^*.$$

- Xiao-Qiu-Yin (2014), the generalized McMullen maps:

$$f_{a,b}(z) = z^n + \frac{a}{z^n} + b, \quad \text{where } n \geq 2,\ a, b \in \mathbb{C}^*.$$

Carpet Julia sets in other families

- Fu-Y. (2015):

$$h_\lambda(z) = \frac{z^n(z^{2n} - \lambda^{n+1})}{z^{2n} - \lambda^{3n-1}},$$

where $n \geq 2$ and $\lambda \in \mathbb{C}^* - \{\lambda : \lambda^{2n-2} = 1\}$.

- Hu-Jimenez-Muzician (2012) and Hu-Muzician-Xiao (2018),
The regularly ramified rational maps (for example):

$$f_\lambda(z) = f_\lambda^{(2,3,4)}(z) = \frac{\lambda z^4(z^4 - 1)^4}{(z^4+1)^2(z^4 - (1+\sqrt{2})^4)^2(z^4 - (1-\sqrt{2})^4)^2}.$$

- Wang-Y.-Zhang-Liao (2019):

$$f_\lambda(z) = \frac{z^{2n} - \lambda^{3n+1}}{z^n(z^{2n} - \lambda^{n-1})},$$

where $n \geq 2$ and $\lambda \in \mathbb{C}^* - \{\lambda : \lambda^{2n+2} = 1\}$.

A criterion to generate carpet Julia sets

Theorem (Y., Proc. AMS, 2018)

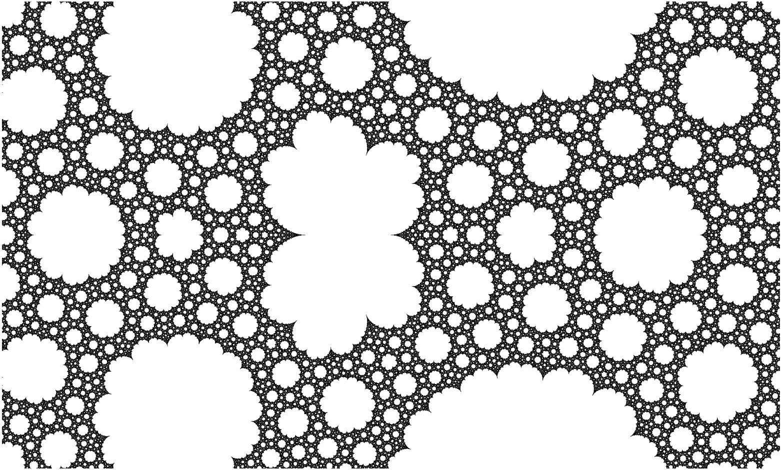
Let f be a rational map satisfying the following conditions:

- *The Julia set of f is connected;*
- *There exist two different Fatou components U and V of f such that $f(U) = U$ and $f^{-1}(U) = U \cup V$;*
- *If W is a connected component of $f^{-1}(V)$, then $\deg(f|_W) > \deg(f|_U)$; and*
- *The intersection of any critical orbit with U is non-empty; In particular, f has degree at least 3.*

Then the Julia set of f is a Sierpiński carpet.

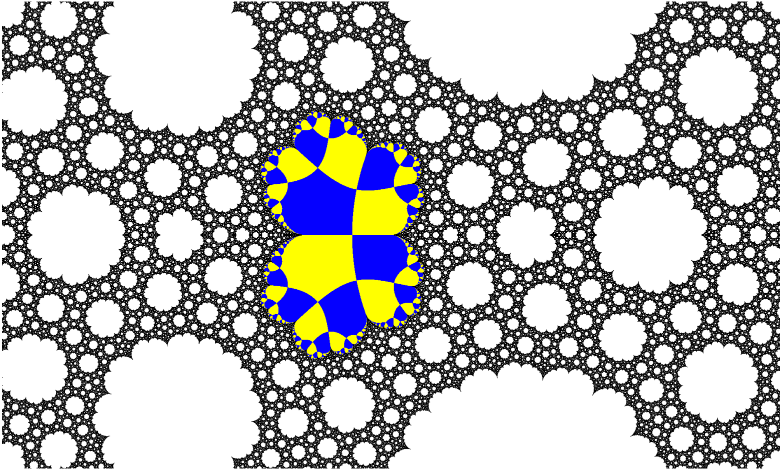
Applications: McMullen family, Morosawa-Pilgrim family, generating parabolic carpet Julia sets etc.

A parabolic carpet Julia set



The Julia set of $g(z) = -\dfrac{289\,(87z^2 - 20z + 120)}{11\,(z-18)^3}$

A parabolic carpet Julia set



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Results on quadratic polynomials

Some known results on $P_c(z) = z^2 + c$:

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Lyubich, Shishikura (1991): such Julia sets have zero area.

Theorem (Fu-Y., Math. Z., 2019)

- *There exist carpet Julia sets with positive area.*
- *There exist carpet Julia sets with Hausdorff dimension two but with zero area.*

Some known results:

1. Barański-Wardal (2015): For $f_{\lambda,n}(z) = z^n + \lambda/z^n$ with $n \geq 2$, there exists $\lambda = \lambda(n)$ such that $J(f_{\lambda,n})$ is a Sierpinski carpet, and

$$\lim_{n\to\infty} \dim_H(J(f_{\lambda,n})) = 1.$$

Bounds and consequence

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Theorem (Fu-Y., 2019)

There exist carpet Julia sets with Hausdorff dim. equaling to any given $s \in (1,2)$.

The sharpness of the result?

Let f be a rational map whose Julia set is a Sierpinski carpet. If f has

1. (Przytycki, 2006) an attracting basin U; or
2. (based on Zdunik, 1990) a parabolic basin U,

then $\dim(J(f)) \geq \dim(\partial U) > 1$.

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How about the Siegel case?

Theorem (Cheraghi-DeZotti-Y., preprint, 2019)

Suppose that $f(z) = e^{2\pi i\alpha}z + z^2$ has a Siegel disk whose boundary does not contain the critical point. If α is of sufficiently high type, then the post-critical set of f has Hausdorff dimension two.

Carpet Julia sets with dim greater than 1

Theorem (Fu-Y., 2019)

Let f be a rational map having a Sierpiński carpet Julia set. Then $\dim_H(J(f)) > 1$ *if any one of the following conditions holds:*

1. *f contains an attracting basin;*
2. *f contains a parabolic basin; or*
3. *f is renormalizable and the small filled Julia set contains a sufficiently high type quadratic Siegel disk.*

Remark: If a Sierpiński carpet rational map contains a Siegel disk, then the boundary of this Siegel disk contains no critical points.

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3. *f is renormalizable and the small filled Julia set contains a sufficiently high type quadratic Siegel disk.*

Remark: If a Sierpiński carpet rational map contains a Siegel disk, then the boundary of this Siegel disk contains no critical points.

We conjecture that all carpet Julia sets have Hausdorff dim strictly great than 1.

Carpet Julia sets with dim greater than 1

Theorem (Fu-Y., 2019)

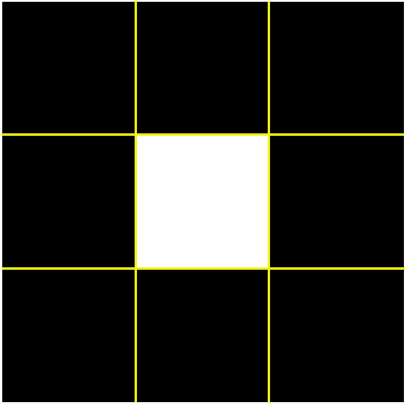
Let f be a rational map having a Sierpiński carpet Julia set. Then $\dim_H(J(f)) > 1$ if any one of the following conditions holds:

1. *f contains an attracting basin;*
2. *f contains a parabolic basin; or*
3. *f is renormalizable and the small filled Julia set contains a sufficiently high type quadratic Siegel disk.*

Remark: If a Sierpiński carpet rational map contains a Siegel disk, then the boundary of this Siegel disk contains no critical points.

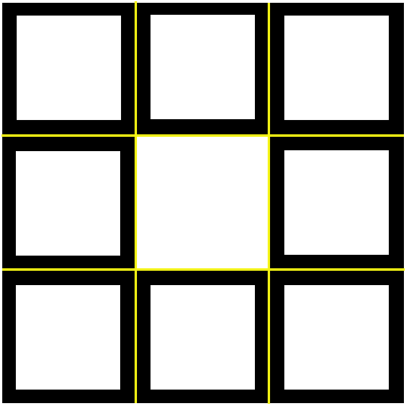
We conjecture that all carpet Julia sets have Hausdorff dim strictly great than 1. But ...

Sierpinski carpets with dimension one



Constructing a Sierpinski carpet with dimension one
(Courtesy of Huojun Ruan)

Sierpinski carpets with dimension one



Constructing a Sierpinski carpet with dimension one
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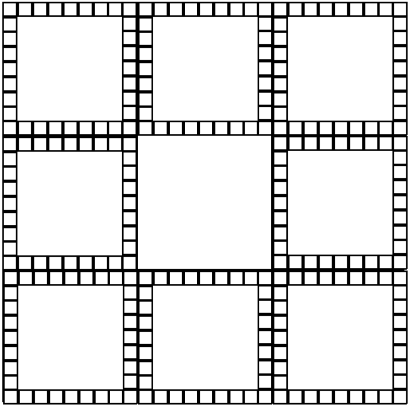
Sierpinski carpets with dimension one



Constructing a Sierpinski carpet with dimension one
(Courtesy of Huojun Ruan)

THANK YOU FOR YOUR ATTENTION !