Rational maps with smooth degenerate Herman rings

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DYNAMICS, TEICHMULLER THEORY AND THEIR RELATED TOPICS

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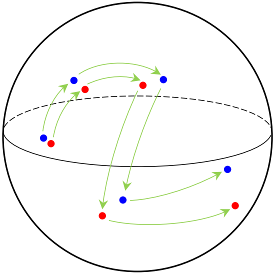
$f : X \to X$

$$x \mapsto f(x) \mapsto f(f(x)) \mapsto f(f(f(x))) \mapsto \cdots \mapsto f^{\circ n}(x) \mapsto \cdots$$

where X is a complex manifold, f is holomorphic.



Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map.

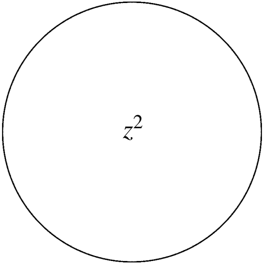
The **Fatou set** (or stable set) of f:

$$F(f) := \{z \in \widehat{\mathbb{C}} : \{f^{\circ n}\}_{n \in \mathbb{N}} \text{ is equicontinuous at } z\}.$$

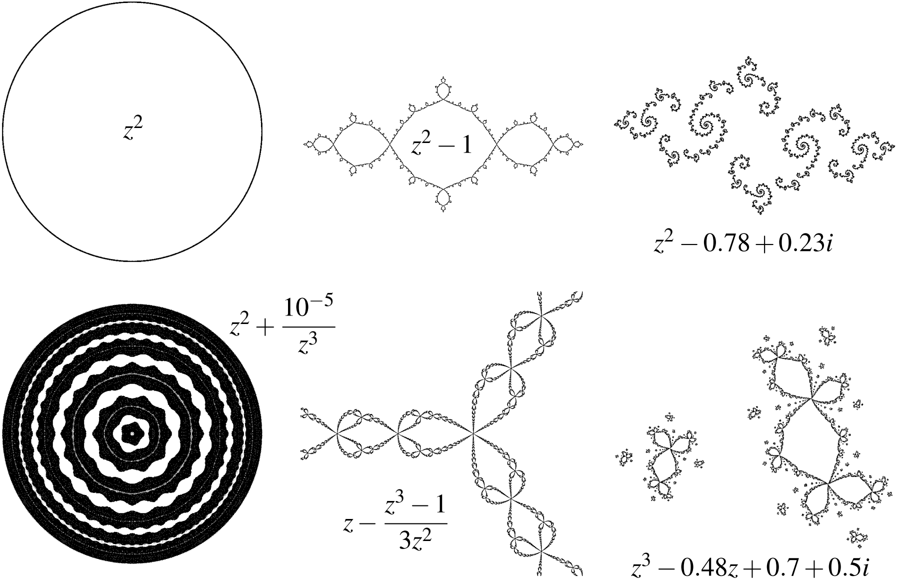
The **Julia set** (or chaotic set) $J(f) := \widehat{\mathbb{C}} \setminus F(f)$.

Each connected component of $F(f)$ (resp. $J(f)$) is called a **Fatou** (resp. **Julia**) **component**.

Julia / Fatou sets



Julia / Fatou sets



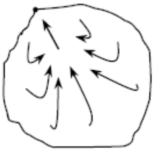
Sullivan's eventually periodic theorem

Theorem (Sullivan, 1985)

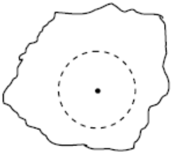
The Fatou components of all rational maps are eventually periodic.

Classification of periodic Fatou components of rational maps:

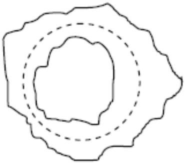
(super-)attracting basin
1884, 1904



parabolic basin
1897



Siegel disk
1942



Herman ring
1979

Shishikura (1987): HR can be obtained from SD through qc surgery, vice versa.

Siegel disk and continued fractions

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, f non-linear holo., $f(z) = e^{2\pi i \alpha} z + \mathscr{O}(z^2)$.

The *maximal* region in which f is conjugate to $R_\alpha(\zeta) = e^{2\pi i \alpha}\zeta$ is a simply connected domain Δ_f called the **Siegel disk** of f centered at 0.

$$\begin{array}{ccc} \mathbb{D}_r & \xrightarrow{R_\alpha} & \mathbb{D}_r \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \Delta_f & \xrightarrow{f} & \Delta_f \end{array}$$

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Let

$$\alpha = [0; a_1, a_2, \cdots, a_n, \cdots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

be the **continued fraction expansion** of α. Then $\frac{p_n}{q_n} := [0; a_1, a_2, \cdots, a_n] \to \alpha$, where p_n, q_n are coprime positive integers.

Siegel-Brjuno-Yoccoz

Theorem (Siegel, 1942)

The holomorphic germ f has a Siegel disk at 0 if $\alpha \in \mathscr{D} = \bigcup_{\kappa \geqslant 2} \mathscr{D}_\kappa$, where

$$\mathscr{D}_\kappa = \left\{ \alpha : \sup_{n \geqslant 1} \left\{ \frac{a_{n+1}}{q_n^{\kappa-2}} \right\} < \infty \right\}.$$

Remark: α is called **bounded type** if $\alpha \in \mathscr{D}_2$, i.e. $\sup_n \{a_n\} < \infty$.

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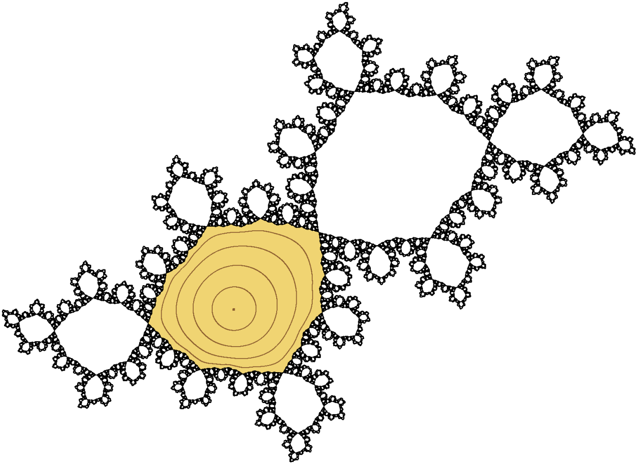
Theorem (Yoccoz, 1988)

(1) If $f(z) = e^{2\pi i \alpha} z + z^2$ has a Siegel disk at 0, then $\alpha \in \mathscr{B}$;
(2) If $\alpha \notin \mathscr{B}$, then $\exists$ a holomorphic germ f which does not have a Siegel disk at 0.

Siegel disk



The Siegel disk of $f(z) = e^{2\pi i\alpha}z + z^2$, where $\alpha = \dfrac{\sqrt{5}-1}{2} = [0; 1, 1, 1, \cdots]$

Arnold-Herman-Yoccoz

Theorem (Arnold, 1965)

Let $\alpha \in \mathscr{D}$ and $\sigma > 1$. There exists $\varepsilon > 0$ s.t. if

- *$f : \mathbb{S}^1 \to \mathbb{S}^1$ is a homeomorphism with rotation number $\rho(f) = \alpha$; and*
- *f can be extended analytically and univalently to $\{z : 1/\sigma < |z| < \sigma\}$ and satisfies $|f(z) - e^{2\pi i\alpha}z| < \varepsilon$ there,*

then f is conformally conjugate to R_α in $\{z : 1/\sqrt{\sigma} < |z| < \sqrt{\sigma}\}$.

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Theorem (Herman, 1979)

Let $\mathscr{H}$ be the set of all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ s.t. every orientation-preserving analytic circle diffeomorphism of rotation number α is analytically conjugate to R_α. Then $\mathscr{D} \subsetneq \mathscr{H}$.

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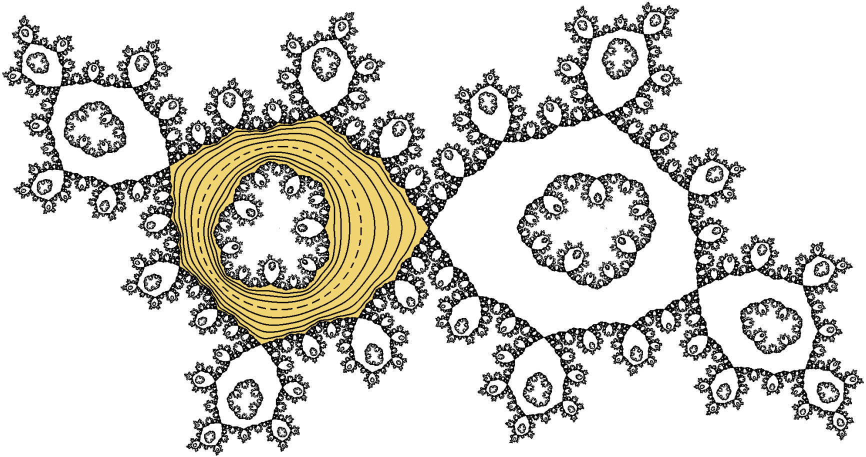
Theorem (Yoccoz, 2002)

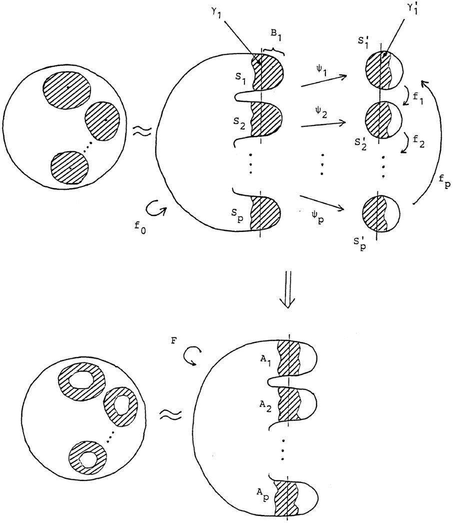
(1) If $\alpha \in \mathscr{H}$, *then every analytic circle diffeo.* f *with* $\rho(f) = \alpha$ *is analytically linearizable;*
(2) If $\rho \notin \mathscr{H}$, *then* $\exists$ *an analytic circle diffeo.* f *with* $\rho(f) = \alpha$ *which is not analytically linearizable.*

Herman ring



The Herman ring of $f(z) = e^{2\pi i t} z^2 \dfrac{z-4}{1-4z}$, where $\rho(f) = \dfrac{\sqrt{5}-1}{2} = [0; 1, 1, 1, \cdots]$

Shishikura's construction of HR by qc surgery



Origin

Fatou, 1920[1]:

> Il nous resterait à étudier les courbes analytiques invariantes par une transformation rationnelle et dont l'étude est intimement liée à celle des fonctions étudiées dans ce Chapitre. Nous espérons y revenir bientôt.

> It would remain for us to study the **invariant analytic curves** of a rational transformation and whose study is intimately linked to that of the functions studied in this chapter. We hope to return there soon.

Motivation: decomposing dynamics.

[1] P. Fatou, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **48** (1920), 208–314.

Progresses

Theorem (Azarina, 1989)

Suppose $f(\gamma) \subset \gamma$, where f is ***entire*** *and γ is a Jordan* ***analytic*** *curve. Then either*

- *γ is a circle and f is conformally conjugate to $z \mapsto z^n$ with $n \in \mathbb{Z}$; or*
- *γ is a level curve of the linearizing function of some Siegel disk.*

Invariant analytic curves under **rational maps**:

Circles, level curves of linearizing functions of Siegel disks or Hermann rings.

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Invariant analytic curves under **rational maps**:

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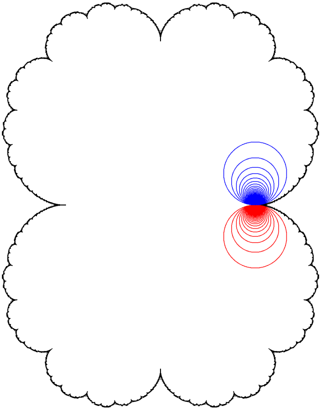
Theorem (Eremenko, 2012)

There are Jordan analytic curves which are not circles and invariant under the ***Lattès maps*** *(Julia sets $= \widehat{\mathbb{C}}$), and moreover, the restriction on each of these curves is* ***not*** *a homeomorphism.*

Invariant smooth curves

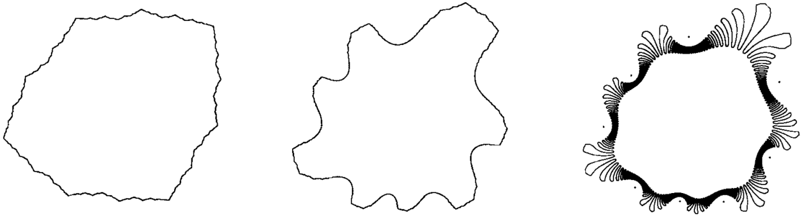
Level curves (C^1-smooth) of attracting Fatou coordinates in **parabolic basins**:



Invariant smooth curves

Boundaries of **Siegel disks**:

- C^∞-smooth (Pérez-Marco 1997; Avila-Buff-Chéritat 2004, 2020; Geyer 2008)
- C^n but not C^{n+1}; C^0 but not Hölder (Buff-Chéritat 2007)

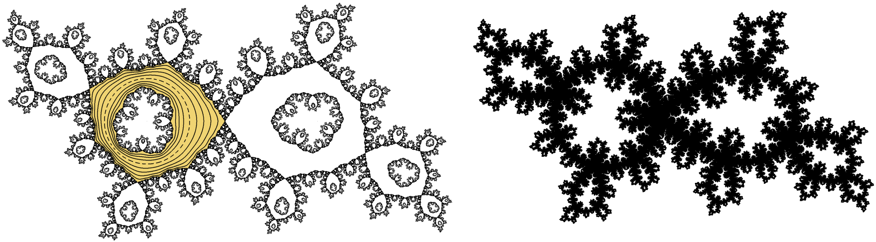


Construction by perturbation

Boundaries of **Herman rings**:

- C^∞-smooth (Buff, unpublished; Avila 2003)

Herman rings and degeneration



Invariant circles of Blaschke products $z \mapsto e^{2\pi it} z^2 \frac{z-a}{1-az}$, where $a = 4, 3$

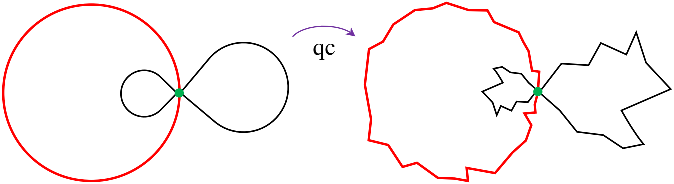
Definition (Degenerate Herman ring)

A Jordan curve $\gamma \subset \widehat{\mathbb{C}}$ is called a **degenerate Herman ring** of a meromorphic function f if γ is **not a spherical circle** and satisfies the following properties:

- γ is contained in the Julia set of f;
- γ is not a boundary component of any Siegel disk or Herman ring of f;
- $f(\gamma) = \gamma$ and $f : \gamma \to \gamma$ is conjugate to an irrational rotation.

Eremenko's question

A degenerate Herman ring by quasiconformal deformation:



From $z \mapsto e^{2\pi i t} z^2 \dfrac{z-3}{1-3z}$ to $z \mapsto \dfrac{\lambda z + a z^2 (z-3)}{1 + (\frac{\lambda}{a} - 3) z} (0 < |\lambda| < 1).$

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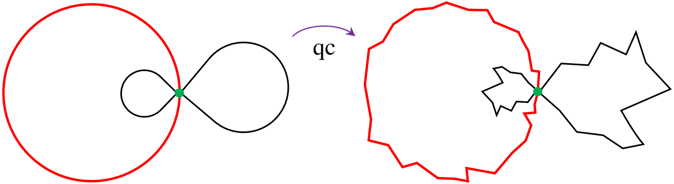

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Question (Eremenko, 2020)

Do there exist (C^∞-)**smooth** degenerate Herman rings?

Main result

Theorem (Y., arXiv 2022)

There exist cubic rational maps having a smooth degenerate Herman ring.

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Main ingredients in the proof:

(1) Siegel-to-Herman qc surgery by Shishikura;

(2) Construction of smooth Siegel disks by Avila-Buff-Chéritat;

(3) Control of the loss of Lebesgue measure of quadratic filled-in Julia sets by Buff-Chéritat;

(4) Rigidity of the cubic maps having bounded type Herman rings.

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Remark

(1) Gives another proof of Avila's result on the existence of smooth Herman rings.

(2) The rotation number of the above degenerated Herman ring is of Brjuno type.

Area of Julia sets

Fatou, 1919[2]:

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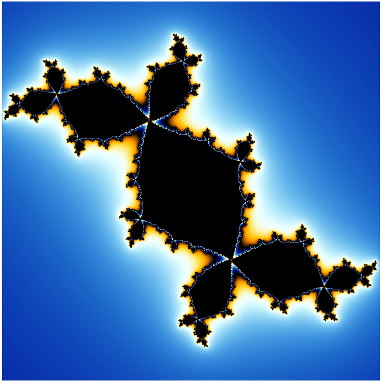
substitution inverse, et telle que la substitution donnée la transforme en une autre qui lui soit complètement intérieure, de manière qu'elle fasse partie du domaine d'attraction du point double, si en outre, sur les courbes antécédentes de C, on a à partir d'un certain rang

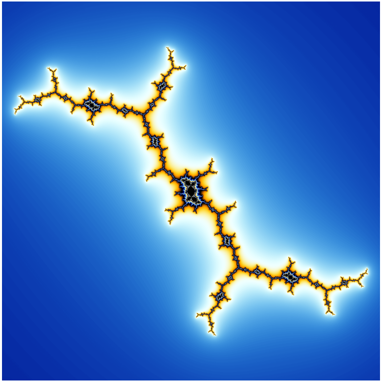
$$|R'(z)| > k > 1,$$

le domaine total du point double a pour frontière un ensemble parfait partout discontinu; cet ensemble est de mesure linéaire nulle si k est supérieur au degré d de $R(z)$, de mesure superficielle nulle si $k > \sqrt{d}$.

[2] P. Fatou, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **47** (1919), 161–271.

Polynomial-like renormalization





Area of Julia sets

Lebesgue measure of nowhere dense Julia sets of rational maps:

Zero area: Very fruitful results

- (Lyubich, Shishikura, 1991): not ∞- renorm. without Siegel, Cremer points

Positive area:

- (Buff-Chéritat, 2012, Ann. Math.): Siegel, Cremer, ∞-satellite renorm. with unbounded combinatorics
- (Avila-Lyubich, 2022, Ann. Math.): ∞-primitive renorm. with bounded combinatorics
- (Dudko-Lyubich, 2018): ∞-satellite renorm. with bounded combinatorics

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Theorem (Y., arXiv 2022)

*There exist cubic **rational maps** having a nowhere dense Julia set of **positive area** for which these maps have no Siegel disks, no Cremer points, no Herman rings, and are not renormalizable.*

Intuitively, “pasting” two quadratic Siegel polynomials along their Siegel boundaries.

Alternative path to Eremenko's question

Lim (arXiv, 2023) gives another method of constructing **non-trivial** degenerate Herman rings (i.e., not by a qc deformation of Blaschke products), based on the study of a prior bounds of bounded type Herman rings.

The degenerate Herman rings constructed there are **quasi-circles** passing through critical points (hence are not smooth).

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Question (Eremenko, 2012)

Does there exist a Jordan **analytic** invariant curve of a rational map, different from a circle, which is mapped onto itself homeomorphically and intersects the Julia set?

Question

Do there exist **analytic** degenerate Herman rings?

Further questions

Questions

(1) Does there exist a smooth degenerate Herman ring whose rotation number is **not of Brjuno type**?

(2) Does there exist a degenerate Herman ring which is **not a quasi-circle**?

(3) (Eremenko) Do there exist smooth degenerate Herman rings for **transcendental meromorphic** functions?

(4) (folk) Do there exist at most finitely renormalizable **polynomials** without irrationally indifferent periodic points whose Julia sets have positive area?

Thank you for your attention !