Meromorphic functions with smooth degenerate Herman rings

Fei YANG

Nanjing University, China

On geometric complexity of Julia sets - VI

Będlewo

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Origin

Fatou, 1920[1]:

Il nous resterait à étudier les courbes analytiques invariantes par une transformation rationnelle et dont l'étude est intimement liée à celle des fonctions étudiées dans ce Chapitre. Nous espérons y revenir bientôt.

> It would remain for us to study the **invariant analytic curves** of a rational transformation and whose study is intimately linked to that of the functions studied in this chapter. We hope to return there soon.

Motivation: decomposing dynamics.

[1] P. Fatou, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **48** (1920), 208–314.

Siegel disk and continued fractions

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, f non-linear holo., $f(z) = e^{2\pi i\alpha} z + \mathscr{O}(z^2)$.

The *maximal* region in which f is conjugate to $R_\alpha(\zeta) = e^{2\pi i\alpha}\zeta$ is a simply connected domain Δ_f called the **Siegel disk** of f centered at 0 (i.e., locally linearizable):

$$\begin{array}{ccc} \Delta_f & \xrightarrow{\ f\ } & \Delta_f \\ \Big\downarrow \varphi & & \Big\downarrow \varphi\text{, conformal} \\ \mathbb{D} & \xrightarrow{\ R_\alpha\ } & \mathbb{D} \end{array}$$

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$$\begin{array}{ccc} \Delta_f & \xrightarrow{f} & \Delta_f \\ \downarrow \varphi & & \downarrow \varphi,\ \text{conformal} \\ \mathbb{D} & \xrightarrow{R_\alpha} & \mathbb{D} \end{array}$$

Let

$$\alpha = [a_0; a_1, a_2, \cdots, a_n, \cdots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

be the **continued fraction expansion** of α, where $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}^+$ for all $n \geqslant 1$. Then $a_0 + \frac{p_n}{q_n} := [a_0; a_1, a_2, \cdots, a_n] \to \alpha$ as $n \to \infty$, where $p_n, q_n \in \mathbb{N}^+$ are coprime.

Siegel-Brjuno-Yoccoz

Theorem (Siegel, 1942)

The holomorphic germ f is locally linearizable at 0 if $\alpha \in \mathscr{D} = \bigcup_{\kappa \geqslant 2} \mathscr{D}_\kappa$, where

$$\mathscr{D}_\kappa = \left\{ \alpha \in \mathbb{R} \setminus \mathbb{Q} : \sup_{n \geqslant 1} \left\{ \frac{a_{n+1}}{q_n^{\kappa-2}} \right\} < \infty \right\}.$$

Remark: α is called **bounded type** if $\alpha \in \mathscr{D}_2$, i.e. $\sup_n \{a_n\} < \infty$.

Theorem (Brjuno, 1965)

The holomorphic germ f is locally linearizable at 0 if $\alpha \in \mathscr{B}$, where

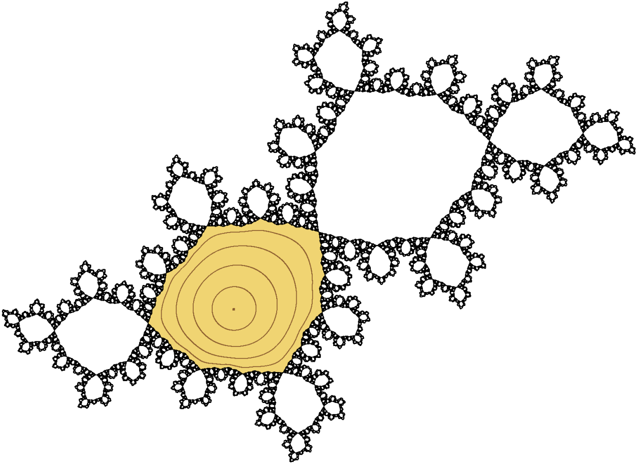
$$\mathscr{B} = \left\{ \alpha \in \mathbb{R} \setminus \mathbb{Q} : \sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty \right\}.$$

Remark: $\mathscr{D} \subsetneq \mathscr{B}$.

Theorem (Yoccoz, 1988)

If $f(z) = e^{2\pi i \alpha} z + z^2$ is locally linearizable at 0, then $\alpha \in \mathscr{B}$.

A Siegel disk



The Siegel disk of $f(z) = e^{2\pi i\alpha}z + z^2$, where $\alpha = \dfrac{\sqrt{5}-1}{2} = [0; 1, 1, 1, \cdots]$

Arnold-Herman-Yoccoz

Theorem (Arnold, 1965)

Let $\alpha \in \mathscr{D}$ and $\sigma > 1$. There exists a small $\varepsilon > 0$ such that if

- $f : \mathbb{S}^1 \to \mathbb{S}^1$ *is a homeomorphism with rotation number* $\rho(f) = \alpha$*; and*
- f *can be extended analytically and univalently to* $\{z : 1/\sigma < |z| < \sigma\}$ *and satisfies* $|f(z) - e^{2\pi i\alpha} z| < \varepsilon$ *there,*

then f *is conformally conjugate to* $R_\alpha(\zeta) = e^{2\pi i\alpha}\zeta$ *in* $\{z : 1/\sqrt{\sigma} < |z| < \sqrt{\sigma}\}$.

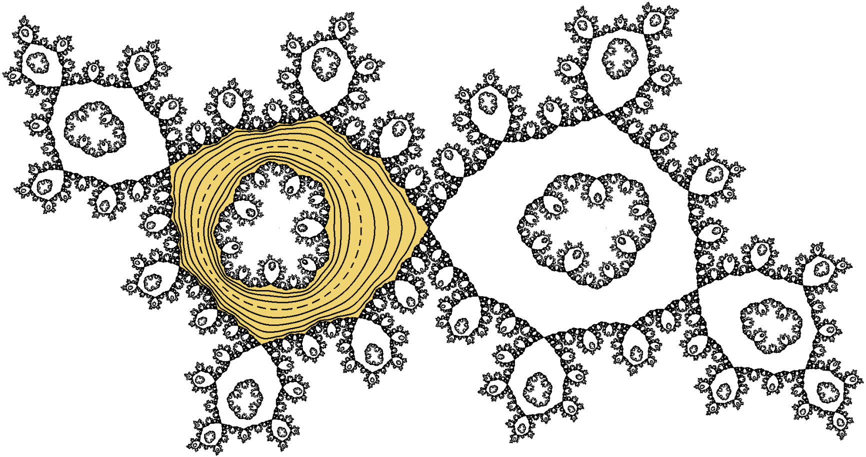
Theorem (Herman, 1979)

Let $\mathscr{H}$ *be the set of all* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *s.t. every orientation-preserving analytic circle diffeomorphism of rotation number* α *is analytically conjugate to* R_α*. Then* $\mathscr{D} \subsetneq \mathscr{H}$.

Theorem (Yoccoz, 2002)

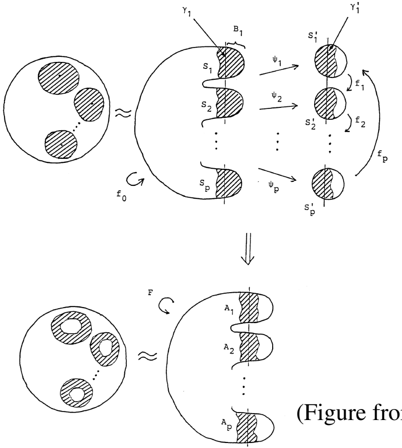
If $\alpha \notin \mathscr{H}$*, then* $\exists$ *an analytic circle diffeo.* f *with* $\rho(f) = \alpha$ *which is not analytically linearizable.*

A Herman ring



The Herman ring of $f(z) = e^{2\pi i t} z^2 \dfrac{z-4}{1-4z}$, where $t = 0.6151732...$ s.t. $\rho(f) = \dfrac{\sqrt{5}-1}{2} = [0; 1, 1, 1, \cdots]$

Shishikura's construction of HR by qc surgery



(Figure from [Shi87])

Shishikura (1987): HR can be obtained from SD through qc surgery, and vice versa.

Invariant analytic Jordan curves

Invariant analytic curves under **rational maps**:

Circles, level curves of linearizing functions of Siegel disks and Hermann rings.

Theorem (Azarina, 1989)

*Suppose $f(\gamma) \subset \gamma$, where f is **entire** and γ is a Jordan **analytic** curve. Then either*

- *γ is a circle and f is conformally conjugate to $z \mapsto z^n$ with $n \geqslant 1$; or*
- *γ is a level curve of the linearizing function of some Siegel disk.*

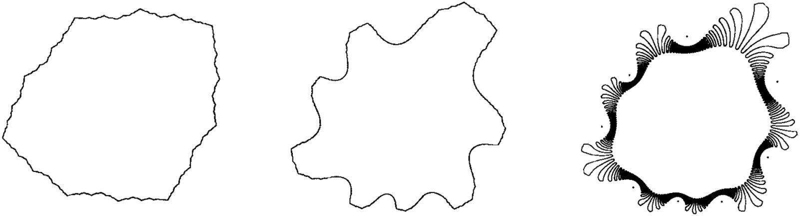
Theorem (Eremenko, 2012)

*There are Jordan analytic curves (algebraic and non-algebraic) which are not circles and invariant under the **Lattès maps** (Julia sets $= \widehat{\mathbb{C}}$), and moreover, the restriction on each of these curves is **not** a homeomorphism.*

Invariant smooth Jordan curves

Boundaries of **Siegel disks**:

- C^∞-smooth (Pérez-Marco 1997; Avila-Buff-Chéritat 2004, 2020)
- C^n but not C^{n+1}; C^0 but not Hölder (Buff-Chéritat 2007)

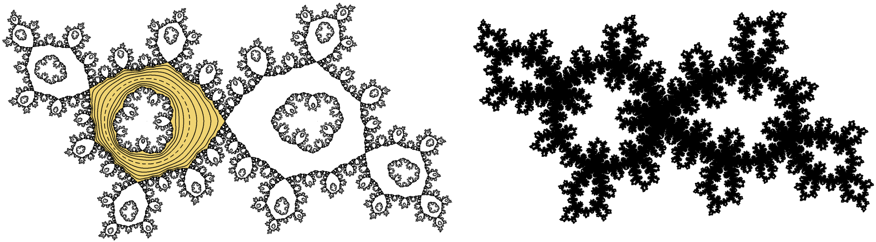


Construction by perturbation (Figures from [ABC04])

Boundaries of **Herman rings** (of cubic Blaschke products):

- C^∞-smooth (Buff, unpublished; Avila 2003)

Herman rings and degeneration



Invariant circles of Blaschke products $z \mapsto e^{2\pi it} z^2 \frac{z-a}{1-az}$, where $a = 4, 3$

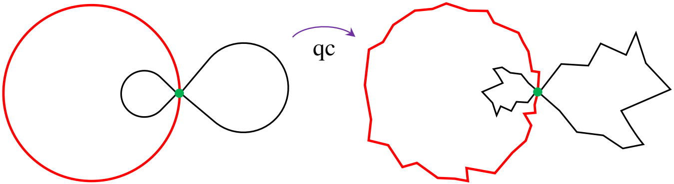
Definition (Degenerate Herman ring / Herman curve)

A Jordan curve $\gamma \subset \widehat{\mathbb{C}}$ is called a **degenerate Herman ring / Herman curve** of f if γ is **not a spherical circle** and satisfies the following properties:

- γ is contained in the Julia set of f;
- γ is not a boundary component of any Siegel disk or Herman ring of f;
- $f(\gamma) = \gamma$ and $f : \gamma \to \gamma$ is conjugate to an irrational rotation.

Eremenko's question

A degenerate Herman ring coming from the deformation of a Blaschke product:



$$\text{From} \quad z \mapsto e^{2\pi i t} z^2 \frac{z-3}{1-3z} \quad \text{to} \quad z \mapsto \frac{\lambda z + a z^2 (z-3)}{1 + (\frac{\lambda}{a} - 3) z} (0 < |\lambda| < 1).$$

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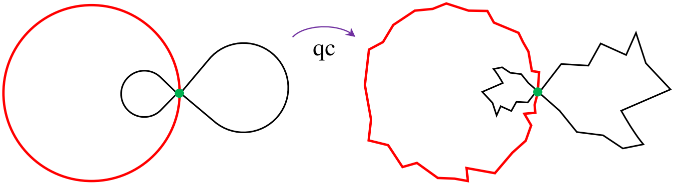

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Question (Eremenko, 2020)

Does there exist (C^∞-)**smooth** degenerate Herman ring?

Eremenko's question

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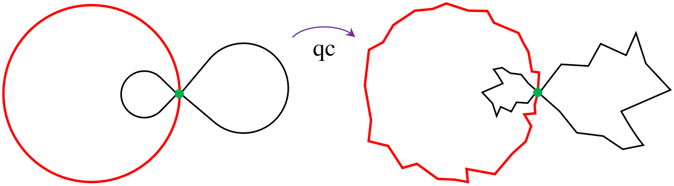

From $z \mapsto e^{2\pi it} z^2 \dfrac{z-3}{1-3z}$ to $z \mapsto \dfrac{\lambda z + a z^2 (z-3)}{1 + (\frac{\lambda}{a} - 3) z} (0 < |\lambda| < 1)$.

Question (Eremenko, 2020)

Does there exist (C^∞-)**smooth** degenerate Herman ring?

Lim (2024) proved that the right Jordan curve above is C^1-smooth.

Smooth degenerate Herman rings

Theorem (Y., 2024)

There exist cubic rational maps having a smooth degenerate Herman ring.

Main ingredients in the proof:

(1) Classical Siegel-to-Herman qc surgery by Shishikura (1987);

(2) Construction of smooth Siegel disks by Avila-Buff-Chéritat (2004);

(3) Control of the loss of Lebesgue measure of quadratic filled-in Julia sets by Buff-Chéritat (2012);

(4) Rigidity of the cubic maps having bounded type Herman rings.

A sketch of the proof

$P_\alpha(z) = e^{2\pi i\alpha} z + z^2$ has a **Siegel disk** Δ_α at 0 for bounded type α.

conformal radius r_α of Δ_α: $\exists 1$ conformal map $\phi_\alpha : \mathbb{D}_{r_\alpha} \to \Delta_\alpha$ s.t. $\phi_\alpha(0) = 0$, $\phi'_\alpha(0) = 1$ and $\phi_\alpha \circ R_\alpha = P_\alpha \circ \phi_\alpha$.

A sketch of the proof

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- Pasting P_α and $P_{-\alpha}$ together to obtain a **quasi-regular map** F_α, and $\exists$ a normalized **quasiconformal mapping** $\Phi_\alpha : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ s.t.

$$Q_\alpha(z) := \Phi_\alpha \circ F_\alpha \circ \Phi_\alpha^{-1}(z) = bz^2 \frac{z-a}{1 - \frac{2a-3}{a-2} z}$$

  has a Herman ring A_α whose bdy components contain crit pts 1 and $\frac{a(a-2)}{2a-3}$ resp.;

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- $\alpha \in \mathscr{D}_2$, $0 < r < r_\alpha$ and $\theta \in [0, 2\pi)$ determine the **unique** (a, b): $\mathrm{mod}(A_\alpha) = \frac{1}{\pi} \log \frac{r_\alpha}{r}$, and θ is the *conformal angle* of two crit pts. Denote $Q_{a,b} = Q_{\alpha,r,\theta}$;

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- Choosing **suitable** bounded type $\alpha_n \to \alpha$ s.t.

$$r_{\alpha_n} \to r_0 \in (r, r_\alpha), \quad F_{\alpha_n} \to F_\alpha, \quad \Phi_{\alpha_n} \to \Phi_\alpha \quad \text{and} \quad Q_{\alpha_n,r,\theta} \to Q_{\alpha,r,\theta};$$

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- Pasting P_α and $P_{-\alpha}$ together to obtain a **quasi-regular map** F_α, and $\exists$ a normalized **quasiconformal mapping** $\Phi_\alpha : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ s.t.

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- $\alpha \in \mathscr{D}_2$, $0 < r < r_\alpha$ and $\theta \in [0, 2\pi)$ determine the **unique** (a, b): $\mathrm{mod}(A_\alpha) = \frac{1}{\pi}\log\frac{r_\alpha}{r}$, and θ is the *conformal angle* of two crit pts. Denote $Q_{a,b} = Q_{\alpha,r,\theta}$;
- Choosing **suitable** bounded type $\alpha_n \to \alpha$ s.t.

$$r_{\alpha_n} \to r_0 \in (r, r_\alpha), \quad F_{\alpha_n} \to F_\alpha, \quad \Phi_{\alpha_n} \to \Phi_\alpha \quad \text{and} \quad Q_{\alpha_n,r,\theta} \to Q_{\alpha,r,\theta};$$

- For $\alpha \in \mathscr{D}_2$, $0 < r < r_\alpha$ and $\theta \in (0, 2\pi)$, $\exists Q_{a',b'}$ close to $Q_{a,b} = Q_{\alpha,r,\theta}$ s.t. $Q_{a',b'}$ has a smooth degenerate Herman ring.

ABC's control on conformal radii

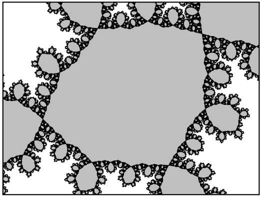
Lemma (Avila-Buff-Chéritat, 2004)

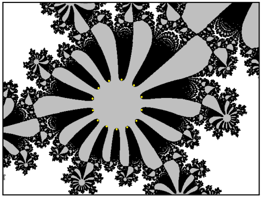
For any Brjuno number $\alpha := [0; a_1, a_2, \cdots, a_n, \cdots]$, any bounded type number $\beta := [0; t_1, t_2, \cdots, t_n, \cdots]$ and any $0 < r_0 < r_\alpha$, let

$$\alpha_n := [0; a_1, a_2, \cdots, a_n, A_n, t_1, t_2, t_3, \cdots],$$

where $A_n := \lfloor (r_\alpha / r_0)^{q_n} \rfloor$. Then

1. $\alpha_n \to \alpha$ and $r_{\alpha_n} \to r_0$ as $n \to \infty$;
2. For any $\varepsilon > 0$, if n is large enough, then P_{α_n} has a repelling cycle which is ε-close to $\phi_\alpha(\mathbb{T}_{r_0})$ in the Hausdorff metric.





The proof is based on **sector renormalization** techniques by Yoccoz (1995) and **parabolic explosion** techniques by Chéritat (2001).

BC's control on the loss of area

High type numbers:

$$\mathrm{HT}_N := \left\{\alpha = [a_0; a_1, a_2, \cdots, a_n, \cdots] \in \mathbb{R} \setminus \mathbb{Q} \mid a_n \geqslant N \text{ for all } n \geqslant 1\right\}.$$

$N \geqslant 1$ is large: s.t. the **near-parabolic renormalization** operator can be acted infinitely many times on P_α whenever $\alpha \in \mathrm{HT}_N$ (Inou-Shishikura, 2008).

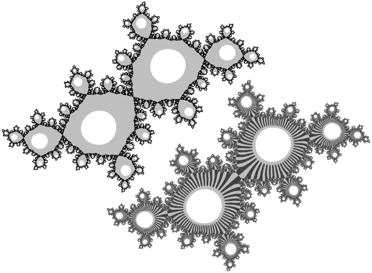
Lemma (Buff-Chéritat, 2012)

For sufficiently large $N \geqslant 1$, let $\alpha := [0; a_1, a_2, \cdots, a_n, \cdots] \in \mathrm{HT}_N$ be a Brjuno number. For any $0 < \rho < r_0 < r_\alpha$ and $n \geqslant 1$, let

$$\alpha_n := [0; a_1, a_2, \cdots, a_n, A_n, N, N, N, \cdots],$$

where $A_n := \lfloor (r_\alpha / r_0)^{q_n} \rfloor$. Then $\forall \varepsilon > 0$, if n is large, then

$$\mathrm{area}(L_{\alpha_n}(\rho)) \geqslant (1 - \varepsilon)\mathrm{area}(L_\alpha(\rho)).$$



$\Delta_\alpha(\rho) := \phi_\alpha(\mathbb{D}_\rho)$
$L_\alpha(\rho) := \{z \in K_\alpha : \forall k \geqslant 0, P_\alpha^{\circ k}(z) \notin \Delta_\alpha(\rho)\}$

Alternative path to Eremenko's question

Lim (2023) gives another method of constructing general examples of **non-trivial** degenerate Herman rings (i.e., not by a qc deformation of Blaschke products), based on the study of **near-degenerate regime** and **a prior bounds** of bounded type Herman rings.

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	rotation number	critical pt	regularity	measure of J
Lim	bounded type	on	quasi-circles	NILF
Y.	Brjuno$\backslash$ Herman	off	C^∞	positive area

Table: Two types of degenerate Herman rings

Intuitively, the degenerate HR is obtained by "pasting" two Siegel polynomials along their Siegel boundaries.

Transcendental meromorphic case

Question (Eremenko, 2020)

Does there exist (C^∞-)**smooth** degenerate Herman ring for **transcendental** meromorphic functions?

Theorem (Y., 2025)

There exist transcendental meromorphic functions having a smooth degenerate Herman ring.

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Main idea:

- Pasting $E_\alpha(z) = e^{2\pi i\alpha} z e^z$ and $P_{-\alpha}(z) = e^{-2\pi i\alpha} z + z^2$ together to obtain a **qr map** F_α, and $\exists$ a normalized **qc** $\Phi_\alpha : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ s.t.

$$Q_\alpha(z) := \Phi_\alpha \circ F_\alpha \circ \Phi_\alpha^{-1}(z) = \frac{bz^2 e^z}{z - \frac{a(1+a)}{2+a}}$$

  has a Herman ring whose bdy components contain crit pts a and $-\frac{2(1+a)}{2+a}$ resp.;
- Study the continuity of F_α, Φ_α, Q_α (w.r.t. α) and the rigidity of $Q_\alpha = Q_{a,b}$.

Unified control on conformal radii and area

IS_α: Inou-Shishikura's class of holo. maps with the form $f_\alpha(z) = e^{2\pi i\alpha} z + \mathscr{O}(z^2)$.
Remark: $E_\alpha(z) = e^{2\pi i\alpha} z e^z \in IS_\alpha$.
r^f_α: the conformal radius of the Siegel disk of $f_\alpha \in QIS_\alpha := IS_\alpha \cup \{P_\alpha\}$.

Lemma

For any Brjuno number $\alpha := [0; a_1, a_2, \cdots, a_n, \cdots] \in \mathrm{HT}_N$, and any $0 < \kappa < 1$, let

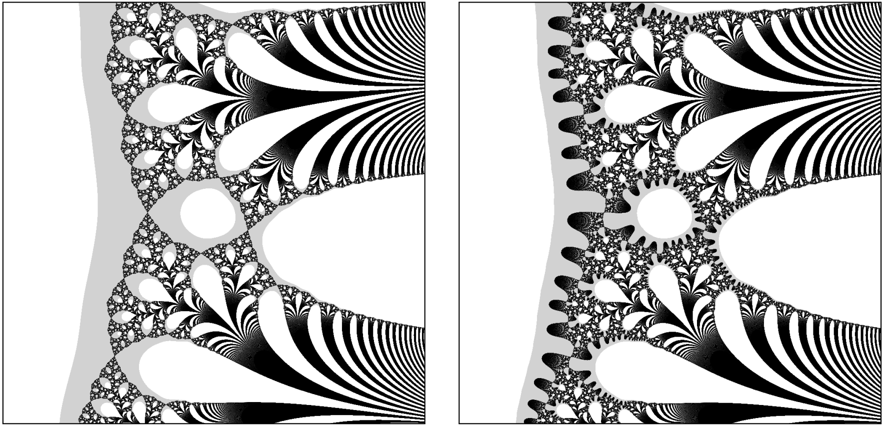
$$\alpha_n := [0; a_1, a_2, \cdots, a_n, A_n, N, N, N, \cdots],$$

where $A_n := \lfloor \kappa^{-q_n} \rfloor$. Then for any $\varepsilon > 0$ and $f_\alpha \in QIS_\alpha$, if n is large enough, then

1. $|\alpha_n - \alpha| < \varepsilon$ and $|r^f_{\alpha_n} - \kappa r^f_\alpha| < \varepsilon$;
2. f_{α_n} has a repelling cycle which is ε-close to $\phi^f_\alpha(\mathbb{T}_{\kappa r^f_\alpha})$ in the Hausdorff metric.

The proof is based on Cheraghi-Chéritat's result (2015) and **near-parabolic renormalization** techniques.

Loss of spherical measure



The loss of spherical measure: $\text{area}_{\widehat{\mathbb{C}}}(L^E_{\alpha_n}(\rho)) \geqslant (1-\varepsilon)\text{area}_{\widehat{\mathbb{C}}}(L^E_{\alpha}(\rho))$ for $E_\alpha(z) = e^{2\pi i\alpha} z e^z$.

A fact: $\text{area}(J(E_\alpha)) = 0$ for bounded type α.

Smooth transcendental Herman rings

The first example of Herman rings of transendental meromorphic function defined on $\mathbb{C}$ was constructed by Zheng (2000). See also Domínguez-Fagella (2004).

Based on the similar argument as above, we also prove the existence of transcendental meromorphic functions having **smooth Herman rings**. Such Herman rings are automatically asymmetric, which cannot be obtained by Avila's argument.

Questions

Question (Eremenko, 2012)

Does there exist a Jordan **analytic** invariant curve of a rational map, different from a circle, which is mapped onto itself homeomorphically and intersects the Julia set?

Question (Eremenko, 2023)

Does there exist **analytic** degenerate Herman ring?

Questions

(1) Does there exist a smooth degenerate Herman ring whose rotation number is **not of Brjuno type**?

(2) Does there exist a degenerate Herman ring which is **not a quasi-circle**?

Thank you for your attention !

Dziękuję !