# Julia sets with positive area and Mañé's conjecture

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Around the Mandelbrot set:

A conference celebrating the 60th birthday of **Mitsuhiro Shishikura**

Kyoto University

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# Buff and Chéritat, at ICM 2010


Proceedings of the International Congress of Mathematicians
Hyderabad, India, 2010

## Quadratic Julia Sets with Positive Area

Xavier Buff* and Arnaud Chéritat†

**Abstract**

We recently proved the existence of quadratic polynomials having a Julia set with positive Lebesgue measure. We present the ideas of the proof and the techniques involved.

**Mathematics Subject Classification (2010).** Primary 37F50; Secondary 37F25.

Xavier Buff and Arnaud Chéritat (Aug 20, 2010)

# Buff and Chéritat, at ICM 2010

They wrote in their paper:

The proofs are based on

- McMullen's results [McM98] regarding the measurable density of the filled-in Julia set near the boundary of a Siegel disk with bounded type rotation number,
- Chéritat's techniques of parabolic explosion [Ché00] and Yoccoz's renormalization techniques [Yoc95] to control the shape of Siegel disks,
- Inou and Shishikura's results [IS] to control the post-critical sets of perturbations of polynomials having an indifferent fixed point.

# Looking for invariant class

In the note “The hunt for Julia sets with positive measure (2009)” by Chéritat, he wrote:

“We lacked first **an invariant class**, second an argument to turn this into a proof of my conjectures. An invariant class was found **by Inou and Shishikura** in 2003.”

“The title of the talk I gave at Hubbard’s 60th birthday conference (2005) was

‘Are there Julia sets with positive measure?’

Four months later I gave a talk in Denmark where the first two words were interchanged and the question mark became a period.”

# Begining

Fatou, 1919[1]:

— 257 —

substitution inverse, et telle que la substitution donnée la transforme en une autre qui lui soit complètement intérieure, de manière qu'elle fasse partie du domaine d'attraction du point double, si en outre, sur les courbes antécédentes de C, on a à partir d'un certain rang

$$|R'(z)| > k > 1,$$

le domaine total du point double a pour frontière un ensemble parfait partout discontinu; cet ensemble est de mesure linéaire nulle si $k$ est supérieur au degré $d$ de R $(z)$, de mesure superficielle nulle si $k > \sqrt{d}$.

[1] P. Fatou, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **47** (1919), 161–271.

# Area of Julia sets

For polynomials,

$$\begin{aligned} &\text{Zero area of Julia sets} \\ \Rightarrow\ &\text{No invariant line field Conjecture} \\ \Rightarrow\ &\text{Hyperbolic density Conjecture} \end{aligned}$$

**Zero area**: Very fruitful results, especially for quadratic polynomials:

- (Douady-Hubbard, Lyubich 1980s):
  Geometrically finite;
- (Lyubich, Shishikura 1991):
  Neither $\infty$- renorm nor have Siegel, Cremer points;
- (Petersen 1996, McMullen 1998, Yampolsky 1999, Petersen-Zakeri 2004):
  Siegel disks with almost all rotation numbers;
- (Yarrington 1995, Avila-Lyubich 2008, A. Dudko-Sutherland 2020):
  Some $\infty$- renorm.

**Conjecture (Douady, 1990s)**

There exist quadratic Julia sets with positive area.

**Positive area**:

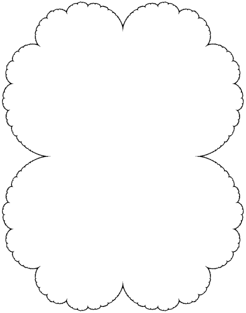
1. (Buff-Chéritat 2012): Siegel, Cremer, $\infty$-satellite renorm with unbounded combinatorics
2. (Avila-Lyubich 2022): $\infty$-primitive renorm with bounded combinatorics
3. (D. Dudko-Lyubich 2019): $\infty$-satellite renorm with bounded combinatorics

Remark:

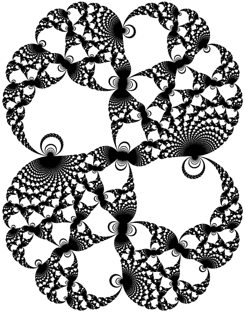
- ①, ② rely on Inou-Shishikura's near-parabolic renorm, while ③ relies on D. Dudko-Lyubich-Selinger's pacman renorm;
- ① (probably) does not have locally connected Julia sets while ②, ③ have;
- The parameters in ① (probably) have zero H-dim while ② have positive.

# Parabolic bifurcations

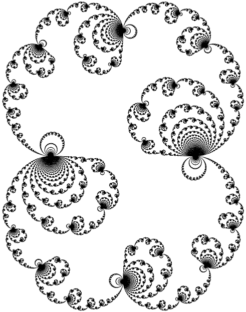
Douady (1990s) proved that the Julia set **does not move continuously** at parabolic parameters. One of the important phenomenon is **parabolic bifurcation**.



$z^2 + \frac{1}{4}$



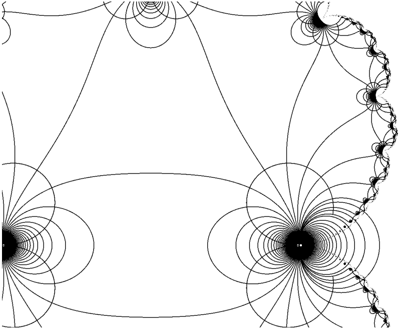
$z^2 + \frac{1}{4} + \varepsilon_1$



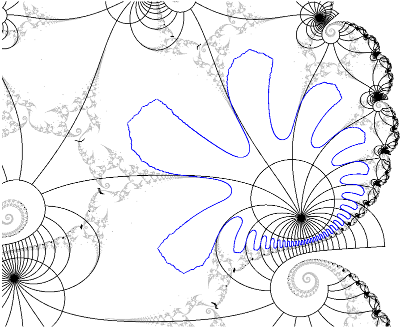
$z^2 + \frac{1}{4} + \varepsilon_2$

# Parabolic bifurcations

Although the Julia set **does not move continuously** at parabolic parameters, it turns out that the (perturbed) Fatou coordinate does (restricted on some truncated chessboard).



$z^2 + \frac{1}{4}$



$z^2 + \frac{1}{4} + \varepsilon$

# Developments

The main tools to analyze such bifurcation are **Fatou coordinates** and **horn maps**, which were developed by:

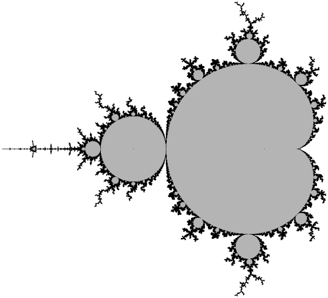
1. Douady-Hubbard (1984-85): landing of external rays at the M-set (Orsay notes), the straightening of polynomial-like maps;
2. Lavaurs (1989): the non-local connectivity of the connectedness locus of cubic polynomials;
3. Douady (1994): the discontinuity of Julia sets;
4. Shishikura (1998): the Hausdorff dim of $\partial$M (**an invariant class**);
5. Yampolsky (2003): cylinder renormalization for critical circle maps;
6. Inou-Shishikura (2006): near-parabolic renormalization (**a new invariant class**);

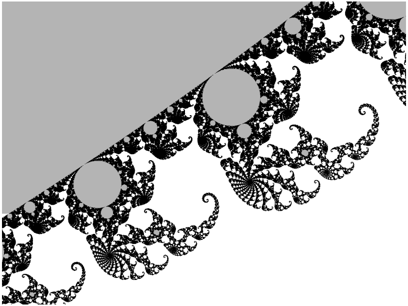
……

# Hausdorff dim of the the boundary of M-set

Denote $p_c(z) = z^2 + c$, where $c \in \mathbb{C}$. The **Mandelbrot set** is

$$\mathrm{M} := \{c \in \mathbb{C} : \lim_{n\to\infty} p_c^{\circ n}(0) \neq \infty\}.$$





**Theorem (Shishikura, 1991)**

- $\text{H-dim}(J(p_c)) = 2$ *for generic* $c \in \partial \mathrm{M}$;
- $\text{H-dim}(\partial \mathrm{M}) = 2$.

# Hausdorff dim of the the boundary of M-set

Idea of the proof:

1. perturb parabolic periodic points;
2. transfer the dim result from dynamical planes to parameter plane.

A class was introduced:

$$\mathscr{F}_0 := \left\{ f : Dom(f) \to \mathbb{C} \;\middle|\; \begin{array}{l} 0 \in Dom(f) \text{ open} \subset \mathbb{C}, f \text{ is holo. in } Dom(f), \\ f(z) = z + O(z^2), f : Dom(f) \setminus \{0\} \to \mathbb{C}^* \text{ is a} \\ \text{branched covering with a } \textbf{unique critical value}, \\ \text{all critical points are of } \textbf{local degree 2} \end{array} \right\}.$$

$\mathscr{R}_0(\mathscr{F}_0) \subset \mathscr{F}_0$, where $\mathscr{R}_0$ is the **parabolic renormalization** operator.

# Parabolic and near-parabolic renormalizations

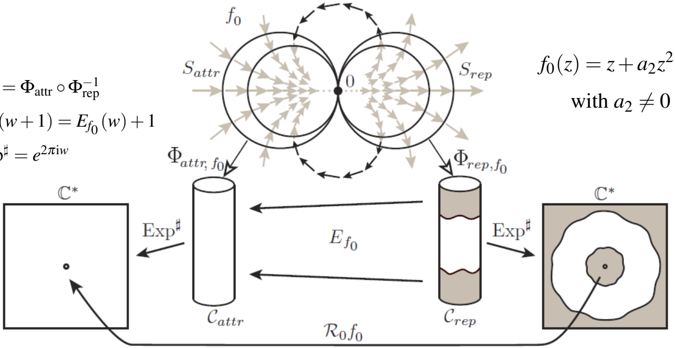
$E_{f_0} = \Phi_{\rm attr} \circ \Phi_{\rm rep}^{-1}$

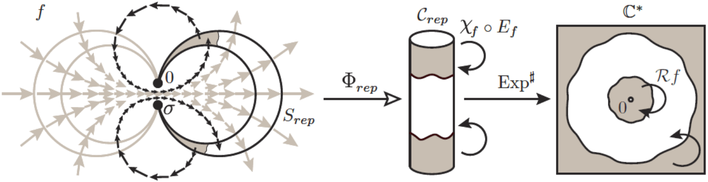
$E_{f_0}(w+1) = E_{f_0}(w) + 1$

$\mathrm{Exp}^\sharp = e^{2\pi i w}$

$f_0(z) = z + a_2 z^2 + \cdots$

with $a_2 \neq 0$

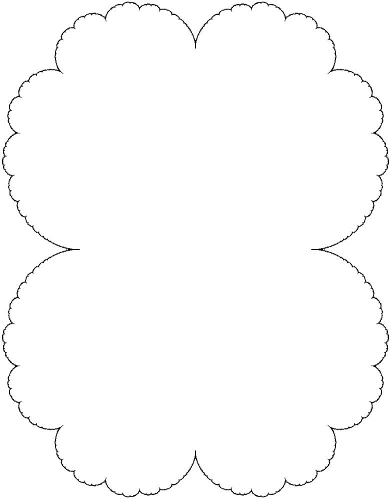



$f(z) = e^{2\pi i \alpha} z + O(z^2)$ with small $\alpha$, $|\arg(\alpha)| < \frac{\pi}{4}$

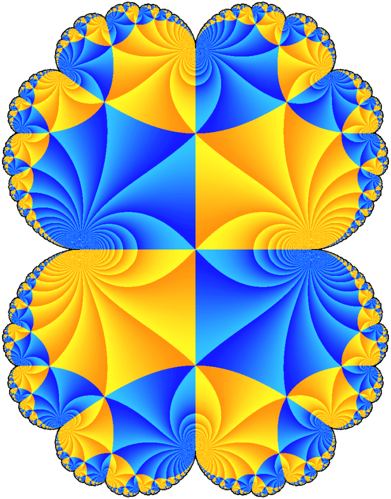
[Pictures by Inou-Shishikura]

# Structure under the parabolic renormalization



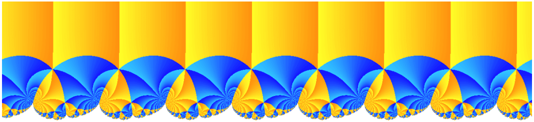
$z \mapsto z + z^2$

# Structure under the parabolic renormalization

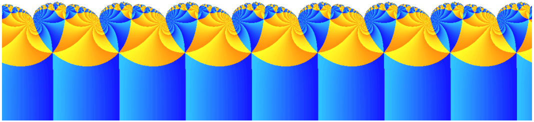


$z \mapsto z + z^2$

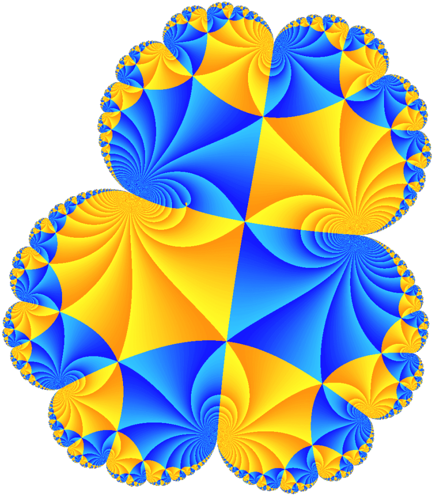
# Structure under the parabolic renormalization



under Fatou coordinate

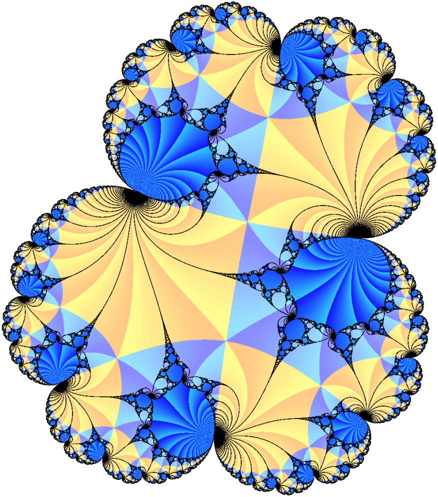


# Structure under the parabolic renormalization



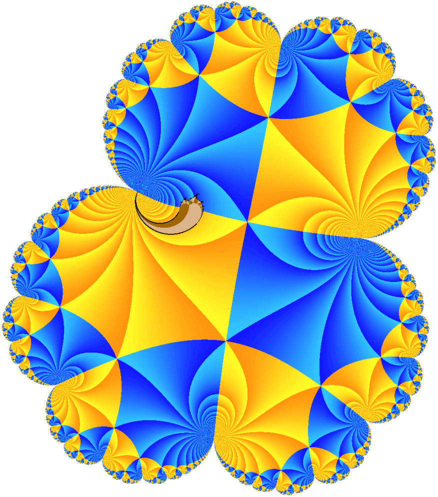
under $\mathrm{Exp}^\sharp$

# Structure under the parabolic renormalization



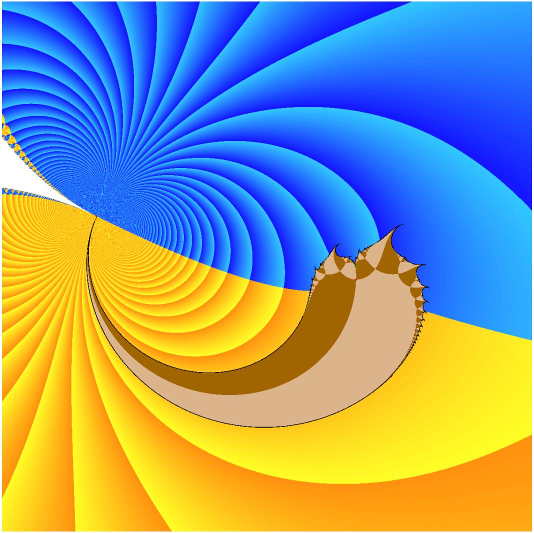
$\mathscr{R}_0(z+z^2)$

# Structure under the parabolic renormalization



$\mathscr{R}_0(z+z^2)$

# Structure under the parabolic renormalization



zoom of
$\mathscr{R}_0(z+z^2)$

[Pictures by Chéritat]

# Why define new class?

For a near-parabolic map $f = e^{2\pi i\alpha} h$ with $h \in \mathscr{F}_0$, one can identify $f$ with $(\alpha, h)$.

The **near-parabolic renormalization** can be expressed as a skew product:

$$\mathscr{R} : (\alpha, h) \mapsto (-\tfrac{1}{\alpha}, \mathscr{R}_\alpha h),$$

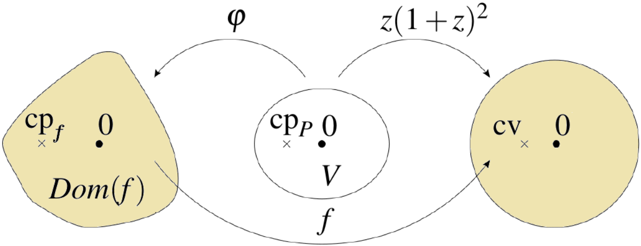
where $\mathscr{R}_\alpha$ is the **renormalization in the fiber direction** and $(\mathscr{R}_\alpha h)(z) = z + O(z^2)$.

Unfortunately $\mathscr{R}_\alpha(\mathscr{F}_0) \not\subset \mathscr{F}_0$ for $\alpha \neq 0$.

# Inou-Shishikura's class

Let $P(z) = z(1+z)^2$, and $V$ a Jordan domain of $\mathbb{C}$ containing 0 and the critical point $cp_P = -1/3$. Define

$$IS_0 := \left\{ f = P \circ \varphi^{-1} : \varphi(V) \to \mathbb{C} \;\middle|\; \begin{array}{l} \varphi : V \to \mathbb{C} \text{ is univalent,} \\ \varphi(z) = z + O(z^2) \text{ and} \\ \varphi \text{ has a q.c. extension to } \mathbb{C} \end{array} \right\}.$$



# Invariance and hyperbolicity of the operator

## Theorem (Inou-Shishikura, 2006)

*There exists a Jordan disk $V \subset \mathbb{C}$ containing $0$ and $cp_P$ such that*

- $\mathscr{R}_0(IS_0) \subset IS_0$. *Moreover,* $\exists$ *a simply connected domain* $V' \subset \mathbb{C}$ *containing* $\overline{V}$ *s.t.*

$$\forall f = P \circ \varphi^{-1} \in IS_0, \quad \mathscr{R}_0 f = P \circ \psi^{-1} \in IS_0,$$

  *where* $\psi$ *extends to a univalent function from* $V'$ *to* $\mathbb{C}$.
- *Identifying* $IS_0$ *with the Teichmüller space of* $\mathbb{C} \setminus \overline{V}$, *the operator* $\mathscr{R}_0$ *is a* ***uniform contraction*** *with respect to the Teichmüller metric.*

The proof of the invariance of $\mathscr{R}_0$ relies on very complicated calculations.

# Renormalization in the fiber direction

Near-parabolic renormalization $\mathscr{R}$ can be expressed as $\mathscr{R} : (\alpha, h) \mapsto (-\frac{1}{\alpha}, \mathscr{R}_\alpha h)$.

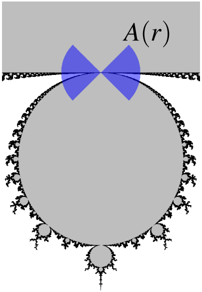
**Theorem (Inou-Shishikura, 2006)**

*$\exists$ a small $r > 0$ such that if*
$\alpha \in A(r) := \{\alpha \in \mathbb{C} : 0 < |\alpha| \leqslant r$ *and* $|\mathrm{Re}\,\alpha| > |\mathrm{Im}\,\alpha|\}$,

1. *$\mathscr{R}_\alpha(IS_0) \subset IS_0$, and $\mathscr{R}_\alpha$ has the same properties as $\mathscr{R}_0$.*
2. *$\mathscr{R}$ can be acted in $A(r) \times IS_0$, and can be* ***iterated infinitely many times*** *in $\mathrm{HT}_N \times IS_0$, where $N \geqslant 1/r$ and*

$$\mathrm{HT}_N := \{[0; a_1, a_2, \cdots] \in (0,1) \setminus \mathbb{Q} \mid \forall n \geqslant 1, a_n \geqslant N\}$$

*is a set of* ***high type*** *numbers.*



$\alpha$-plane

The proof relies on the continuity of horn maps and the compactness of $IS_0$.

## Continuity of the post-critical sets

Denote

$$QIS_0 := IS_0 \cup \left\{z+z^2, \frac{z}{(1-z)^2}\right\} \quad \text{and} \quad QIS_\alpha := \{e^{2\pi i\alpha} f_0 \mid f_0 \in QIS_0\}.$$

Let $\Delta(f)$ be the **Siegel disk** of $f \in QIS_\alpha$ at 0 if any, and $\Delta(f) = \{0\}$ otherwise.

The **postcritical set** of $f \in QIS_\alpha$:

$$PC(f) := \overline{\bigcup_{k \geqslant 1} f^{\circ k}(cp_f)}$$

### Theorem (Cheraghi, 2019)

*Let $f_\alpha, \alpha \in [0,1]$, be a continuous family of maps s.t. for $\alpha \in \mathrm{HT}_N$, $f_\alpha \in QIS_\alpha$. Then, $\forall\, \alpha_0 \in \mathrm{HT}_N$ and $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $\forall\, \alpha \in \mathrm{HT}_N$ with $|\alpha - \alpha_0| < \delta, PC(f_\alpha) \cup \Delta(f_\alpha)$ is contained in $\varepsilon$-neighborhood of $PC(f_{\alpha_0}) \cup \Delta(f_{\alpha_0})$.*

The above result is also true for complex $\alpha$;

Buff-Chéritat: bounded type real $\alpha$.

# Maps in Inou-Shishikura's class

**New** class:

$$IS_0 := \left\{ f = P \circ \varphi^{-1} : \varphi(V) \to \mathbb{C} \;\middle|\; \begin{array}{l} \varphi : V \to \mathbb{C} \text{ is univalent,} \\ \varphi(z) = z + O(z^2) \text{ and} \\ \varphi \text{ has a q.c. extension to } \mathbb{C} \end{array} \right\}.$$

**Old** class:

$$\mathscr{F}_0 := \left\{ f : Dom(f) \to \mathbb{C} \;\middle|\; \begin{array}{l} 0 \in Dom(f) \text{ open} \subset \mathbb{C}, f \text{ is holo. in } Dom(f), \\ f(z) = z + O(z^2), f : Dom(f) \setminus \{0\} \to \mathbb{C}^* \text{ is a} \\ \text{branched covering with a unique critical value,} \\ \text{all critical points are of local degree 2} \end{array} \right\}.$$

## Proposition (Inou-Shishikura, 2006)

$\mathscr{F}_0 \setminus \{z + z^2, \dfrac{z}{(1-z)^2}\} \subsetneq IS_0.$

# Julia sets of positive area

## Theorem (Y., 2023)

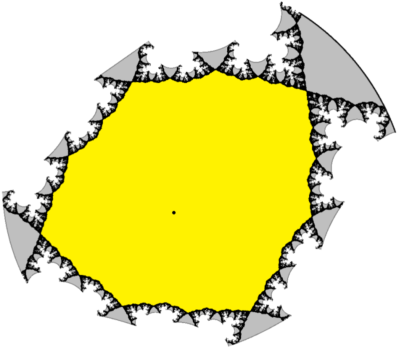
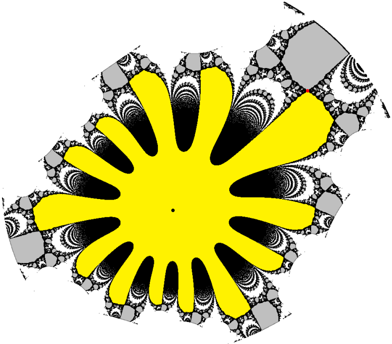
*For any meromorphic function $f:\mathbb{C}\to\widehat{\mathbb{C}}$ in $\mathscr{F}_0$, there exists $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ such that $f_\alpha=e^{2\pi i\alpha}f$ has a Cremer point (resp. Siegel disk) and a Julia set of positive area.*

In fact, we prove $\mathrm{area}(J_\alpha(\delta))>0$ for small $\delta>0$, where

$$J_\alpha(\delta):=J(f_\alpha)\cap K_\alpha(\delta) \text{ is a 'bounded' Julia set,}$$
$$K_\alpha(\delta):=\big\{z\in V_\alpha(\delta):\forall k\geqslant 0,\, f_\alpha^{\circ k}(z)\in V_\alpha(\delta)\big\},$$
$$V_\alpha(\delta):=\big\{z\in\mathbb{C}:d\big(z,PC(f_\alpha)\cup\Delta(f_\alpha)\big)<\delta\big\}.$$

Ingredients in the proof:

- Extend McMullen's measurable density result to more meromorphic functions;
- Control the shape of Siegel disks (assume high type) by near-parabolic renormalization techniques;
- Cheraghi's result on the continuity of post-critical sets.

Control the loss of the measure in a small neighborhood of the post-critical set.

## Corollary

Suppose $f$ is one of the following:

1. **Polynomials**: $z \mapsto z(1+z)^n$ for $n \geqslant 1$, and $z \mapsto z(1+z)^2(1+\frac{1}{2}z)$;
2. **Rationals**: $z \mapsto z/(1-z)^n$ for $n \geqslant 2$, and $z \mapsto z(1-z)^3/\big(1-\frac{8}{9}z\big)$;
3. **Transcendental entire**: $z \mapsto ze^z$, and $z \mapsto \sin(z)$;
4. **Transcendental meromorphic**: $z \mapsto ze^z/(1+az)^3$, where $a < -1$ is chosen such that the map has exactly one critical value in $\mathbb{C} \setminus \{0\}$.

Then there exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $e^{2\pi i \alpha} f$ has a Cremer point (resp. Siegel disk) and a ('bounded') Julia set of positive area.

Remark:

- Every meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ in $\mathscr{F}_0$ is **not** polynomial-like renorm;
- Polynomial case: Qiao-Qu (2020) for $n = 2$, X. Zhang (2022) for $n \geqslant 21$;
- In transcendental case, McMullen (1987) proved the positive area for fast escaping set;
- $\mathscr{F}_0$ contains uncountably many topologically inequivalent transcendental entire functions (Chéritat, 2006).

# Ergodicity

A rational map $f$ is called **ergodic** with respect to the Lebesgue measure if $\text{area}(X) = 0$ or $\text{area}(\widehat{\mathbb{C}} \setminus X) = 0$ whenever $X = f^{-1}(X)$ for a measurable subset $X \subset \widehat{\mathbb{C}}$.

Rees (1986) proved that there exists a large set of ergodic rational maps with respect to the Lebesgue measure whose Julia sets are $\widehat{\mathbb{C}}$.

Mañé raised the following conjecture[2], which was also asked as a question by Sullivan[3] and Lyubich[4]:

**Conjecture (Mañé, 1985)**

Every rational map $f$ with $J(f) = \widehat{\mathbb{C}}$ is ergodic with respect to the Lebesgue measure.

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[2] R. Mañé, *On the instability of Herman rings*, Invent. Math. **81** (1985), no. 3, 459–471.

[3] D. A. Brannan and W. K. Hayman, *Research problems in complex analysis*, Bull. Lond. Math. Soc. **21** (1989), no. 1, 1–35.

[4] M. Lyubich, *Measure and dimension of Julia sets*, Problems in holomorphic dynamics, edited by B. Bielefeld and M. Lyubich, arXiv: 9205209, 1992.

## Theorem (Y., 2023)

*There exists a quadratic rational map $f$ having two Cremer cycles for which $J(f) = \widehat{\mathbb{C}}$ and $f$ is **not ergodic** with respect to the Lebesgue measure.*

Idea of the proof:

Consider a quadratic parabolic rational map having two parabolic fixed points:

$$f_0(z) = \frac{-z+z^2}{1+z}.$$

$\mathscr{R}_0(f_0)$ can be defined near $\infty$, and $\mathscr{R}_0(f_0^{\circ 2})$ can be defined near 0.

There exists a small **perturbation region** $\Lambda$ of $f_0$ in $\mathrm{Rat}_2$ s.t.

- $\forall f \in \Lambda$, $\mathscr{R}(f)$ can be defined near $\infty$, and $\mathscr{R}(f^{\circ 2})$ can be defined near 0;
- $\exists f_1 \in \Lambda$, s.t. $f_1$ has a fixed Siegel point at $\infty$, a 2-cycle of Siegels containing 0;
- $\exists f \in \Lambda$ very close to $f_1$ such that $f$ has a Cremer fixed point at $\infty$, a 2-cycle of Cremer points containing 0, and a Julia set equaling $\widehat{\mathbb{C}}$ for which $f$ is not ergodic.

# More examples

**Theorem (Y., 2023)**

*There exists a quadratic rational map having a* ***parabolic basin*** *whose boundary has positive area.*

Such a quadratic rational map has a parabolic fixed point with multiplier 1 and a Cremer cycle of period two.

Petersen-Roesch (2021): Parabolic Mandelbrot set;

Roesch-Yin (2011): No invariant line fields on the boundary of parabolic basin of polynomials.

# Harvests based on Inou-Shishikura's class

for $z^2 + c$ and high type

**Area and dim** of Julia sets and post-critical sets:

1. Buff-Chéritat (2012): Quadratic Julia sets with positive area (three types)
2. Avila-Lyubich (2022): Quadratic Feigenbaum Julia sets with positive area
3. Cheraghi (2013, 2019): Zero area of post-critical set
4. Cheraghi-DeZotti-Y. (2020): H-dim of post-critical set for non-Herman type

**Topology** of post-critical sets:

1. Cheraghi (2017), Shishikura-Y. (2018): Siegel boundaries and hedgehogs
2. Cheraghi-Pedramfar (2019): Complex Feigenbaum phenomena

Other applications:

1. Cheraghi-Shishikura (2015): MLC at unbounded type infinitely satellite renormalization pts
2. Cheraghi-Chéritat (2015): Marmi-Moussa-Yoccoz conjecture
3. Avila-Cheraghi (2018): Statistical properties (uniquely ergodic on the post-critical set) and small cycles
4. Y. (2022): Smooth degenerate Herman rings

……

# Developments and questions

Extend to higher local degrees:

- (Chéritat 2022): Invariant class for unicritical maps (No numerical calculations)
- (Y. 2022) Invariant class for cubic unicritical maps (follow Inou-Shishikura)

Developments:

- Extend high type to more rotation numbers ([D. Dudko-Lyubich 2022], [Kapiamba 2022])
- Extend unicritical to multi-critical values ([Chéritat-Petersen 2016])

**Questions**

(1) (Avila-Lyubich 2015) Does there exist $c \in \mathbb{R}$ s.t. $\text{area}(J(z^2+c)) > 0$?

(2) (Chéritat 2009) Does there exist a rational map having a Cremer Julia set with area zero?


どうもありがとうございます! Thank you very much! 非常感谢!