SELF-SIMILARITY OF THE BOUNDARY OF HIGH TYPE SIEGEL DISKS IN THE QUADRATIC FAMILY

ARNAUD CHÉRITAT AND FEI YANG

Abstract. Let α be an irrational number of sufficiently high type. Suppose that f is a map in the Inou-Shishikura class such that $f(0)=0$ and $f'(0)=e^{2\pi\mathrm{i}\alpha}$. We prove that the boundary of the Siegel disk of f is self-similar at the critical point if α is of bounded type. In particular, this result can be applied to the quadratic polynomial $P_\alpha(z)=e^{2\pi\mathrm{i}\alpha}z+z^2$, the exponential map $E_\alpha(z)=e^{2\pi\mathrm{i}\alpha}ze^z$ and the sine family $S_{\alpha/2}(z)=e^{\pi\mathrm{i}\alpha}\sin(z)$.

Contents

1. Introduction

Let f be a non-linear holomorphic function such that $f(0)=0$ and $f'(0)=e^{2\pi\mathrm{i}\alpha}$, where $0<\alpha<1$ is an irrational number. We say that f is *locally linearizable* at the fixed point 0 if there exists a holomorphic function defined near 0 which conjugates f to the *rigid rotation* $R_\alpha(z)=e^{2\pi i\alpha}z$. The maximal region in which f is conjugate to the rigid rotation is a simply connected domain Δ_f called the *Siegel disk* of f centered at 0. Let

$$[0;a_1,a_2,\cdots]:=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots}}}$$

be the *continued fraction expansion* of an irrational $\alpha\in(0,1)\smallsetminus\mathbb{Q}$. If α is of *bounded type*, i.e. $\sup_{n\geq 1}a_n<+\infty$ then according to [Sie42] such f

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is locally linearizable near 0 and hence f has a Siegel disk centered at the origin.

Since the dynamical behavior in the Siegel disk is simple, one is mainly concerned by the properties of its boundary. For the boundaries of Siegel disks, many people had observed the universality and self-similarity near the critical points. Originally, this phenomenon was studied numerically in [MN83], [Wid83] and [MP87]. Later, McMullen proved that the Siegel disks of quadratic polynomials are self-similar at the critical points if the rotation number is a *quadratic irrational*, i.e. if the continued fraction expansion of the rotation number is pre-periodic [McM98]. In [LP08], the regularity of the boundaries of the Siegel disks and the scaling exponents were studied numerically. A more precise analysis on the similarity factor has also been carried out in [BH99] and [Gai15].

In [IS08], Inou and Shishikura introduced a renormalization operator $\mathcal{R}$ and a compact class $\mathcal{F}$ that is invariant under $\mathcal{R}$. All the maps in $\mathcal{F}$ have a special covering structure, have a neutral fixed point at the origin, and possess a unique simple critical point in their domain of definition. The renormalization operator $\mathcal{R}$ assigns a new map $\mathcal{R}f \in \mathcal{F}$ to a given map of $f \in \mathcal{F}$ that is obtained by considering the return map to a sector landing at the origin. As a return map, one iterate of $\mathcal{R}f$ corresponds to many iterates of $f \in \mathcal{F}$. To study very large iterates of f near 0, one hopes to repeat this process infinitely many times. However, to iterate $\mathcal{R}$ infinitely many times at some f, their scheme requires the rotation number α, where $f'(0) = e^{2\pi \mathrm{i} \alpha}$, to be of *high type*, that is, α belongs to

$$\mathrm{HT}_N := \{\alpha = [0; a_1, a_2, \cdots] \in (0,1) \setminus \mathbb{Q} \mid a_n \geq N \text{ for all } n \geq 1\}$$

for some large constant[1] $N \in \mathbb{N}$.

1.1. **Statement of the main results.**

In this article, we consider the self-similarity of the boundaries of the Siegel disks for the quadratic polynomials and the maps in the Inou-Shishikura class. In order to use Inou-Shishikura's renormalization scheme, we restrict the rotation numbers of the Siegel disks to be of high type.

Theorem 1.1. *Let f be a map in the Inou-Shishikura class such that $f(0) = 0$ and $f'(0) = e^{2\pi \mathrm{i}\alpha}$. If α is of bounded type and it is of sufficiently high type, then the boundary of the Siegel disk of f is a Jordan curve contained in the domain of definition, containing the critical point and self-similar at the critical point.*

On the one hand, note that the self-similarity of the boundaries of the Siegel disks was known only for those rotation numbers which are quadratic irrationals. Our Theorem 1.1 indicates that the self-similarity also exists for more general rotation numbers. On the other hand, there is a large class of analytic maps defined on $\mathbb{C}$ or $\widehat{\mathbb{C}}$ that have a restriction (or its variation) which belongs to the Inou-Shishikura class. In particular, we have the following corollary.

[1] The precise value of N is not known. Inou and Shishikura think that the value of N is likely to be no less than 20. However, it is conjectured that a variation of the class and renormalization may be defined for $N = 1$.

Corollary 1.2. *If α is of bounded type and is of sufficiently high type, then the boundaries of the Siegel disks of*

$$P_\alpha(z) = e^{2\pi\mathrm{i}\alpha}z + z^2,\ E_\alpha(z) = e^{2\pi\mathrm{i}\alpha}ze^z \text{ and } S_{\alpha/2}(z) = e^{\pi\mathrm{i}\alpha}\sin(z),$$

are Jordan curves and they are self-similar at their corresponding critical points $-e^{2\pi\mathrm{i}\alpha}/2$, -1 and $\pm\pi/2$ respectively.

It was proved by Douady and Herman in 1980s that the boundaries of the bounded type Siegel disks of quadratic polynomials P_α are Jordan curves (actually are quasicircles) containing the critical point $-e^{2\pi\mathrm{i}\alpha}/2$ (see [Dou87] and [Her87]). Geyer proved that the boundary of the Siegel disk of E_α is a Jordan curve containing the critical point -1 for the irrational numbers of bounded type [Gey01], and Zhang proved that the bounded type boundary of the Siegel disk of S_α is a Jordan curve containing exactly two critical points $\pm\pi/2$ [Zha05]. Actually, for almost all irrational numbers, Petersen and Zakeri proved that the boundary of the Siegel disk of P_α is a Jordan curve containing the critical point $-e^{2\pi\mathrm{i}\alpha}/2$ [PZ04], and Zhang proved that the boundary of the Siegel disk of S_α is a Jordan curve containing exactly two critical points $\pm\pi/2$ [Zha16]. However, these results are all based on the surgery: quasiconformal or trans-quasiconformal. It was proved in [AL15, Proposition 5.11] (see also [Yam08]) that the boundaries of the bounded type Siegel disks of the maps in the Inou-Shishikura class are Jordan curves. However, their proofs rely on the result of the quadratic polynomials due to Douady and Herman, hence also based on the quasiconformal surgery. In this article, we will give a proof of Theorem 1.1 and hence Corollary 1.2 that avoids the use of surgery.

For the self-similarity of the boundaries of the Siegel disks of quadratic polynomials, the proof in [McM98] relies on quasiconformal surgery and the local connectivity of the whole Julia sets [Pet96]. We will also prove the result of the self-similarity but not use the surgery. Although the self-similarity of the boundaries of the Siegel disks of quadratic polynomials was known, the corresponding result of the exponential maps and the sine family was not clear. Hence our result gives a little inspiration on this problem.

One can show that the restriction of E_α on a special domain is contained in the Inou-Shishikura class (after normalizing the critical value, see Proposition 7.2). On the other hand, nor the quadratic polynomial P_α and the sine function $S_{\alpha/2}$ are contained in the Inou-Shishikura class. However for P_α the first renormalization $\mathcal{R}P_\alpha$ is. For $S_{\alpha/2}$, we prove that its variation belongs to the Inou-Shishikura class by considering its semi-conjugacy $\widetilde{S}_\alpha$ (see Proposition 7.3). Note that in Corollary 1.2 we make a statement on $S_{\alpha/2}$ but not S_α since 2α is not necessarily a sufficiently high type irrational although if α is[2]. See Figure 1 for an example of a Julia set and a Siegel disk in the family P_α, with α a quadratic irrational.

Let $p_n/q_n = [0; a_1, \cdots, a_n]$ be the convergents of $\alpha = [0; a_1, a_2, \cdots]$, where $n \geq 1$ and p_n, q_n are coprime to each other. The following theorem indicates that the boundaries of the Siegel disks of the maps in the Inou-Shishikura

[2]For example, if $\alpha = [0; 999, 999, 999, 999, \cdots]$ is of some sufficiently high type, but $2\alpha = [0; 499, 1, 1, 499, 1998, \cdots]$ is not.

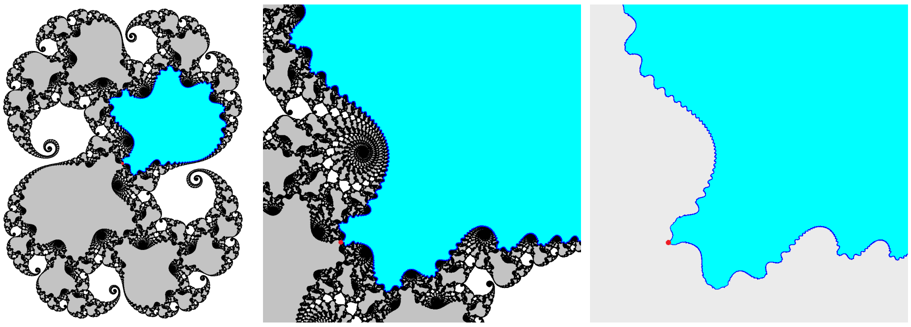

Figure 1: The Julia set (black part) and the Siegel disk (cyan part) of P_α, where $\alpha = [0; 30, 30, \cdots, 30, \cdots]$. The two pictures on the right are the zooms of the left one near the critical point (more and more deeper and the widths of this two pictures are 0.533 and 0.040 respectively). The boundary of the Siegel disk (blue part) is self-similar at the critical point (the red point). The light gray parts are the union of all the iterated preimages of the Siegel disk and the white parts are contained in the basin of infinity (The outside part of the Siegel disk in the rightmost picture is removed because of the complexity of the calculation). See Figure 7 for the Siegel disk of P_α with a non-quadratic irrational α.

class have also the dynamical self-similarity if the rotation numbers are of quadratic irrational.

Theorem 1.3. *Let $\alpha = [0; a_1, a_2, \cdots] \in \mathrm{HT}_N$ such that $a_{n+s} = a_n$ for $n \geq n_0$, where $n_0 \geq 1$ and $s \geq 1$. Suppose that f is a map in the Inou-Shishikura class such that $f(0) = 0$ and $f'(0) = e^{2\pi \mathrm{i}\alpha}$. Then there exists an integer N_0 such that if $N \geq N_0$, then*

$$\left| \frac{f^{\circ q_{n+s}}(\mathrm{cp}_f) - \mathrm{cp}_f}{f^{\circ q_n}(\mathrm{cp}_f) - \mathrm{cp}_f} \right| \underset{n\to\infty}{\longrightarrow} \lambda,$$

where cp_f is the unique critical point of f and $0 < \lambda < 1$ is a constant independent on f. Moreover, the limit stated above holds also for $f = P_\alpha$, $f = E_\alpha$ and[3] *$f = S_{\alpha/2}$.*

The universal nature of the "scaling ratio" stated in Theorem 1.3 was observed partially by many others. This is because this property can be explained if one could prove the hyperbolicity of renormalization in an appropriate functional class and the Inou-Shishikura class is exactly a such class. For example, see [Yam08] for managing the Inou-Shishikura class with golden-like rotation numbers. We will also deal with the rotation numbers covering this, but our method is quite different.

1.2. **Sketch of the proof.** Let us first briefly sketch the idea of the proof of the self-similarity in Theorem 1.1. For each f in the Inou-Shishikura class

[3]The case for the sine family $f = S_{\alpha/2}$ is a little special: the limit holds if $f = S_{\alpha/2}$, $\mathrm{cp}_f = \pi/2$ or $-\pi/2$, and the numbers (q_n) are still correspond to the continued fraction expansion of α but not $\alpha/2$. See §7 for details.

$\mathcal{IS}_0$ with $f(0)=0$ and $f'(0)=1$, we define a set $\mathcal{C}_f$ in the attracting petal $\mathcal{P}_f$ of f such that

$$\Phi_f(\mathcal{C}_f)=\{\zeta\in\mathbb{C}:1/2\leq \operatorname{Re}\zeta\leq 3/2 \text{ and } -2<\operatorname{Im}\zeta\leq 2\},$$

where Φ_f is the Fatou coordinate on $\mathcal{P}_f$ normalized by $\Phi_f(-\frac{4}{27})=1$ and $-\frac{4}{27}$ is the critical value of f. We prove that the closure $\overline{\mathcal{C}}_f$ is compactly contained in $\Omega:=\mathbb{E}\mathrm{xp}\circ\Phi_f(\operatorname{int}\mathcal{C}_f)$, where $\mathbb{E}\mathrm{xp}(\zeta)=-\frac{4}{27}s(e^{2\pi\mathrm{i}\zeta})$ is the modified exponential map and $s(z)=\overline{z}$ is the complex conjugacy (see Proposition 4.1). For each f in the Inou-Shishikura class $\mathcal{IS}_\alpha$ with $f(0)=0$ and $f'(0)=e^{2\pi\mathrm{i}\alpha}$ with sufficiently high type $\alpha\in\mathrm{HT}_N$, we define a similar set $\mathcal{C}_f$ as above which is contained in the perturbed petal (see (2.9)). By the pre-compactness of $\mathcal{IS}_0$ and the continuity (see Proposition 2.4), we know that $\overline{\mathcal{C}}_f$ is compactly contained in Ω for all $f\in\mathcal{IS}_\alpha$ since α is sufficiently small. The above statements are also true for the normalized quadratic polynomial Q_α (see (2.3) for the definition).

Suppose that $f_*\in\mathcal{IS}_\alpha$ is a renormalization fixed point such that $\mathcal{R}f_*=f_*$ (this requires that α has continued fractional expansion $\alpha=[0;a_1,a_1,\cdots]$ and the similar arguments can be applied to renormlization periodic points). For simplifying the notations, we put the star '$*$' at the subscript of the notations that correspond to f_*. Note that $\mathcal{C}_*$ contains the critical value $-\frac{4}{27}$. We use $\mathcal{C}_*^{-1}$ to denote the component of $f_*^{-1}(\mathcal{C}_*)$ that contains the unique critical point cp_* of f_*. By using $\mathcal{C}_{\mathcal{R}f_*}=\mathcal{C}_*$ and $\overline{\mathcal{C}}_f\Subset\Omega$, we can construct an anti-holomorphic map $\Upsilon:\mathcal{C}_*^{-1}\to\mathbb{C}$ such that (i) $\Upsilon(\overline{\mathcal{C}}_*^{-1})$ is compactly contained in the interior of $\mathcal{C}_*^{-1}$; (ii) $\Upsilon(\mathrm{cp}_*)=\mathrm{cp}_*$; and (iii) Υ maps the component of $\partial\Delta_*\cap\mathcal{C}_*^{-1}$ containing cp_* (which is a Jordan arc) to a strict subset of itself, where Δ_* is the Siegel disk of f_*. By Schwarz's Lemma (for anti-holomorphic maps), it means that the boundary of the Siegel disk of f_* is self-similar about the critical point cp_* (see Theorem 4.3). Moreover, the scaling function Υ is anti-holomorphic (or holomorphic if f_* is periodic and the period is an even number).

Next, we transfer the self-similarity of the boundary of the Siegel disk of f_* to all the maps in $\mathcal{IS}_\alpha\cup\{Q_\alpha\}$. Taking advantage of the fact that all the maps in $\mathcal{IS}_\alpha\cup\{Q_\alpha\}$ are attracted to f_* exponentially fast in a suitable metric proved by Inou and Shishikura, we prove that the boundary of the Siegel disk of each $f\in\mathcal{IS}_\alpha\cup\{Q_\alpha\}$ is self-similar at the critical point when one zooms. This allows us to prove the self-similarity when the rotation numbers are of quadratic irrationals.

For the case of non-quadratic irrational case, suppose that $\alpha=[0;\,a_1,\,a_2,\,\cdots]\in\mathrm{HT}_N$ is of bounded type. For $n\geq 1$, let $\alpha_n=[0;a_{n+1},a_{n+2},\cdots]$ be the irrational numbers defined inductively in (2.13) and $f_n:=\mathcal{R}^{\circ n}f$ the sequence of n-th renormalization defined in (2.14). According to Inou-Shishikura (see Theorem 6.1), the sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $h_n:=f_n(e^{-2\pi\mathrm{i}\alpha_n}z)\in\mathcal{IS}_0\cup\{Q_0\}$, respectively, have convergent subsequences $(\alpha_{n_k})_{k\in\mathbb{N}}$ and $(h_{n_k})_{k\in\mathbb{N}}$. We will use this fact to prove the self-similarity of the boundaries of the non-quadratic irrational Siegel disks at the critical points by adopting the similar method as the quadratic irrational case (see §6).

In order to prove that the boundary of the Siegel disk in Theorem 1.1 is a Jordan curve containing the critical point, we will construct a sequence of continuous maps γ_n defined from the unit circle to a set which contains the first several points of the critical orbit of f. Then, by using an estimation on the post-critical set which was proved by Cheraghi in [Che10] (see Proposition 3.3), we prove that (γ_n) converges uniformly to a limit function whose image is exactly the post-critical set of f. Finally, a simple argument shows that the boundary of the Siegel disk is a Jordan curve and is exactly the closed post-critical set of f.

1.3. **Some observations.** The Inou-Shishikura invariant class defined in [IS08] have several applications. The first remarkable application is that Buff and Chéritat used it as one of the main tools to prove the existence of quadratic polynomials with positive area [BC12]. Recently, Cheraghi and his coauthors have found several other important applications. In [Che10] and [Che13], Cheraghi developed several elaborate analytic techniques based on the Inou-Shishikura results. The tools in [Che10] and [Che13] have led to part of the recent major progresses on the dynamics of quadratic polynomials. For example, the Feigenbaum Julia sets with positive area (which is very different from the examples in [BC12]) have been found in [AL15], the Marmi-Moussa-Yoccoz conjecture for rotation numbers of high type has been proved in [CC15], the local connectivity of the Mandelbrot set at some infinitely satellite renormalizable points was proved in [CS15], some statistical properties of the dynamics of quadratic polynomials was depicted in [AC12] etc.

Recently, Chéritat generalized the parabolic and near-parabolic theory to all the unicritical case for any finite degrees [Ché14]. Therefore, there is a hope to prove the self-similarity of the boundaries of the high type Siegel disks of unicritical polynomials at the critical points. However, the method of the proof needs to be changed substantially since parts of our proof in this article rely on some specific estimations in [IS08]. It is worth noting that the second author has established the near-parabolic theory for a class of maps with local degree three by following Inou and Shishikura's work [Yan15]. Hence the idea of the proofs in this article can be used completely similarly to deal with the cubic unicritical case.

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Notations. We use $\mathbb{N}$, $\mathbb{N}^+$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ to denote the sets of all natural numbers (including 0), positive integers, integers, rational numbers, real numbers and complex numbers, respectively. The Riemann sphere and the unit disk are denoted by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ respectively. A disk in the complex plane is denoted by $\mathbb{D}(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ and $\overline{\mathbb{D}}(a, r)$ is its closure. We use $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ to denote the punctured complex plane. For $x \geq 0$, we use $\lfloor x \rfloor$ to denote the integer part of x. Let A, B be two bounded subsets in $\mathbb{C}$. We say that A is *compactly contained* in B if the closure of A is contained in the interior of B, which we denote it by $A \Subset B$.

2. Inou-Shishikura class, (near-) parabolic renormalization

We summarize in this section some results of the Inou-Shishikura's parabolic and near-parabolic renormalization theory in [IS08] and [BC12] which will be used in this article. Some statements can be also found in [Shi98], [Shi00], [Che10] and [Che13].

2.1. **The invariant class and parabolic renormalization.** Let $P(z) := z(1+z)^2$. This cubic polynomial has a parabolic fixed point at 0 with multiplier 1. One can check easily that P has a critical point $\mathrm{cp}_P := -1/3$ which is mapped to the critical value $\mathrm{cv}_P := -4/27$. It has also another critical point -1, which is mapped to 0. Consider the ellipse

$$E := \left\{ x + y\mathrm{i} \in \mathbb{C} : \left(\frac{x+0.18}{1.24}\right)^2 + \left(\frac{y}{1.04}\right)^2 \leq 1 \right\} \tag{2.1}$$

and define[4]

$$U := \psi_1(\widehat{\mathbb{C}} \setminus E), \text{ where } \psi_1(z) := -\frac{4z}{(1+z)^2}. \tag{2.2}$$

The domain U contains the parabolic fixed point 0 and the critical point cp_P but $(-\infty,-1] \cap \overline{U} = \emptyset$ [IS08, §5.A]. For a given function f, we denote its domain of definition by U_f. Following [IS08, §4], we define a class of maps[5]

$$\mathcal{IS}_0 := \left\{ f = P \circ \varphi^{-1} : U_f \to \mathbb{C} \;\middle|\; \begin{array}{l} 0 \in U_f \text{ open in } \mathbb{C},\ \varphi : U \to U_f \\ \text{is conformal},\ \varphi(0) = 0,\ \varphi'(0) \\ = 1 \text{ and } \varphi \text{ has a quasiconformal} \\ \text{extension to } \mathbb{C} \end{array} \right\}.$$

Each map in this class has a parabolic fixed point at the origin, a unique critical point at $\mathrm{cp}_f := \varphi(-1/3) \in U_f$ and a unique critical value at $\mathrm{cv} := -4/27$, which is independent of f.

Since the (near) parabolic renormalization of quadratic polynomials will be also considered, for convenience, we normalize the quadratic polynomial $P_\alpha(z) = e^{2\pi\mathrm{i}\alpha} z + z^2$ with $\alpha \in \mathbb{R}$ to

$$Q_\alpha(z) = e^{2\pi\mathrm{i}\alpha} z + \frac{27}{16} e^{4\pi\mathrm{i}\alpha} z^2 \tag{2.3}$$

such that all Q_α has the same critical value $\mathrm{cv} = -4/27$. In particular, $Q_\alpha = Q_0 \circ R_\alpha$, where $R_\alpha(z) = e^{2\pi\mathrm{i}\alpha} z$.

Proposition 2.1 (Leau-Fatou and Inou-Shishikura). *For all $f \in \mathcal{IS}_0 \cup \{Q_0\}$, there exist two domains $\mathcal{P}_{attr,f}$, $\mathcal{P}_{rep,f} \subset U_f$ and two univalent maps $\Phi_{attr,f} : \mathcal{P}_{attr,f} \to \mathbb{C}$, $\Phi_{rep,f} : \mathcal{P}_{rep,f} \to \mathbb{C}$ such that*

(a) *$\mathcal{P}_{attr,f}$ and $\mathcal{P}_{rep,f}$ are bounded by piecewise analytic curves and compactly contained in U_f, $\mathrm{cp}_f \in \partial\mathcal{P}_{attr,f}$ and $\partial\mathcal{P}_{attr,f} \cap \partial\mathcal{P}_{rep,f} = \{0\}$.*

(b) *The image $\Phi_{attr,f}(\mathcal{P}_{attr,f})$ is a right half plane and $\Phi_{rep,f}(\mathcal{P}_{rep,f})$ is a left half plane.*

(c) *$\Phi_{s,f}(f(z)) = \Phi_{s,f}(z) + 1$ for all $z \in \mathcal{P}_{s,f}$ such that both sides of the equation are defined, where $s = attr, rep$.*

[4] The domain U is denoted by V in [IS08]. But we intend to use the same notation as in [Che10] and [Che13]. Similar consideration will be applied to U' later.

[5] The class $\mathcal{IS}_0$ is denoted by $\mathcal{F}_1$ in [IS08].

Normalization of $\Phi_{attr,f}$ and $\Phi_{rep,f}$. The univalent map $\Phi_{attr,f}$ and the domain $\mathcal{P}_{attr,f}$ in Proposition 2.1 are called an *attracting Fatou coordinate* and an *attracting petal* of f respectively. Since $\Phi_{attr,f}$ is unique up to an additive constant, we *normalize* it by $\Phi_{attr,f}(\mathrm{cp}_f) = 0$. Therefore, we have $\Phi_{attr,f}(\mathcal{P}_{attr,f}) = \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > 0\}$.

The attracting Fatou coordinate $\Phi_{attr,f}$ can be naturally extended to the immediate attracting basin $\mathcal{A}_{attr,f}$ of 0. Specifically, for $z \in \mathcal{A}_{attr,f}$ such that $f^{\circ k}(z) \in \mathcal{P}_{attr,f}$ with $k \geq 0$, define

$$\Phi_{attr,f}(z) := \Phi_{attr,f}(f^{\circ k}(z)) - k.$$

The univalent map $\Phi_{rep,f}$ and the domain $\mathcal{P}_{rep,f}$ in Proposition 2.1 are called a *repelling Fatou coordinate* and a *repelling petal* of f. Since $\Phi_{rep,f}$ is also unique up to an additive constant, we *normalize* it by

$$\Phi_{attr,f}(z) - \Phi_{rep,f}(z) \to 0 \text{ when } z \to 0, \tag{2.4}$$

where z is contained in $\mathcal{A}_{attr,f} \cap \mathcal{P}_{rep,f}$ such that $\operatorname{Im}\Phi_{attr,f}(z) \to +\infty$ as $z \to 0$.

Definition (See Figure 2)**.** For $f \in \mathcal{IS}_0 \cup \{Q_0\}$, define

$$\begin{aligned} \mathcal{C}_f &:= \left\{ z \in \mathcal{P}_{attr,f} \;\middle|\; \begin{array}{l} 1/2 \leq \operatorname{Re}\Phi_{attr,f}(z) \leq 3/2 \\ \text{and } -2 < \operatorname{Im}\Phi_{attr,f}(z) \leq 2 \end{array} \right\}, \text{ and} \\ \mathcal{C}_f^\sharp &:= \left\{ z \in \mathcal{P}_{attr,f} \;\middle|\; \begin{array}{l} 1/2 \leq \operatorname{Re}\Phi_{attr,f}(z) \leq 3/2 \\ \text{and } 2 \leq \operatorname{Im}\Phi_{attr,f}(z) \end{array} \right\}. \end{aligned} \tag{2.5}$$

By the normalization of $\Phi_{attr,f}$, we have $\mathrm{cv} = -4/27 \in \operatorname{int}\mathcal{C}_f$ and $0 \in \partial\mathcal{C}_f^\sharp$.

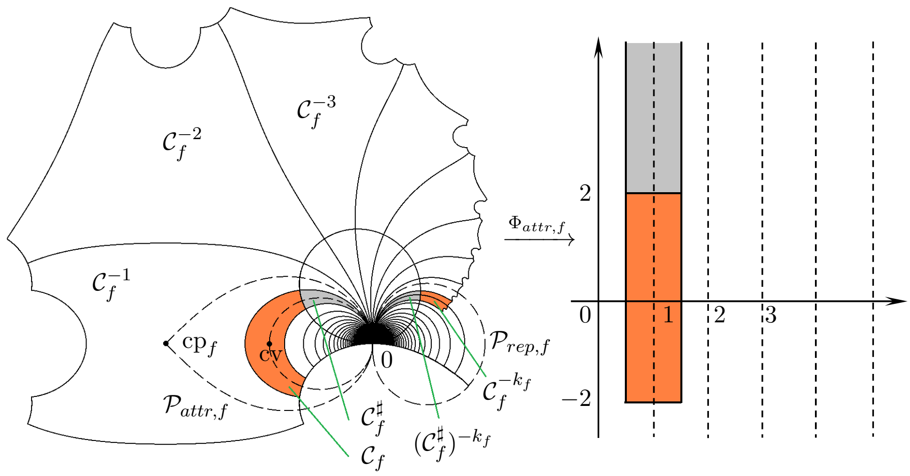

Figure 2: Left: The two Fatou petals $\mathcal{P}_{attr,f}$ and $\mathcal{P}_{rep,f}$ (whose boundaries are depicted by the dashed lines) and $\mathcal{C}_f \cup \mathcal{C}_f^\sharp$ with its images and preimages. It can be clearly observed that $\mathcal{C}_f \cup \mathcal{C}_f^\sharp \subset \mathcal{P}_{attr,f}$ and $\mathcal{C}_f^{-k_f} \cup (\mathcal{C}_f^\sharp)^{-k_f} \subset \mathcal{P}_{rep,f}$. Right: The image of the attracting petal $\mathcal{P}_{attr,f}$ under the attracting Fatou coordinate $\Phi_{attr,f}$. The parabolic fixed point 0 of f is the intersection of $\overline{\mathcal{P}}_{attr,f}$ and $\overline{\mathcal{P}}_{rep,f}$.

Proposition 2.2 ([BC12, Result of Inou-Shishikura, p. 703] and [BC12, Lemma 8, p. 705], see Figure 2)**.** *There exist two positive integers* $\boldsymbol{k}$ *and* $\boldsymbol{k}''$ *such that for all* $f \in \mathcal{IS}_0 \cup \{Q_0\}$*, there exists a positive integer* $k_f \leq \boldsymbol{k}''$ *such that*[6]

(a) *For all* $k \geq 0$*, the unique connected component* $(\mathcal{C}_f^\sharp)^{-k}$ *of* $f^{-k}(\mathcal{C}_f^\sharp)$ *that contains* 0 *in its closure is relatively compact in* U_f *and* $f^{\circ k} : (\mathcal{C}_f^\sharp)^{-k} \to \mathcal{C}_f^\sharp$ *is an isomorphism; the unique connected component* $\mathcal{C}_f^{-k}$ *of* $f^{-k}(\mathcal{C}_f)$ *that intersects* $(\mathcal{C}_f^\sharp)^{-k}$ *is relatively compact in* U_f *and* $f^{\circ k} : \mathcal{C}_f^{-k} \to \mathcal{C}_f$ *is a covering of degree* 2 *ramified over* cv.

(b) *The repelling petal* $\mathcal{P}_{rep,f}$ *of* f *defined in Proposition 2.1 can be chosen so that* $\Phi_{rep,f}(\mathcal{P}_{rep,f}) = \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta < -\boldsymbol{k}\}$*.*

(c) k_f *is the* smallest *positive integer such that for all* $k \geq k_f$*, one has* $\mathcal{C}_f^{-k} \cup (\mathcal{C}_f^\sharp)^{-k} \subset \{z \in \mathcal{P}_{rep,f} : \operatorname{Re}\Phi_{rep,f}(z) < -\boldsymbol{k} - 1/2\}$*.*

Definition (Parabolic renormalization)**.** For $f \in \mathcal{IS}_0 \cup \{Q_0\}$, define

$$S_f := \mathcal{C}_f^{-k_f} \cup (\mathcal{C}_f^\sharp)^{-k_f}.$$

Then consider the map

$$\Phi_{attr,f} \circ f^{\circ k_f} \circ \Phi_{rep,f}^{-1} : \Phi_{rep,f}(S_f) \to \mathbb{C}.$$

This map commutes with the translation by one. Hence it projects by the *modified exponential map*

$$\mathbb{E}\mathrm{xp}(\zeta) = -\frac{4}{27} s\,(e^{2\pi \mathrm{i}\zeta}) \tag{2.6}$$

to a well-defined holomorphic map $\mathcal{R}_0 f$ which is defined on a set punctured at the origin, where $s : z \mapsto \overline{z}$ is the complex conjugacy. It can be checked that 0 is a removable singularity of $\mathcal{R}_0 f$. By the normalization of $\Phi_{attr,f}$ and $\Phi_{rep,f}$ in (2.4), it follows that $\mathcal{R}_0 f(0) = 0$ and $(\mathcal{R}_0 f)'(0) = 1$, i.e., the origin is again a parabolic fixed point of $\mathcal{R}_0 f$ with multiplier 1. The map $\mathcal{R}_0 f$ is called in this article the *parabolic renormalization* of f.

Recall that $P(z) = z(1+z)^2$ is the cubic polynomial defined at the beginning of this subsection. Define a domain

$$U' := P^{-1}(\mathbb{D}(0, \tfrac{4}{27} e^{4\pi})) \smallsetminus ((-\infty, -1] \cup \overline{B}), \tag{2.7}$$

where B is the connected component of $P^{-1}(\mathbb{D}(0, \frac{4}{27} e^{-4\pi}))$ that contains -1. By an explicit calculation, one can prove that $\overline{U} \subset U'$ (see [IS08, Proposition 5.2]).

Theorem 2.3 ([IS08, Main Theorem 1(c)])**.** *For any* $f \in \mathcal{IS}_0 \cup \{Q_0\}$*, the parabolic renormalization* $\mathcal{R}_0 f$ *is well-defined so that a restriction of* $\mathcal{R}_0 f = P \circ \psi^{-1}$ *belongs to* $\mathcal{IS}_0$*. Moreover,* ψ *extends to a univalent function from* U' *to* $\mathbb{C}$*.*

[6] The results in [BC12, Lemma 8, p. 705] hold for $V_f = \{z \in \mathcal{P}_{attr,f} : 0 < \operatorname{Re}\Phi_{attr,f}(z) < 2 \text{ and } 0 < \operatorname{Im}\Phi_{attr,f}(z)\}$ and $W_f = \{z \in \mathcal{P}_{attr,f} : 0 < \operatorname{Re}\Phi_{attr,f}(z) < 2 \text{ and } -2 < \operatorname{Im}\Phi_{attr,f}(z) < 2\}$ (see [BC12, Definition 7, p. 703]). Therefore, the results hold for $\mathcal{C}_f$ and $\mathcal{C}_f^\sharp$.

Note that the quadratic polynomial $Q_0(z) = z + \frac{27}{16}z^2$ does not belong to the Inou-Shishikura class, but its first parabolic renormalization does (see [IS08, §3]). The renormalized map $\mathcal{R}_0 f = P \circ \psi^{-1} : \psi(U') \to \mathbb{D}(0, \frac{4}{27}e^{4\pi})$ is surjective but not proper.

2.2. **Near-parabolic renormalization.** For $\alpha \in \mathbb{R}$, define

$$\mathcal{IS}_\alpha = \{f(z) = f_0(e^{2\pi \mathrm{i}\alpha} z) : e^{-2\pi\mathrm{i}\alpha} \cdot U_{f_0} \to \mathbb{C} \mid f_0 \in \mathcal{IS}_0\}. \tag{2.8}$$

If A is a subset of $\mathbb{R}$, we denote by $\mathcal{IS}_A$ the set

$$\mathcal{IS}_A := \bigcup_{\alpha \in A} \mathcal{IS}_\alpha.$$

We need to consider a sequence of functions converging to a limiting function and sometimes we will use the term that a function is "close" to another. Hence the definition of a neighborhood of a function is needed.

Definition (Neighborhood of a function)**.** For a given function f, a *neighborhood* of f is

$$\mathcal{N} = \mathcal{N}(f; K, \varepsilon) = \left\{ g : U_g \to \widehat{\mathbb{C}} \;\middle|\; K \subset U_g \text{ and } \sup_{z \in K} d_{\widehat{\mathbb{C}}}(g(z), f(z)) < \varepsilon \right\},$$

where $d_{\widehat{\mathbb{C}}}$ denotes the spherical distance, K is a compact set contained in U_f and $\varepsilon > 0$. A sequence (f_n) is called to *converge* to f *uniformly on compact sets* if for any neighborhood $\mathcal{N}$ of f, there exists $n_0 > 0$ such that $f_n \in \mathcal{N}$ for all $n \geq n_0$.

We just defined a "neighborhood basis" which thus defines a topology (which is called the *compact-open topology*). However, this topology is not nice: there are non-closed points (for any extension g of a function f, the map g belongs to all neighborhoods of f).

If $f \in \cup_{\alpha \in [0,1)} \mathcal{IS}_\alpha \cup \{Q_\alpha\}$, we denote by $\alpha_f \in [0, 1)$ the rotation number of f at the origin, i.e. the real number $\alpha_f \in [0, 1)$ such that $f'(0) = e^{2\pi\mathrm{i}\alpha_f}$. If $\alpha_f > 0$ is small, besides the origin, the map f has another fixed point $\sigma_f \neq 0$ near 0 in U_f. The fixed point σ_f depends continuously on f (see [BC12, Lemma 9, p. 707]).

Proposition 2.4 ([BC12, Proposition 12, p. 707], see Figure 3)**.** *There exist two positive integers*[7] $\boldsymbol{k}$, $\hat{\boldsymbol{k}}$ *and a constant* $\varepsilon_1 > 0$ *satisfying* $\lfloor \frac{1}{\varepsilon_1} \rfloor - \boldsymbol{k} > 1$, *such that for all* $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ *with* $\alpha \in (0, \varepsilon_1]$, *there exist a Jordan domain* $\mathcal{P}_f \subset U_f$ *and a univalent map* $\Phi_f : \mathcal{P}_f \to \mathbb{C}$, *such that*

(a) $\mathcal{P}_f$ *contains* cv *and it is bounded by two arcs joining* 0 *and* σ_f, *and there is a branch of argument defined on* $\mathcal{P}_f$ *such that*

$$\sup_{z, z' \in \mathcal{P}_f} |\arg(z) - \arg(z')| \leq 2\pi \hat{\boldsymbol{k}}.$$

(b) $\Phi_f(\mathrm{cv}) = 1$; $\Phi_f(\mathcal{P}_f) = \{\zeta \in \mathbb{C} : 0 < \mathrm{Re}\,\zeta < \lfloor \frac{1}{\alpha_f} \rfloor - \boldsymbol{k}\}$ *with* $\mathrm{Im}\,\Phi_f(z) \to +\infty$ *as* $z \to 0$ *and* $\mathrm{Im}\,\Phi_f(z) \to -\infty$ *as* $z \to \sigma_f$;

[7] Note that the constant $\boldsymbol{k}$ has appeared in Proposition 2.2. They can be seen as a same constant since we can assume that they are both large enough.

If $z \in \mathcal{P}_f$ *and* $\operatorname{Re}\Phi_f(z) < \lfloor \frac{1}{\alpha_f} \rfloor - \boldsymbol{k} - 1$, *then* $f(z) \in \mathcal{P}_f$ *and* $\Phi_f(f(z)) = \Phi_f(z) + 1$.

(c) *If* (f_n) *is a sequence of maps in* $\cup_{\alpha\in(0,\varepsilon_1]}\mathcal{IS}_\alpha \cup \{Q_\alpha\}$ *converging to a map* $f_0 \in \mathcal{IS}_0 \cup \{Q_0\}$, *then any compact set* $K \subset \mathcal{P}_{attr,f_0}$ *is contained in* $\mathcal{P}_{f_n}$ *for* n *large enough and the sequence* (Φ_{f_n}) *converges to* Φ_{attr,f_0} *uniformly on* K*; Moreover, any compact set* $K \subset \mathcal{P}_{rep,f_0}$ *is contained in* $\mathcal{P}_{f_n}$ *for* n *large enough and the sequence* $(\Phi_{f_n} - \frac{1}{\alpha_{f_n}})$ *converges to* Φ_{rep,f_0} *uniformly on* K.

Proposition 2.4 was proved in [BC12] only for the Inou-Shishikura class. However, when $f = Q_\alpha$ with sufficiently small $\alpha > 0$, the existence of the domain $\mathcal{P}_f$ and the coordinate $\Phi_f : \mathcal{P}_f \to \mathbb{C}$ satisfying the properties in the above proposition is rather classical (see [Shi00]). The map Φ_f in Proposition 2.4 is called the *perturbed Fatou coordinate* of f and $\mathcal{P}_f$ is called a *perturbed petal*. Sometimes we omit the word "perturbed" for convenience.

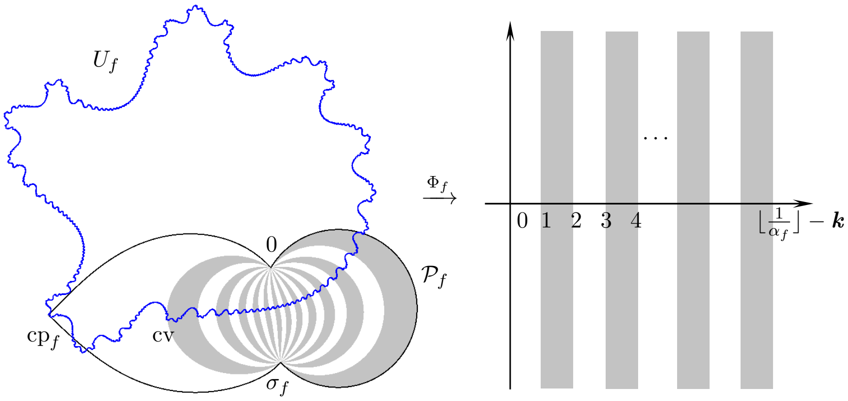

Figure 3: The perturbed Fatou coordinate Φ_f and its domain of definition $\mathcal{P}_f$. The image of $\mathcal{P}_f$ under Φ_f has been colored by the same color on the right. The blue set on the left depicts the forward orbit of the critical point cp_f. In this figure, the rotation number is chosen as $\alpha_f = [0; 30, 30, \cdots, 30, \cdots]$. Figure 4 is also.

Definition (See Figure 4)**.** Let $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_1]$, where ε_1 is the constant in Proposition 2.4. Similar to the case in parabolic renormalization, we define

$$\begin{aligned} \mathcal{C}_f &:= \left\{ z \in \mathcal{P}_f \;\middle|\; \begin{array}{l} 1/2 \le \operatorname{Re}\Phi_f(z) \le 3/2 \\ \text{and } -2 < \operatorname{Im}\Phi_f(z) \le 2 \end{array} \right\}, \text{ and} \\ \mathcal{C}_f^\sharp &:= \left\{ z \in \mathcal{P}_f \;\middle|\; \begin{array}{l} 1/2 \le \operatorname{Re}\Phi_f(z) \le 3/2 \\ \text{and } 2 \le \operatorname{Im}\Phi_f(z) \end{array} \right\}. \end{aligned} \tag{2.9}$$

Note that $\mathrm{cv} = -4/27 \in \operatorname{int}\mathcal{C}_f$ and $0 \in \partial\mathcal{C}_f^\sharp$.

The following statement is very similar to Proposition 2.2.

Proposition 2.5 ([BC12, Proposition 13, p. 713], see Figure 4)**.** *There exist a constant $\varepsilon_2 \in (0, \varepsilon_1]$ and a positive integer*[8] $\boldsymbol{k}''$ *such that for any* $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ *with* $\alpha \in (0, \varepsilon_2]$, *there exists a positive integer* $k_f \leq \boldsymbol{k}''$ *such that*

(a) *For all* $0 \leq k \leq k_f$, *the unique connected component* $(\mathcal{C}_f^\sharp)^{-k}$ *of* $f^{-k}(\mathcal{C}_f^\sharp)$ *that contains* 0 *in its closure is relatively compact in* U_f *and* $f^{\circ k} : (\mathcal{C}_f^\sharp)^{-k} \to \mathcal{C}_f^\sharp$ *is an isomorphism; the unique connected component* $\mathcal{C}_f^{-k}$ *of* $f^{-k}(\mathcal{C}_f)$ *that intersects* $(\mathcal{C}_f^\sharp)^{-k}$ *is relatively compact in* U_f *and* $f^{\circ k} : \mathcal{C}_f^{-k} \to \mathcal{C}_f$ *is a covering of degree* 2 *ramified over* cv*.*

(b) k_f *is the* smallest *positive integer such that* $\mathcal{C}_f^{-k_f} \cup (\mathcal{C}_f^\sharp)^{-k_f} \subset \{z \in \mathcal{P}_f : 0 < \operatorname{Re} \Phi_f(z) < \lfloor \frac{1}{\alpha_f} \rfloor - \boldsymbol{k} - \frac{1}{2}\}$.

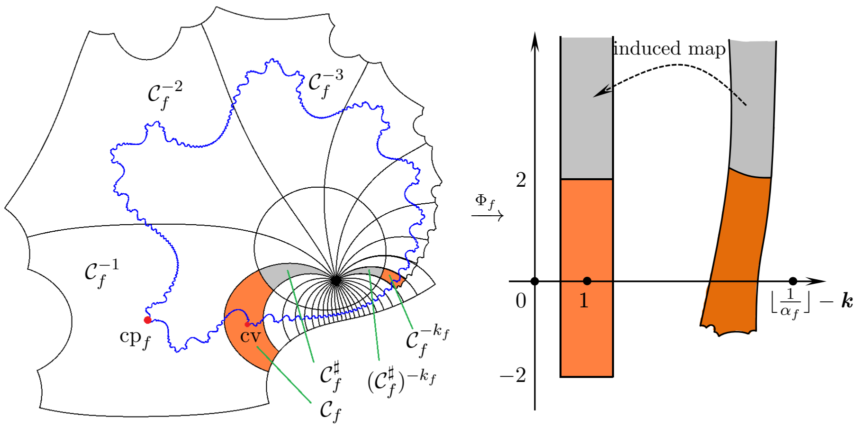

Figure 4: Left: The sets $\mathcal{C}_f$, $\mathcal{C}_f^\sharp$ and some of their preimages. The blue set depicts the forward orbit of the critical point cp$_f$. Right: The images of $\mathcal{C}_f \cup \mathcal{C}_f^\sharp$ and S_f under the perturbed Fatou coordinate Φ_f. It also shows how the near-parabolic renormalization map is induced.

Definition (Near-parabolic renormalization[9], see Figure 4)**.** For $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_2]$, define

$$S_f := \mathcal{C}_f^{-k_f} \cup (\mathcal{C}_f^\sharp)^{-k_f}.$$

Then consider the map

$$\Phi_f \circ f^{\circ k_f} \circ \Phi_f^{-1} : \Phi_f(S_f) \to \mathbb{C}.$$

[8] The constant $\boldsymbol{k}''$ also appears in Proposition 2.2. They can be seen as a same constant since we can assume that they are both large enough.

[9] Near-parabolic renormalization is also called *cylinder renormalization*, which was introduced by Yampolsky in the study of analytic circle homeomorphisms with a critical point.

This map commutes with the translation by one[10]. Hence it projects by the modified exponential map $\mathbb{E}\mathrm{xp}(\zeta)$ to a well-defined holomorphic map $\mathcal{R}f$ which is defined on a set punctured at zero, where $\mathbb{E}\mathrm{xp}$ is defined in (2.6). One can check that $\mathcal{R}f$ extends across zero and satisfies $\mathcal{R}f(0) = 0$ and $(\mathcal{R}f)'(0) = e^{2\pi \mathrm{i}/\alpha_f}$. The map $\mathcal{R}f$ is called the *near-parabolic renormalization* of f.

If necessary, enlarging the constant $\boldsymbol{k}$ in Proposition 2.4 slightly (hence the constant $\boldsymbol{k}''$ in Proposition 2.5 should be enlarged accordingly), we can assume that the set

$$\widetilde{\mathcal{P}}_f := \mathcal{P}_f \cup \mathcal{C}_f^{-1} \cup (\mathcal{C}_f^{\sharp})^{-1} \tag{2.10}$$

is simply connected. By Propositions 2.4(a), 2.5 and the pre-compactness of the class $\cup_{\alpha\in(0,\varepsilon_2]}\mathcal{IS}_\alpha$, there exists an integer $\boldsymbol{k}' \geq \hat{\boldsymbol{k}}$ such that

$$\sup_{z,z'\in\widetilde{\mathcal{P}}_f} |\arg(z) - \arg(z')| \leq 2\pi \boldsymbol{k}' \tag{2.11}$$

for any continuous branch of the argument function defined on $\widetilde{\mathcal{P}}_f$.

For two maps $f = P \circ \varphi_f^{-1}$ and $g = P \circ \varphi_g^{-1}$ in $\mathcal{IS}_0$, the *Teichmüller distance* between f and g is defined as

(2.12)

$$\mathrm{d_{Teich}}(f,g) := \inf \left\{ \log K \;\middle|\; \begin{array}{l} K \text{ is the dilatation of the quasi-} \\ \text{conformal map } \hat{\varphi}_g \circ \hat{\varphi}_f^{-1}, \text{ where } \hat{\varphi}_f \\ \text{and } \hat{\varphi}_g \text{ are quasiconformal extentions} \\ \text{of } \varphi_f \text{ and } \varphi_g \text{ onto } \mathbb{C}, \text{ respectively} \end{array} \right\}.$$

This metric is inherited from the one to one correspondence between $\mathcal{IS}_0$ and the Teichmüller space of $\mathbb{C} \setminus \overline{U}$. It is known that the Teichmüller space with the above metric is a complete metric space. The convergence in this metric implies the uniform convergence on compact sets. The complex structure inherited from the space of the Schwarzian derivatives S_{φ_f} makes $\mathcal{IS}_0$ isomorphic to the universal Teichmüller space as a complex manifold of infinite dimension (see [IS08, §6]).

Since each $f \in \mathcal{IS}_\alpha$ can be uniquely written as $f(z) = f_0(e^{2\pi\mathrm{i}\alpha}z) : e^{-2\pi\mathrm{i}\alpha} \cdot U_{f_0} \to \mathbb{C}$ with $f_0 \in \mathcal{IS}_0$, there is a natural projection

$$\pi : \mathcal{IS}_\alpha \to \mathcal{IS}_0,\ \pi(f) = f_0, \text{ where } f_0(z) = f(e^{-2\pi\mathrm{i}\alpha}z).$$

Recall that U' is a simply connected domain defined in (2.7) which satisfies $\overline{U} \subset U'$.

Theorem 2.6 ([IS08, Main Theorem 3]). *There exists a constant $\varepsilon_0 \in (0,\varepsilon_2]$ such that for all $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0,\varepsilon_0]$, the near-parabolic renormalization $\mathcal{R}f$ is well-defined so that*

(a) $\mathcal{R}f = P \circ \psi^{-1} \in \mathcal{IS}_{1/\alpha}$. *Moreover, ψ extends to a univalent function from U' to $\mathbb{C}$.*

[10]The commutation occurs also on a subset A_f of the boundary of S_f. The fact that $\mathcal{R}f$ is holomorphic on $\mathbb{E}\mathrm{xp}(A_f)$ is not automatic: it comes from the fact that the map, before projection, had a holomorphic extension to a neighborhood of $\Phi_f(S_f)$ or at least a neighborhood of the part of the boundary where the commutation occurs (However, we think it can be extended to a neighborhood of the closure of S_f in $\mathbb{C}$).

(b) *There exists a constant $0 < \varrho < 1$, such that for all $f, g \in \mathcal{IS}_\alpha$ with $\alpha \in (0, \varepsilon_0]$, we have*

$$\mathrm{d}_{\mathrm{Teich}}(\pi \circ \mathcal{R}(f), \pi \circ \mathcal{R}(g)) \leq \varrho\, \mathrm{d}_{\mathrm{Teich}}(\pi(f), \pi(g)).$$

Recall that HT_N is the collection of the high type irrational numbers introduced in the introduction with each coefficient of the continued fractional expansion at least N.

Theorem 2.7 ([IS08, Corollary 4.1 and Main Theorems 2 and 3])**.** *Let $\varepsilon_0 > 0$ be the constant in Theorem 2.6. Then the following holds:*

(a) *There exists a unique $f_0^* \in \mathcal{IS}_0$ such that $\mathcal{R}_0 f_0^* = f_0^*$. For every $f \in \mathcal{IS}_0 \cup \{Q_0\}$, $\mathcal{R}_0^{\circ n} f \to f_0^*$ as $n \to \infty$ exponentially fast under the metric $\mathrm{d}_{\mathrm{Teich}}$.*

(b) *Let $\alpha = [0; a_1, a_2, \cdots] \in \mathrm{HT}_N$ such that $N \geq 1/\varepsilon_0$ and $a_{n+s} = a_n$ for $n \geq n_0$, where $n_0 \geq 1$ and $s \geq 1$. Let $\beta = [0; \hat{a}_1, \hat{a}_2, \cdots]$ such that $\hat{a}_n = a_{n+n_0-1}$ for $n \geq 1$. Then there exists a unique $f_\beta^* \in \mathcal{IS}_\beta$ such that $\mathcal{R}^{\circ s} f_\beta^* = f_\beta^*$. For every $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$, then $\pi \circ \mathcal{R}^{\circ(ns)} f \to \pi(f_\beta^*)$ exponentially fast under the metric $\mathrm{d}_{\mathrm{Teich}}$ as $n \to \infty$.*

For more information on the study of (near) parabolic renormalization fixed point, see [Yam08] and [LY14].

2.3. **Renormalization tower and orbit relations.** In the rest of this article, we always assume that N is sufficiently large so that $N \geq 1/\varepsilon_0$, where $\varepsilon_0 > 0$ is the constant introduced in Theorem 2.6. Let $\alpha = [0; a_1, a_2, \cdots] \in \mathrm{HT}_N$. Define $\alpha_0 := \alpha$, and inductively for $n \geq 1$, define the sequence of real numbers $\alpha_n \in (0, 1)$ as

$$\alpha_n = \frac{1}{\alpha_{n-1}} - \lfloor \frac{1}{\alpha_{n-1}} \rfloor, \text{ where } n \geq 1. \tag{2.13}$$

Then each α_n has continued fraction expansion $[0; a_{n+1}, a_{n+2}, \cdots]$. By definition, we have $\alpha_n \in (0, \varepsilon_0]$ for $n \in \mathbb{N}$.

Let $\alpha \in \mathrm{HT}_N$ and $f_0 \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$. By Theorem 2.6, we can inductively define the sequence of maps

$$f_n := \mathcal{R} f_{n-1} : U_{f_n} \to \mathbb{C} \text{ with } n \geq 1. \tag{2.14}$$

Let $U_n := U_{f_n}$ be the domain of definition of f_n for $n \geq 0$. Then for all n, we have

$$f_n : U_n \to \mathbb{C},\ f_n(0) = 0,\ f_n'(0) = e^{2\pi \mathrm{i} \alpha_n} \text{ and } \mathrm{cv}_{f_n} = -4/27.$$

For $n \geq 0$, let $\Phi_n := \Phi_{f_n}$ denote the Fatou coordinate of $f_n : U_n \to \mathbb{C}$ defined on the perturbed petal $\mathcal{P}_n := \mathcal{P}_{f_n}$. To simplify the technical details of the proofs, we assume further that

$$N \geq \boldsymbol{k} + 2\boldsymbol{k}' + 2, \tag{2.15}$$

where $\boldsymbol{k}$ and $\boldsymbol{k}'$ are constants introduced in Proposition 2.4 and (2.11) respectively. The reason to do this is to make the set $\Phi_n(\mathcal{P}_n)$ wide enough to contain a set defined later[11]. Note that $\widetilde{\mathcal{P}}_n := \widetilde{\mathcal{P}}_{f_n}$ defined in (2.10) is simply

[11]The assumption (2.15) will be used in the proofs of Lemma 3.1 and Proposition 4.4.

connected. By (2.11) and assumption on N, there is an anti-holomorphic inverse branch[12] of $\mathbb{E}\mathrm{xp}$:

$$\chi_n : \widetilde{\mathcal{P}}_n \to \Phi_{n-1}(\mathcal{P}_{n-1}), \tag{2.16}$$

where $n \geq 1$. There are several choices of this map but we fix any one of them such that

$$\operatorname{Re} \chi_n(\widetilde{\mathcal{P}}_n) \subset [0, \boldsymbol{k}' + 1]. \tag{2.17}$$

We will fix this choice in the rest of this article. Now define

$$\psi_n := \Phi_{n-1}^{-1} \circ \chi_n : \widetilde{\mathcal{P}}_n \to \mathcal{P}_{n-1}. \tag{2.18}$$

Then each ψ_n extends continuously to $0 \in \partial \widetilde{\mathcal{P}}_n$ by mapping it to 0.

For $n \geq 1$, we define the composition

$$\Psi_n := \psi_1 \circ \psi_2 \circ \cdots \circ \psi_n : \widetilde{\mathcal{P}}_n \to \mathcal{P}_0 \subset U_0. \tag{2.19}$$

For every $n \geq 0$, let $\mathcal{C}_n := \mathcal{C}_{f_n}$ and $\mathcal{C}_n^\sharp := \mathcal{C}_{f_n}^\sharp$ be the corresponding sets for f_n defined in (2.9). Let $k_n := k_{f_n}$ be the positive integer defined in Proposition 2.5 such that

$$S_n^0 := S_{f_n} = \mathcal{C}_n^{-k_n} \cup (\mathcal{C}_n^\sharp)^{-k_n} \subset \{z \in \mathcal{P}_n : 0 < \operatorname{Re} \Phi_n(z) < \lfloor \tfrac{1}{\alpha_n} \rfloor - \boldsymbol{k} - \tfrac{1}{2}\}.$$

Each irrational number $\alpha = [0; a_1, a_2, \cdots] \in (0,1)$ can be approximated by the rational numbers $p_n/q_n := [0;\ a_1,\ \cdots,\ a_n]$, where p_n and q_n are coprime to each other. Define

$$\mathcal{P}_n' := \mathcal{C}_n^{-1} \cup (\mathcal{C}_n^\sharp)^{-1} \cup \{z \in \mathcal{P}_n : 0 < \operatorname{Re} \Phi_n(z) < \lfloor \tfrac{1}{\alpha_n} \rfloor - \boldsymbol{k} - 1\}. \tag{2.20}$$

Lemma 2.8. *For every $n \geq 1$, we have*

(a) *For every $z \in \mathcal{P}_n'$, $f_{n-1}^{\circ \lfloor 1/\alpha_{n-1} \rfloor} \circ \psi_n(z) = \psi_n \circ f_n(z)$ and $f_0^{\circ q_n} \circ \Psi_n(z) = \Psi_n \circ f_n(z)$;*

(b) *For every $z \in S_n^0$, $f_{n-1}^{\circ(k_n \lfloor 1/\alpha_{n-1} \rfloor + 1)} \circ \psi_n(z) = \psi_n \circ f_n^{\circ k_n}(z)$ and $f_0^{\circ(k_n q_n + q_{n-1})} \circ \Psi_n(z) = \Psi_n \circ f_n^{\circ k_n}(z)$.*

The first parts of the two statements in Lemma 2.8 were proved in [Che10, Lemma 2.2], and the second parts were proved in [BC12, Propositions 14 and 15, pp. 716 and 718] and [Che10, Lemma 2.3]. It was proved in [Che10, Lemma 2.2] that Lemma 2.8(a) holds for $\{z \in \mathcal{P}_n : 0 < \operatorname{Re} \Phi_n(z) < \lfloor \frac{1}{\alpha_n} \rfloor - \boldsymbol{k} - 1\}$. Actually, it is true for $z \in \mathcal{P}_n'$ since we have assumed that $\widetilde{\mathcal{P}}_n$ defined in (2.10) is simply connected.

3. The boundaries of the Siegel disks are Jordan curves

It was known that the boundary of the Siegel disk of the quadratic polynomial $P_\alpha(z) = e^{2\pi i \alpha} z + z^2$ is a Jordan curve containing the critical point if α is of bounded type ([Dou87] and [Her87]). In this article we need that the boundary of the Siegel disk of $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ is a Jordan curve and is the closure of the post-critical set if α is of high type and bounded type. Actually, this follows by holomorphic motion arguments from the quadratic

[12] A better notation for the inverse of of $\mathbb{E}\mathrm{xp}$ is $\mathbb{L}\mathrm{og}$, but this notation will be saved and used in §4.2 and §5.3.

polynomial [AL15, Proposition 5.11], thus eventually relies on quasiconformal surgery. We present here a proof that does not rely on that, but instead on some fine estimates to be found in [Che10].

3.1. **The critical orbit is infinite.** We still assume that N is large enough such that $N \geq 1/\varepsilon_0$. Recall that HT_N is the set of high type irrationals defined in the introduction. For $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in \mathrm{HT}_N$, the *post-critical set* of f is defined as $\mathcal{PC}(f) = \overline{\cup_{k=1}^\infty f^{\circ k}(\mathrm{cp}_f)}$, where cp_f is the unique critical point of f. The following lemma shows that $\mathcal{PC}(f)$ consists of infinitely many points.

Lemma 3.1. *Let $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in \mathrm{HT}_N$. For any positive integers $k \neq l$, we have $f^{\circ k}(\mathrm{cp}_f) \neq f^{\circ l}(\mathrm{cp}_f)$.*

Proof. Let $f_0 := f$ and inductively we define the sequence of maps $f_n := \mathcal{R} f_{n-1}$ with $n \geq 1$ as in (2.14). In the rest of the proof, we use 'n' to replace 'f_n' in the subscript of all the notations as before. Moreover, we use $c_0 := \mathrm{cp}_{f_0}$ to denote the unique critical point of f_0.

For every $n \geq 0$, let $z_n \in \mathcal{P}_n$ be the unique point such that $\Phi_n(z_n) = 2$ (by (2.15), such point exists). Note that $\Phi_n(-\frac{4}{27}) = 1$. It follows from Lemma 2.8(a) that

$$\Psi_n(z_n) = f_0^{\circ q_n}(\Psi_n(-\tfrac{4}{27})),$$

where $\Psi_n : \widetilde{\mathcal{P}}_n \to \mathcal{P}_0$ is the map defined in (2.19). According to (2.17), we know that $\mathrm{Re}\,\Phi_0(\Psi_n(z_n))$ and $\mathrm{Re}\,\Phi_0(\Psi_n(-\frac{4}{27}))$ are both contained in $[0, \boldsymbol{k}' + 1]$. In particular, according to the definition of near-parabolic renormalization, $\Phi_0(\Psi_n(-\frac{4}{27}))$ is an integer contained in $[1, \boldsymbol{k}']$ since $\Psi_n(-\frac{4}{27}) \in \mathrm{int}\,\Psi_n(\mathcal{P}_n)$. Therefore, there exists an integer $1 \leq l_n \leq \boldsymbol{k}'$ such that

$$\Psi_n(-\tfrac{4}{27}) = f_0^{\circ l_n}(c_0) \text{ and } \Psi_n(z_n) = f_0^{\circ(l_n+q_n)}(c_0).$$

We claim that $\{f_0^{\circ(l_n+q_n)}(c_0)\}_{n\in\mathbb{N}}$ is an infinite set. Otherwise, there exist two integers $m > n$ such that $\Psi_m(z_m) = \Psi_n(z_n)$, where $z_m \in \mathcal{P}_m$ and $z_n \in \mathcal{P}_n$. By the definition of Ψ_n, we have $\Psi_m = \Psi_n \circ \psi_{n+1} \circ \cdots \circ \psi_m$, where ψ_{n+1} is defined in (2.18) satisfying $\psi_{n+1}^{-1}(z_n) = \mathbb{E}\mathrm{xp} \circ \Phi_n(z_n) = \mathbb{E}\mathrm{xp}(2) = -\frac{4}{27}$. Therefore, we have

$$\begin{aligned} 2 &= \Phi_m(z_m) = \Phi_m \circ \Psi_m^{-1} \circ \Psi_m(z_m) = \Phi_m \circ \Psi_m^{-1} \circ \Psi_n(z_n) \\ &= \Phi_m \circ \psi_m^{-1} \circ \cdots \circ \psi_{n+1}^{-1} \circ \Psi_n^{-1}(\Psi_n(z_n)) = \Phi_m \circ \psi_m^{-1} \circ \cdots \circ \psi_{n+1}^{-1}(z_n) \\ &= \Phi_m(-\tfrac{4}{27}) = 1. \end{aligned}$$

This is a contradiction. Therefore, $\{f_0^{\circ(l_n+q_n)}(c_0)\}_{n\in\mathbb{N}}$ is an infinite set and hence the critical orbit of $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ is infinite. □

Remark. Note that Inou and Shishikura proved that the critical point cp_f can be iterated infinitely many times under $f \in \mathcal{IS}_\alpha$ (see [IS08, Corollary 4.2]). However, they did not give a proof that the critical orbit is infinite (i.e. not eventually periodic).

3.2. **The boundary of the Siegel disk is a Jordan curve.** Let $[0; a_1, a_2, \cdots]$ be the continued fraction expansion of $\alpha \in \mathrm{HT}_N$. For $n \geq 1$, let

$$\frac{p_n}{q_n} = [0; a_1, \cdots, a_n] := \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

be the convergents of α, where p_n and q_n are coprime to each other. For example, $q_1 = a_1$ and $q_2 = a_1 a_2 + 1$, $\cdots$. For convenience, we set $q_0 := 1$. Then one can check that $q_{n+1} = a_{n+1} q_n + q_{n-1}$ for $n \geq 1$.

Let $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle on which acts by the rigid rotation $R_\alpha(z) = e^{2\pi \mathrm{i} \alpha} z$. The order and the combinatorial structure of the forward orbit of a point in $\mathbb{S}^1$ is subtle but well understood. We denote $z_k := R_\alpha^{\circ k}(1)$ for $k \in \mathbb{N}$. In particular, $z_0 = 1$. In the following, we use $[z_i, z_j]$ to denote the closed shorter arc in $\mathbb{S}^1$ connecting z_i and z_j. The following lemma can be obtained by a straightforward analysis of continued fractions. One can see [Mil06, Appendix C] for reference.

Lemma 3.2. *For each $n \geq 1$, the set of points $\mathcal{O}_n := \{z_0, z_1, \cdots, z_{q_n}\}$ divides the unit circle into $q_n + 1$ subarcs (with disjoint interiors). Each subarc $I_k^n = [z_i, z_j]$ with $0 \leq k \leq q_n$ belongs to one of the following four cases:*

(1) $I_k^n = [z_0, z_{q_n}]$ *and* $(\operatorname{int} I_k^n) \cap \mathcal{O}_{n+1} = \emptyset$, *where* $\mathcal{O}_{n+1} = \{z_0,\ z_1,\ \cdots,\ z_{q_{n+1}}\}$;

(2) $I_k^n = [z_{q_{n-1}}, z_0]$ *and* $(\operatorname{int} I_k^n) \cap \mathcal{O}_{n+1} = \{z_{q_n+q_{n-1}}, z_{2q_n+q_{n-1}}, \cdots, z_{(a_{n+1}-1)q_n+q_{n-1}},\ z_{q_{n+1}}\}$;

(3) $I_k^n = [z_{q_{n-1}+b_n}, z_{b_n}]$ *and* $(\operatorname{int} I_k^n) \cap \mathcal{O}_{n+1} = \{z_{q_n+q_{n-1}+b_n}, z_{2q_n+q_{n-1}+b_n}, \cdots,\ z_{(a_{n+1}-1)q_n+q_{n-1}+b_n}\}$, *where* $1 \leq b_n \leq q_n - q_{n-1}$;

(4) $I_k^n = [z_{d_n}, z_{q_n-q_{n-1}+d_n}]$ *and* $(\operatorname{int} I_k^n) \cap \mathcal{O}_{n+1} = \{z_{q_n+d_n},\ z_{2q_n+d_n},\ \cdots,\ z_{a_{n+1}q_n+d_n}\}$, *where* $1 \leq d_n \leq q_{n-1} - 1$.

Definition. Let N be the integer fixed before. Define

$$\mathcal{S}_N := \{\alpha = [0; a_1, a_2, \cdots] \in \mathrm{HT}_N \mid (a_n) \text{ is bounded}\}. \tag{3.1}$$

In order to prove that the boundary of the Siegel disk of $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in \mathcal{S}_N$ is a Jordan curve containing the critical point, we need the following estimation on the post-critical set:

Proposition 3.3 ([Che10, Proposition 4.8])**.** *There are constants $K_0 > 0$ and $0 < \rho < 1$ such that for every $\alpha \in \mathrm{HT}_N$ and every $z \in \mathcal{PC}(f) \cup \{\mathrm{cp}_f\}$ with $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$, one has*

$$|f^{\circ q_n}(z) - z| \leq K_0 \cdot \rho^n.$$

The proof of this proposition is based on the uniform contraction with respect to the hyperbolic metric of some domains in each adjacent levels of the renormalization tower when one goes up the renormalization tower. For the proof and details, see [Che10, §4]. This proposition shows that the critical point and every point in the post-critical set of $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in \mathrm{HT}_N$ are recurrent. We will also need the following result due to Cheraghi:

Proposition 3.4 ([Che10, Propositions 2.4 and 4.9]). *For every $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in \mathrm{HT}_N$, if α is a Brjuno number, then there exists a sequence of domains $\cdots \Subset \Omega^{n+1} \Subset \Omega^n \Subset \cdots \Subset \Omega^1 \Subset \Omega^0 \Subset U_f$ such that*

(a) $\mathrm{int}(\cap_{n=0}^{\infty} \Omega^n) = \Delta_f$, *where Δ_f is the Siegel disk of f; and*
(b) $\partial \Delta_f$ *is contained in the post-critical set of f.*

Although we only need the results of bounded type, Propositions 3.3 and 3.4 apply also to unbounded type numbers. The proofs of them do not use a priori knowledge on quadratic polynomials.

Theorem 3.5. *Let $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in \mathcal{S}_N$. Then the Siegel disk of f is compactly contained in the domain of definition of f and its boundary is a Jordan curve passing through the unique critical point. Moreover, the critical point is recurrent and the post-critical set is exactly the boundary of the Siegel disk.*

Maybe this theorem was known to experts. However, we provide here a proof that does not rely on the quasiconformal surgery, but instead on the two propositions of Cheraghi stated above.

Proof. By Proposition 3.4(a), the Siegel disk of f is compactly contained in the domain of definition of f. We first prove that the post-critical set $\mathcal{PC}(f)$ is locally connected. The idea is to construct a sequence of continuous maps γ_n defined from the unit circle to a set which contains the first $q_n + 1$ points of the critical orbit of f. Then, by using Proposition 3.3, we prove that (γ_n) converges uniformly to a limit function whose image is exactly the post-critical set of f.

Let $c_0 := \mathrm{cp}_f$ be the unique critical point of f. We denote $c_k := f^{\circ k}(c_0)$ for $k \in \mathbb{N}$. In the following, we use $[c_i, c_j]$ (abuse of notations) to denote the closed segment in $\mathbb{C}$ that connects the points c_i and c_j. Recall that $\{z_k = R_\alpha^{\circ k}(1)\}_{k \in \mathbb{N}}$ is the orbit of 1 under $R_\alpha(z) = e^{2\pi \mathrm{i} \alpha} z$ on $\mathbb{S}^1$ and $[z_i, z_j] = I_k^n$ is a subarc that appeared in Lemma 3.2 with $0 \le k \le q_n$. Let $\gamma_n : \mathbb{S}^1 \to \mathbb{C}$ be the map such that
$$\gamma_n([z_i, z_j]) = [c_i, c_j],$$
where γ_n is linear[13] on $[z_i, z_j]$ such that $\gamma_n(z_i) = c_i$ and $\gamma_n(z_j) = c_j$. It is easy to see that γ_n is continuous.

Now we prove that $\gamma_n : \mathbb{S}^1 \to \mathbb{C}$ converges uniformly as $n \to \infty$. It is sufficient to prove that $\{\gamma_n\}$ forms a Cauchy sequence. Since $\alpha = [0; a_1, a_2, \cdots] \in \mathcal{S}_N$ is of bounded type, there exists a constant K' such that $a_n \le K'$ for all $n \ge 1$.

By Lemma 3.2, the finite orbit $\mathcal{O}_n$ divides the unit circle into four kinds of subarcs. Suppose that $I_k^n = [z_0, z_{q_n}]$ belongs to the first case. Then $\gamma_n(I_k^n)$ and $\gamma_{n+1}(I_k^n)$ are both the segment connecting c_0 and c_{q_n}. For any $z \in I_k^n$, we have $\gamma_n(z) = \gamma_{n+1}(z)$ and hence $|\gamma_n(z) - \gamma_{n+1}(z)| = 0$.

Suppose that $I_k^n = [z_{q_{n-1}}, z_0]$ belongs to the second case. Then $\gamma_n(I_k^n)$ is the segment connecting $c_{q_{n-1}}$ and c_0 while $\gamma_{n+1}(I_k^n)$ is a curve consisting of $a_{n+1} + 1$ segments. See Figure 5.

[13] Although $[z_i, z_j]$ is not a segment, but it is easy to convert it to a segment since it is a subarc of $\mathbb{S}^1$. We will see in the proof that the requirement "linear" is not important.

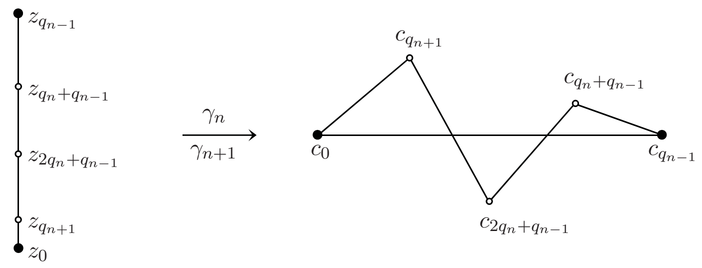

Figure 5: The sketch of the restrictions of the maps γ_n and γ_{n+1}. Both of them are defined on $[z_{q_{n-1}}, z_0]$. In this picture, we set $a_{n+1} = 3$.

For any $z \in I_k^n = [z_{q_{n-1}}, z_0]$, by Proposition 3.3 and the definition of γ_n and γ_{n+1}, we have

$$\begin{aligned}
&|\gamma_n(z) - \gamma_{n+1}(z)| \\
\le\ & |c_0 - c_{q_{n-1}}| + \sum_{\ell=0}^{a_{n+1}-1} |c_{\ell q_n+q_{n-1}} - c_{(\ell+1)q_n+q_{n-1}}| + |c_0 - c_{q_{n+1}}| \\
\le\ & K_0 \cdot \rho^{n-1} + K' \cdot K_0 \cdot \rho^n + K_0 \cdot \rho^{n+1} = K_0(\tfrac{1}{\rho} + K' + \rho) \cdot \rho^n.
\end{aligned}$$

Suppose that $I_k^n = [z_{q_{n-1}+b_n}, z_{b_n}]$ belongs to the third case, where $1 \le b_n \le q_n - q_{n-1}$. Then we will have the same estimation as the second case since

$$\begin{aligned}
&|c_{b_n} - c_{(a_{n+1}-1)q_n+q_{n-1}+b_n}| \\
\le\ & |c_{b_n} - c_{q_{n+1}+b_n}| + |c_{q_{n+1}+b_n} - c_{(a_{n+1}-1)q_n+q_{n-1}+b_n}| \\
\le\ & K_0 \cdot \rho^{n+1} + K_0 \cdot \rho^n.
\end{aligned}$$

Suppose that $I_k^n = [z_{d_n}, z_{q_n-q_{n-1}+d_n}]$ belongs to the fourth case, where $1 \le d_n \le q_{n-1} - 1$. By Proposition 3.3, we have

$$\begin{aligned}
|c_{d_n} - c_{q_n-q_{n-1}+d_n}| &\le |c_{d_n} - c_{q_n+d_n}| + |c_{q_n+d_n} - c_{q_n-q_{n-1}+d_n}| \\
&\le K_0(1 + \tfrac{1}{\rho}) \cdot \rho^n
\end{aligned}$$

and

$$\begin{aligned}
&|c_{a_{n+1}q_n+d_n} - c_{q_n-q_{n-1}+d_n}| \\
\le\ & |c_{q_{n+1}-q_{n-1}+d_n} - c_{q_{n+1}+q_n-q_{n-1}+d_n}| + |c_{q_{n+1}+q_n-q_{n-1}+d_n} - c_{q_n-q_{n-1}+d_n}| \\
\le\ & K_0 \cdot \rho^n + K_0 \cdot \rho^{n+1} = K_0(1+\rho) \cdot \rho^n.
\end{aligned}$$

Therefore, for any $z \in I_k^n = [z_{d_n}, z_{q_n-q_{n-1}+d_n}]$, by Proposition 3.3, we have

$$\begin{aligned}
&|\gamma_n(z) - \gamma_{n+1}(z)| \\
\le\ & |c_{d_n} - c_{q_n-q_{n-1}+d_n}| + \sum_{\ell=0}^{a_{n+1}-1} |c_{\ell q_n+d_n} - c_{(\ell+1)q_n+d_n}| \\
&+ |c_{a_{n+1}q_n+d_n} - c_{q_n-q_{n-1}+d_n}| \\
\le\ & K_0(1 + \tfrac{1}{\rho}) \cdot \rho^n + K' \cdot K_0 \cdot \rho^n + K_0(1+\rho) \cdot \rho^n \\
=\ & K_0(K' + 2 + \rho + \tfrac{1}{\rho}) \cdot \rho^n.
\end{aligned}$$

In summary, we have

$$|\gamma_n(z)-\gamma_{n+1}(z)|\leq K_0(K'+2+\rho+\tfrac{1}{\rho})\cdot\rho^n \text{ for all } z\in\mathbb{S}^1. \tag{3.2}$$

Thus $\gamma_n:\mathbb{S}^1\to\mathbb{C}$ converges uniformly to a continuous map, which we denote by $\gamma:\mathbb{S}^1\to\mathbb{C}$.

Note that $\gamma_n\circ R_\alpha=f\circ\gamma_n$ holds on $\{z_0,z_1,\cdots,z_{q_n-1}\}$. Letting $n\to\infty$, it follows that

$$\gamma\circ R_\alpha=f\circ\gamma \tag{3.3}$$

holds on the forward orbit of z_0 under R_α. Note that the forward orbit of z_0 under R_α is dense in $\mathbb{S}^1$. By the continuity, this means that (3.3) holds on $\mathbb{S}^1$, i.e. the map γ semi-conjugates the dynamics of R_α on $\mathbb{S}^1$ to the dynamics of f on $\gamma(\mathbb{S}^1)$. Since the forward orbit of z_0 under R_α is dense in $\mathbb{S}^1$, then the forward orbit of c_0 under f is dense in $\gamma(\mathbb{S}^1)$. Therefore, $\mathcal{PC}(f)=\gamma(\mathbb{S}^1)=\mathcal{PC}(f)\cup\{c_0\}$ and the post-critical set of f is locally connected.

Let $z\in\partial\Delta_f$ be a point on the boundary of the Siegel disk of f. We denote the forward orbit of z by $\mathcal{O}(z):=\{f^{\circ k}(z):k\in\mathbb{N}\}$. Since $f(\partial\Delta_f)\subset\partial\Delta_f$, it follows that $\mathcal{O}(z)\subset\partial\Delta_f$. By Proposition 3.4(b), we have $\partial\Delta_f\subset\mathcal{PC}(f)$. Note that $\mathcal{O}(z)$ is dense in $\mathcal{PC}(f)$, we have $\overline{\mathcal{O}(z)}=\partial\Delta_f=\mathcal{PC}(f)$, i.e. the post-critical set is exactly the boundary of the Siegel disk. Therefore, $\partial\Delta_f$ is locally connected[14].

Since $\overline{\Delta}_f$ is compactly contained in the domain of definition of f, by the definition of Δ_f, there exists a conformal map $\phi:\mathbb{D}\to\Delta_f$ such that $f\circ\phi(z)=\phi(e^{2\pi\mathrm{i}\alpha}z)$. According to Carathéodory, the map ϕ can be extended continuously to $\phi:\overline{\mathbb{D}}\to\overline{\Delta}_f$ since $\partial\Delta_f$ is locally connected. For each $\theta\in[0,2\pi)$, let $r_\theta:=\{\phi(te^{\mathrm{i}\theta}):0\leq t\leq 1\}$ be the internal ray of Δ_f. Suppose that there are two different rays r_{θ_1} and r_{θ_2} landing at a common point on $\partial\Delta_f$, i.e. $\phi(e^{\mathrm{i}\theta_1})=\phi(e^{\mathrm{i}\theta_2})$. Then $r_{\theta_1}\cup r_{\theta_2}$ is a Jordan curve contained in $\overline{\Delta}_f$. By the Maximum Modulus Principle, $\{f^{\circ n}\}_{n\in\mathbb{N}}$ forms a normal family in the bounded domain $\Omega_{\theta_1,\theta_2}$ that is bounded by $r_{\theta_1}\cup r_{\theta_2}$. This means that $\Omega_{\theta_1,\theta_2}$ is contained in the Fatou set (actually contained in Δ_f). However, by Riesz Brothers' Theorem [Mil06, Theorem A.3, p. 220], ϕ must be a constant. This is a contradiction. Therefore, each point in $\partial\Delta_f$ is the landing point of exactly one internal ray. It follows that $\partial\Delta_f$ is a Jordan curve passing through the unique critical point of f. This ends the proof of Theorem 3.5 and Theorem 1.1 modulo the statement of the selft-similarity. □

4. Similarity of the renormalization periodic points

In the present and next sections, we will study the self-similarity of the boundary of the Siegel disk of $f\in\mathcal{IS}_\alpha\cup\{Q_\alpha\}$, where $\alpha\in(0,\varepsilon_0]$ is a quadratic irrational. According to Theorem 3.5, the unique critical point of f is contained in the boundary of the Siegel disk of f and the post-critical set is equal to the boundary of the Siegel disk of f. We will use the critical orbit to study the properties of the boundary of the Siegel disk of f.

[14] According to [Mil06, Lemma 18.7], the boundary of the Siegel disk of a rational map is a Jordan curve provided it is locally connected.

4.1. **The choice of suitable neighborhoods.** In order to prove the self-similarity of the boundary of the Siegel disk of $f \in \cup_{\alpha\in(0,\varepsilon_0]}\mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with α a quadratic irrational number, we first need to study the properties of the maps in $\mathcal{IS}_0$. In [IS08], besides the class $\mathcal{IS}_0$, Inou and Shishikura also introduced the following class[15] which corresponds to $\mathcal{IS}_0$:

$$\mathcal{IS}_0^Q = \left\{ F = Q\circ\varphi^{-1} \,\middle|\, \begin{array}{l} \varphi : \widehat{\mathbb{C}} \smallsetminus E \to \widehat{\mathbb{C}} \smallsetminus \{0\} \text{ is univalent,} \\ \varphi(\infty)=\infty,\ \varphi'(\infty)=1 \text{ and } \varphi \text{ has} \\ \text{a quasiconformal extension to } \widehat{\mathbb{C}} \end{array} \right\},$$

where E is the ellipse defined in (2.1) and $Q(z) = z(1+\frac{1}{z})^6/(1-\frac{1}{z})^4$ is a rational map. Each map in this class has a parabolic fixed point at ∞, a critical point at $\mathrm{cp}_F := \varphi(5+2\sqrt{6})$ and a critical value at $\mathrm{cv}_Q = 27$ which is independent on F.

By [IS08, Lemma 5.14(a)], P and Q are related by

$$Q = \mathrm{Inv}^{-1}\circ P\circ\psi_1, \text{ where } \mathrm{Inv}(z) = -4/z$$

and $\psi_1(z) = -4z/(1+z)^2$ is defined in (2.2). By [IS08, Proposition 5.3(c)], there exists a one to one correspondence between $\mathcal{IS}_0$ and $\mathcal{IS}_0^Q$, which is related by

$$\begin{aligned}\mathcal{IS}_0 \ni f &= P\circ\varphi^{-1} \mapsto F = \mathrm{Inv}^{-1}\circ f\circ \mathrm{Inv} \\ &= \mathrm{Inv}^{-1}\circ P\circ\psi_1\circ\psi_1^{-1}\circ\varphi^{-1}\circ\mathrm{Inv} = Q\circ\hat\varphi^{-1}\in\mathcal{IS}_0^Q,\end{aligned}$$

with the correspondence $\varphi\mapsto\hat\varphi = \mathrm{Inv}^{-1}\circ\varphi\circ\psi_1$. For $F\in\mathcal{IS}_0^Q$, one has natural definitions of the attracting petal $\mathcal{P}_{attr,F}$, repelling petal $\mathcal{P}_{rep,F}$, attracting Fatou coordinate $\Phi_{attr,F}$ and repelling Fatou coordinate $\Phi_{rep,F}$ etc based on the definitions for $f\in\mathcal{IS}_0$ in §2.1. For example, the attracting Fatou coordinate of F is defined as $\Phi_{attr,F}(z) = \Phi_{attr,f}\circ\mathrm{Inv}(z)$.

Let $f\in\mathcal{IS}_0$. Recall that $\mathcal{C}_f$ and $\mathcal{C}_f^\sharp$ are two sets defined in (2.5). Define

$$\Omega := \mathbb{E}\mathrm{xp}\circ\Phi_f(\mathrm{int}\,\mathcal{C}_f) = \mathbb{D}(0,\tfrac{4}{27}e^{4\pi}) \smallsetminus (\overline{\mathbb{D}}(0,\tfrac{4}{27}e^{-4\pi})\cup[0,+\infty)). \tag{4.1}$$

Proposition 4.1. *For all $f\in\mathcal{IS}_0$, the closure $\overline{\mathcal{C}}_f$ is compactly contained in Ω.*

Proof. In order to prove $\overline{\mathcal{C}}_f \Subset \Omega$, it is convenient to work in the corresponding dynamical plane of $F = \mathrm{Inv}^{-1}\circ f\circ\mathrm{Inv}\in\mathcal{IS}_0^Q$. Define

$$D_0 := \left\{ z\in\mathcal{P}_{attr,F} \,\middle|\, \begin{array}{l} 0 < \mathrm{Re}\,\Phi_{attr,F}(z) < 1 \text{ and} \\ -2 < \mathrm{Im}\,\Phi_{attr,F}(z) < 2 \end{array}\right\}, \text{ and}$$

$$D_0^\sharp := \left\{ z\in\mathcal{P}_{attr,F} \,\middle|\, \begin{array}{l} 0 < \mathrm{Re}\,\Phi_{attr,F}(z) < 1 \text{ and} \\ 2 < \mathrm{Im}\,\Phi_{attr,F}(z) \end{array}\right\}.$$

Let $D_1 := F(D_0)$ and $D_1^\sharp := F(D_0^\sharp)$. By the definition of $\mathcal{C}_f$, we have

$$\mathcal{C}_F := \mathrm{Inv}^{-1}(\mathcal{C}_f) = \left\{ z\in\mathcal{P}_{attr,F} \,\middle|\, \begin{array}{l} 1/2 \le \mathrm{Re}\,\Phi_{attr,F}(z) \le 3/2 \\ \text{and } -2 < \mathrm{Im}\,\Phi_{attr,F}(z) \le 2 \end{array}\right\}.$$

Therefore, $\mathcal{C}_F \subset \overline{D}_0\cup\overline{D}_1$. See Figure 6.

[15] This class is denoted by $\mathcal{F}_1^Q$ in [IS08, p. 17].

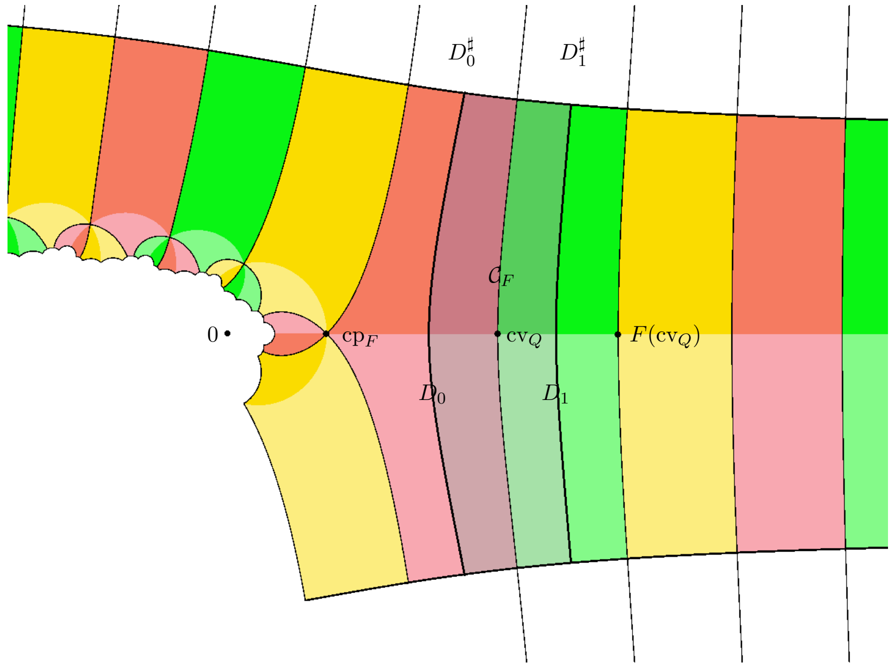

Figure 6: The chessboard of the Fatou coordinate of $F = Q$ ($\varphi = \mathrm{id}$). The set $\mathcal{C}_F = \mathrm{Inv}^{-1}(\mathcal{C}_f)$ is contained in $\overline{D}_0 \cup \overline{D}_1$.

In order to prove that $\mathcal{C}_f$ is compactly contained in Ω, it is sufficient to prove that

$$\overline{D}_0 \cup \overline{D}_1 \subset \mathrm{Inv}^{-1}(\Omega) = \mathbb{D}(0, 27e^{4\pi}) \setminus (\overline{\mathbb{D}}(0, 27e^{-4\pi}) \cup (-\infty, 0]). \tag{4.2}$$

For $z_0 \in \mathbb{C}$ and $0 < \theta < \pi$, we denote the sector

$$\mathbb{V}(z_0, \theta) := \{z \in \mathbb{C} : z \neq z_0, |\arg(z - z_0)| < \theta\}.$$

By [IS08, Proposition 5.6(b)], we have

$$D_1 \subset \mathbb{V}(u_0, \tfrac{2\pi}{3}) \cap \mathbb{D}(\mathrm{cv}_Q, R_1),$$

where $u_0 = \frac{25}{\sqrt{3}}$, $\mathrm{cv}_Q = 27$ and $R_1 = 239$. Since

$$u_0 \sin \tfrac{\pi}{3} = \frac{25}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} > 1 > 0.05 > \frac{2^5}{2^{12}} > \frac{27}{e^{4\pi}},$$

hence $\overline{D_1}$ is disjoint with $\overline{\mathbb{D}}(0, 27e^{-4\pi}) \cup (-\infty, 0]$. On the other hand, for each $z \in \overline{D_1}$, we have

$$|z| \le \mathrm{cv}_Q + R_1 = 266 < 2^4 \cdot 2^{12} < 27e^{4\pi}.$$

This means that (4.2) holds for $\overline{D}_1$.

By [IS08, Proposition 5.7(e)], we have

$$\overline{D}_0 \subset \mathbb{D}(0, R) \setminus (\overline{\mathbb{D}}(0, \rho) \cup (-\infty, 0]),$$

where $R = 266$ and $\rho = 0.05$. Since $27e^{-4\pi} < 0.05$ and $266 < 27e^{4\pi}$, this means that (4.2) holds for $\overline{D}_0$. The proof is finished. □

Let $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_0]$, where $\varepsilon_0 > 0$ is the constant introduced in Theorem 2.6. Recall that $\mathcal{C}_f$ and $\mathcal{C}_f^\sharp$ are defined in (2.9) and Ω is defined in (4.1). By Proposition 4.1 and the continuity stated in Proposition 2.4(c) (see also [Shi00, Proposition 3.2.2(iv)]), we have the following immediate corollary.

Corollary 4.2. *There exists a large integer N_1 such that if $N \geq N_1$, then $\overline{\mathcal{C}}_f$ is compactly contained in Ω for any $f \in \mathcal{IS}_\alpha$ with $\alpha \in \mathrm{HT}_N$.*

4.2. **Similarity of the boundaries of the Siegel disks of renormalization periodic points.** In this section, let[16]

$$\alpha = [0; a_1, a_2, \cdots] = [0; a_1, \cdots, a_s, a_1, \cdots, a_s, \cdots] \in \mathrm{HT}_N \tag{4.3}$$

be a quadratic irrational number such that $N \geq N_1$ and $a_{n+s} = a_n$ for $n \geq 1$, where $s \geq 1$. Recall that $f_\alpha^* \in \mathcal{IS}_\alpha$ is the renormalization periodic point introduced in Theorem 2.7(b). For simplicity, we denote $f_* := f_\alpha^*$, and use $\Phi_* := \Phi_{f_*}$ to denote the perturbed Fatou coordinate defined in Proposition 2.4 and let $k_* := k_{f_*}$ be the positive integer defined in Proposition 2.5. Moreover, we also use '$*$' to replace 'f_*' in the subscript of the notations which we have defined before.

Theorem 4.3. *Let $\alpha \in \mathrm{HT}_N$ be a quadratic irrational defined as in* (4.3)*. Then the boundary of the Siegel disk of $f_* \in \mathcal{IS}_\alpha$ is self-similar about its critical point* cp_*.

Proof. Without loss of generality, we suppose that $s = 1$, i.e. $\mathcal{R}f_* = f_*$. We will see that the arguments for $s \geq 2$ are completely similar.

By the definition of near-parabolic renormalization, the map $\mathcal{R}f_*$ is induced by the map $f_*^{\circ k_*} : \mathcal{C}_*^{-k_*} \cup (\mathcal{C}_*^\sharp)^{-k_*} \to \mathcal{C}_* \cup \mathcal{C}_*^\sharp$ under the map $\mathbb{E}\mathrm{xp} \circ \Phi_*$, i.e. we have[17]

$$\mathcal{R}f_* : \mathbb{E}\mathrm{xp} \circ \Phi_*(\mathcal{C}_*^{-k_*} \cup (\mathcal{C}_*^\sharp)^{-k_*}) \to \mathbb{E}\mathrm{xp} \circ \Phi_*(\mathcal{C}_* \cup \mathcal{C}_*^\sharp).$$

Consider the interior of $\mathcal{C}_*$ and note that $\Omega = \mathbb{E}\mathrm{xp} \circ \Phi_*(\mathrm{int}\,\mathcal{C}_*)$ is defined in (4.1). We have the restriction

$$\mathcal{R}f_* : \mathbb{E}\mathrm{xp} \circ \Phi_*(\mathrm{int}\,\mathcal{C}_*^{-k_*}) \to \Omega. \tag{4.4}$$

By Corollary 4.2, we have $\overline{\mathcal{C}}_{\mathcal{R}f_*} = \overline{\mathcal{C}}_* \Subset \Omega$. By taking the preimage of $\overline{\mathcal{C}}_{\mathcal{R}f_*}$ under $\mathcal{R}f_*$ and noticing that the map in (4.4) is proper, we have

$$\overline{\mathcal{C}}_{\mathcal{R}f_*}^{-1} = (\mathcal{R}f_*)^{-1}(\overline{\mathcal{C}}_{\mathcal{R}f_*}) \Subset \mathbb{E}\mathrm{xp} \circ \Phi_*(\mathrm{int}\,\mathcal{C}_*^{-k_*}).$$

Therefore, $\mathbb{E}\mathrm{xp}$ has an inverse branch $\mathbb{L}\mathrm{og}$ defined on $\overline{\mathcal{C}}_{\mathcal{R}f_*}^{-1}$ (since each $\overline{\mathcal{C}}_f^{-1}$ is simply connected and avoids the origin for every $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$, see Proposition 2.5(a) and Figure 4) such that

$$\mathbb{L}\mathrm{og}(\overline{\mathcal{C}}_{\mathcal{R}f_*}^{-1}) \Subset \Phi_*(\mathrm{int}\,\mathcal{C}_*^{-k_*}). \tag{4.5}$$

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[16]We only deal with the renormalization periodic point and this is sufficient to prove the self-similarity of the quadratic irrational of high type.

[17]Note that $\mathcal{R}f_*$ defined in this way is not contained in the Inou-Shishikura class. However, its restriction on some domain is. Sometimes we omit this difference if there is no confusion. See Theorem 2.6(a).

Hence we can define an anti-holomorphic map[18]

$$\Upsilon(z) := f_*^{\circ(k_*-1)} \circ \Phi_*^{-1} \circ \mathbb{L}\mathrm{og}(z) : \overline{\mathcal{C}}_{\mathcal{R}f_*}^{-1} = \overline{\mathcal{C}}_*^{-1} \to \mathbb{C}. \tag{4.6}$$

Note that the critical point cp_* of f_* is contained in the interior of $\mathcal{C}_*^{-1}$. We claim that the following assertions hold:

(i) $\Upsilon(\overline{\mathcal{C}}_*^{-1}) \Subset \mathrm{int}\,\mathcal{C}_*^{-1}$;

(ii) $\Upsilon(\mathrm{cp}_*) = \mathrm{cp}_*$; and

(iii) $\Upsilon(\Gamma_*) \subset \Gamma_*$, where Γ_* is the component of $\partial\Delta_* \cap \mathcal{C}_*^{-1}$ containing cp_*, where Δ_* is the Siegel disk of f_*. By Theorem 3.5, it follows that Γ_* is a Jordan arc.

Indeed, (i) is the direct corollary of (4.5). For (ii), by the definition of near-parabolic renormalization, the critical point $\mathrm{cp}_{\mathcal{R}f_*}$ is equal to $\mathbb{E}\mathrm{xp} \circ \Phi_*(\widetilde{z})$ and $\widetilde{z} \in \mathrm{int}\,\mathcal{C}_*^{-k_*}$ is the preimage of cp_*. Then (ii) holds since $\mathrm{cp}_{\mathcal{R}f_*} = \mathrm{cp}_*$. For (iii), note that Υ maps the critical orbit of $\mathcal{R}f_*$ to a subset of that of f_* by the first statement of Lemma 2.8(a) and (b). On the other hand, $\partial\Delta_*$ is equal to the closure of the critical orbit of f_* by Theorem 3.5. Hence we have $\Upsilon(\partial\Delta_* \cap \mathcal{C}_*^{-1}) = \Upsilon(\partial\Delta_{\mathcal{R}f_*} \cap \mathcal{C}_{\mathcal{R}f_*}^{-1}) \subset \partial\Delta_*$. In particular, we have $\Upsilon(\Gamma_*) \subset \Gamma_*$.

Since Υ is antiholomorphic and univalent, $\Upsilon(\overline{\mathcal{C}}_*^{-1}) \Subset \mathrm{int}\,\mathcal{C}_*^{-1}$ and cp_* is the fixed point of Υ, we have $0 < |\frac{\partial\Upsilon}{\partial\overline{z}}(\mathrm{cp}_*)| < 1$ by Schwarz's Lemma. If we notice the statements (ii) and (iii) stated above, this means that the boundary of Siegel disk of f_* is self-similar[19] about the critical point cp_*. □

In the proof of Theorem 4.3, we assumed that $\alpha = [0;\, a_1,\, \cdots,\, a_s,\, a_1,\, \cdots,\, a_s,\, \cdots]$ with $s = 1$ and obtained a contracting function Υ which is *anti-holomorphic*. On the other hand, if $s \geq 2$ is an even number, one can construct a contracting function Υ which is *holomorphic*. This is because $\mathbb{E}\mathrm{xp}$ is orientation-reversing and Υ is the composition of s anti-holomorphic maps.

In [McM98], in order to prove the self-similarity of the boundaries of the Siegel disks of quadratic polynomials, McMullen also constructed a corresponding contracting function which is $C^{1+\varepsilon}$ for some $\varepsilon > 0$ (see [McM98, Theorem 7.1]). This means that the renormalization periodic points of $\mathcal{R}$ have some special properties such that one can make the contracting function either holomorphic or anti-holomorphic, so the boundary of the the Siegel disk of f_* has a "stronger" form of self-similarity.

The following proposition indicates that the self-similarity of the boundaries of the Siegel disks is also dynamical:

Proposition 4.4. *The contracting function Υ maps the q_n-th iterate of the critical point to the q_{n+s}-th. In particular, we have*

$$\lambda = \lim_{n\to\infty} \left| \frac{f_*^{\circ q_{n+s}}(\mathrm{cp}_*) - \mathrm{cp}_*}{f_*^{\circ q_n}(\mathrm{cp}_*) - \mathrm{cp}_*} \right|, \tag{4.7}$$

[18]Both f_* and Φ_* are holomorphic while $\mathbb{L}\mathrm{og}$ is anti-holomorphic since $\mathbb{E}\mathrm{xp}$ is. Actually, it is easy to check that Υ is anti-univalent.

[19]A rough definition of the self-similarity can be characterized like this: if we zoom at the critical point cp_* by some "power microscope", one can see that a sequence of the subsets of the boundaries of the Siegel disks (after blowing up) tends to a limit in the Hausdorff topology. See Proposition 5.11 for a more precise statement of the self-similarity.

where $\lambda = |\frac{\partial \Upsilon}{\partial \bar{z}}(\mathrm{cp}_*)|$ *if s is odd and* $\lambda = |\frac{\partial \Upsilon}{\partial z}(\mathrm{cp}_*)|$ *if s is even. Moreover, the convergence is exponentially fast.*

Proof. We continue using the notations in the proof of Theorem 4.3 and also suppose that $s = 1$. Since $\mathcal{R}f_* = f_*$, we use $\widetilde{\mathcal{P}}_*$ to denote the simply connected set defined in (2.10), χ_* the inverse of $\mathbb{E}\mathrm{xp}$ define in (2.16) and $\psi_* = \Phi_*^{-1} \circ \chi_*$ defined in (2.18) for f_* (i.e. the subscript is independent on n). Note that ψ_* is defined on $\widetilde{\mathcal{P}}_*$, $\mathbb{L}\mathrm{og}$ is defined on $\mathcal{C}_*^{-1} \subset \widetilde{\mathcal{P}}_*$, and both ψ_*, $\mathbb{L}\mathrm{og}$ are the inverse branches of $\mathbb{E}\mathrm{xp}$. This means thatχ_* and $\mathbb{L}\mathrm{og}$ only differ by an integer in $\mathcal{C}_*^{-1}$. Thus one can extend $\mathbb{L}\mathrm{og}$ to $\widetilde{\mathcal{P}}_*$ by analytic continuation such that

$$\mathbb{L}\mathrm{og} = \chi_* + m \tag{4.8}$$

for some $m \in \mathbb{Z}$ on $\widetilde{\mathcal{P}}_*$. According to Proposition 2.5(b) and the assumption on N in (2.15), it follows that m is a positive integer. By (4.8) and $\psi_* = \Phi_*^{-1} \circ \chi_*$, we have

$$f_*^{\circ m} \circ \psi_*(z) = \Phi_*^{-1} \circ \mathbb{L}\mathrm{og}(z), \tag{4.9}$$

where $z \in \mathcal{C}_*^{-1}$ (since ψ_* is defined in $\widetilde{\mathcal{P}}_*$ and $\Phi_*^{-1} \circ \mathbb{L}\mathrm{og}$ can be defined in $\mathcal{C}_*^{-1} \subset \widetilde{\mathcal{P}}_*$ by (4.6)).

Recall that $\mathcal{P}_*'$ is the set defined in (2.20) for f_*. By Lemma 2.8(a), the map $\psi_*^{\circ n}$ maps the pair $(z, f_*(z)) = (z, (\mathcal{R}^{\circ n} f_*)(z))$ with $z \in \mathcal{P}_*'$ to $(\psi_*^{\circ n}(z), f_*^{\circ q_n}(\psi_*^{\circ n}(z)))$ and $\psi_*^{\circ(n+1)}$ maps $(z, f_*(z)) = (z, (\mathcal{R}^{\circ(n+1)} f_*)(z))$ to the pair $(\psi_*^{\circ(n+1)}(z),\ f_*^{\circ q_{n+1}}(\psi_*^{\circ(n+1)}(z)))$. Therefore, ψ_* maps the pair $(\psi_*^{\circ n}(z),\ f_*^{\circ q_n}(\psi_*^{\circ n}(z)))$ to the pair $(\psi_*^{\circ(n+1)}(z),\ f_*^{\circ q_{n+1}}(\psi_*^{\circ(n+1)}(z)))$. This means that

$$\psi_* \circ f_*^{\circ q_n}(z) = f_*^{\circ q_{n+1}} \circ \psi_*(z) \tag{4.10}$$

holds for $z \in \psi_*^{\circ n}(\mathcal{P}_*') \subset \mathcal{P}_*$. Note that cv is contained in $\widetilde{\mathcal{P}}_*$ and $\mathbb{E}\mathrm{xp} \circ \Phi_*(\mathrm{cv}) = \mathrm{cv}$. By (2.17), there exists a positive integer $k \leq \boldsymbol{k}' + 1$ such that $f_*^{\circ k}(\mathrm{cp}_*)$, $f_*^{\circ(q_n+k)}(\mathrm{cp}_*) \in \psi_*^{\circ n}(\mathcal{P}_*')$. Therefore, by (4.10), we have

$$\psi_* \circ f_*^{\circ(q_n+k)}(\mathrm{cp}_*) = f_*^{\circ q_{n+1}} \circ \psi_* \circ f_*^{\circ k}(\mathrm{cp}_*). \tag{4.11}$$

By Proposition 3.3, there exists an integer $n' \geq 1$ such that $f_*^{\circ q_n}(\mathrm{cp}_*) \in \mathcal{C}_*^{-1}$ for $n \geq n'$. Suppose that $n \geq n'$. Note that cp_*, $f_*^{\circ q_n}(\mathrm{cp}_*)$, $f_*^{\circ k}(\mathrm{cp}_*)$ and $f_*^{\circ(q_n+k)}(\mathrm{cp}_*)$ are all contained in $\mathcal{P}_*'$. By Lemma 2.8(a), there exists a positive integer l such that

$$\begin{aligned} \psi_* \circ f_*^{\circ k}(\mathrm{cp}_*) &= f_*^{\circ l} \circ \psi_*(\mathrm{cp}_*) \text{ and} \\ \psi_* \circ f_*^{\circ(q_n+k)}(\mathrm{cp}_*) &= f_*^{\circ l} \circ \psi_* \circ f_*^{\circ q_n}(\mathrm{cp}_*). \end{aligned} \tag{4.12}$$

Comparing (4.11) and (4.12), we have

$$f_*^{\circ l} \circ \psi_* \circ f_*^{\circ q_n}(\mathrm{cp}_*) = f_*^{\circ l} \circ f_*^{\circ q_{n+1}} \circ \psi_*(\mathrm{cp}_*). \tag{4.13}$$

Note that $\psi_* \circ f_*^{\circ q_n}(\mathrm{cp}_*)$ and $f_*^{\circ q_{n+1}} \circ \psi_*(\mathrm{cp}_*)$ are both contained in the forward orbit of the critical point cp_* under f_*. By Lemma 3.1, it follows from (4.13) that

$$\psi_* \circ f_*^{\circ q_n}(\mathrm{cp}_*) = f_*^{\circ q_{n+1}} \circ \psi_*(\mathrm{cp}_*). \tag{4.14}$$

By (4.9) and (4.14), we have

$$\begin{aligned}\Phi_*^{-1}\circ\mathbb{L}\mathrm{og}(f_*^{\circ q_n}(\mathrm{cp}_*)) &= f_*^{\circ m}\circ\psi_*\circ f_*^{\circ q_n}(\mathrm{cp}_*) = f_*^{\circ m}\circ f_*^{\circ q_{n+1}}\circ\psi_*(\mathrm{cp}_*)\\ &= f_*^{\circ q_{n+1}}\circ f_*^{\circ m}\circ\psi_*(\mathrm{cp}_*) = f_*^{\circ q_{n+1}}\circ\Phi_*^{-1}\circ\mathbb{L}\mathrm{og}(\mathrm{cp}_*).\end{aligned}$$

By (4.6), it follows that

$$\begin{aligned}&\Upsilon(f_*^{\circ q_n}(\mathrm{cp}_*)) = f_*^{\circ(k_*-1)}\circ\Phi_*^{-1}\circ\mathbb{L}\mathrm{og}(f_*^{\circ q_n}(\mathrm{cp}_*))\\ =\ & f_*^{\circ(k_*-1)}\circ f_*^{\circ q_{n+1}}\circ\Phi_*^{-1}\circ\mathbb{L}\mathrm{og}(\mathrm{cp}_*) = f_*^{\circ q_{n+1}}\circ f_*^{\circ(k_*-1)}\circ\Phi_*^{-1}\circ\mathbb{L}\mathrm{og}(\mathrm{cp}_*)\\ =\ & f_*^{\circ q_{n+1}}\circ\Upsilon(\mathrm{cp}_*) = f_*^{\circ q_{n+1}}(\mathrm{cp}_*).\end{aligned}$$

This means that the contracting function Υ maps the q_n-th iterate of cp_* to the q_{n+1}-th. The map Υ is an anti-holomorphic map and Υ has $\overline{z}$-partial derivative at cp_*. If $s\geq 2$ is even, Υ is holomorphic and differentiable at cp_*. By Proposition 3.3, $f_*^{\circ q_n}(\mathrm{cp}_*)$ converges to cp_* exponentially fast. This means that limit (4.7) holds and the convergence is exponentially fast. □

We will use Proposition 4.4 to study the dynamical similarity of the boundaries of the Siegel disks of the quadratic polynomials and the maps in the Inou-Shishikura class in next section.

5. Transferring the similarity to the maps in the IS class

We have proved the self-similarity of the boundary of the Siegel disk at the critical point of the renormalization periodic point f_* in §4. In this section, we will transfer the corresponding results to the quadratic polynomial and all the maps in the Inou-Shishikura class.

5.1. A distortion lemma of the planar quasiconformal mappings.

Recall that each map in the Inou-Shishikura's class is written as $P\circ\varphi^{-1}$, where P is a cubic parabolic polynomial and φ is a univalent map defined near the origin which can be extended to a planar quasiconformal mapping. In order to estimate some kind of the convergence of the renormalized sequence (see Lemma 5.8), we first prepare a distortion lemma of the planar quasiconformal mappings.

Lemma 5.1. *Suppose that $\varphi:\mathbb{C}\to\mathbb{C}$ is a quasiconformal mapping which is conformal in a neighborhood W of the origin with $\varphi(0)=0$ and $\varphi'(0)=1$. Let μ be the complex dilatation of φ satisfying $\|\mu\|_\infty\leq k_0<1$. Then for any compact subset K of $\mathbb{C}$, there exists a constant $C>0$ depending only on k_0, K and W such that*

$$|\varphi(z)-z|\leq C\cdot\|\mu\|_\infty,\ \text{where } z\in K.$$

The idea of the proof was suggested by Cui Guizhen. Parts of the proof are inspired by [Leh87, Theorem 3.2, p. 72].

Proof. Without loss of generality, we assume that W contains the unit disk $\mathbb{D}$. Otherwise, one can consider a new map $z\mapsto\varphi(az)/a$, where $a\neq 0$ is chosen such that $\{z\in\mathbb{C}:|z|\leq|a|\}\subset W$. For $z\in\mathbb{D}$, the *pre-Schwarzian derivative* of φ at z is defined as

$$pS_\varphi(z):=\varphi''(z)/\varphi'(z).$$

Let $\eta(z)|\mathrm{d}z| = (1-|z|^2)^{-1}|\mathrm{d}z|$ be the Poincaré metric[20] in $\mathbb{D}$. The *norm* of the pre-Schwarzian derivative in $\mathbb{D}$ is defined as

$$\|pS_\varphi\| = \sup_{z\in\mathbb{D}} |pS_\varphi(z)| \cdot \eta(z)^{-1}.$$

Since $\varphi(0)=0$ and $\varphi'(0)=1$, by [Pom75, Lemma 1.3, p. 21], we have

$$\left| z\frac{\varphi''(z)}{\varphi'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \leq \frac{4|z|}{1-|z|^2}, \text{ where } z\in\mathbb{D}.$$

This means that $\|pS_\varphi\| \leq 6$.

If $\|\mu\|_\infty = 0$, then $\varphi:\mathbb{C}\to\mathbb{C}$ is a conformal mapping satisfying $\varphi(0)=0$ and $\varphi'(0)=1$. This means that φ must be the identity and the lemma holds. Thus we can assume that $\|\mu\|_\infty > 0$. Let $t\in\mathbb{C}$ such that $|t| < 1/\|\mu\|_\infty$. For each fixed t, let $\varphi_t:\mathbb{C}\to\mathbb{C}$ be the quasiconformal mapping with complex dilation $t\mu$ which satisfies $\varphi_t(0)=0$ and $\varphi_t'(0)=1$. Then for any fixed $z\in\mathbb{C}$, $\varphi_t(z)$ depends holomorphically on t. Keeping $z\in\mathbb{D}$ fixed, we define a map

$$t\mapsto \psi(t) = pS_{\varphi_t}(z)\cdot\eta(z)^{-1}.$$

Then $\psi(t)$ is holomorphic in $\{t\in\mathbb{C} : |t| < 1/\|\mu\|_\infty\}$ and $|\psi(t)|\leq 6$ since $\|pS_\varphi\|\leq 6$. Note that $\varphi_0(z)=z$ and hence $\psi(0)=0$. Applying Schwarz's lemma to $\hat{t}\mapsto\psi(\hat{t}/\|\mu\|_\infty)/6$ from $\mathbb{D}$ to itself, we have

$$|\psi(t)| \leq 6\,|t|\cdot\|\mu\|_\infty.$$

In particular, if $t=1$, then $\varphi_1=\varphi$ and we have

$$|\psi(1)| = |pS_\varphi(z)|\cdot\eta(z)^{-1} = \left|\frac{\varphi''(z)}{\varphi'(z)}\right|(1-|z|^2) \leq 6\,\|\mu\|_\infty.$$

By Koebe's distortion theorem, for $|z|\leq 1/2$, we have

$$|\varphi'(z)| \leq \frac{1+|z|}{(1-|z|)^3} \leq 12$$

and

$$|\varphi''(z)| \leq \frac{|\varphi'(z)|}{1-|z|^2}\cdot 6\,\|\mu\|_\infty \leq 96\,\|\mu\|_\infty.$$

Therefore, we have

$$\begin{aligned}|\varphi(1/2)-1/2| &= \left|\int_0^{1/2}(\varphi'(z)-1)\mathrm{d}z\right| \\ \leq \int_0^{1/2}|\varphi'(z)-\varphi'(0)||\mathrm{d}z| &= \int_0^{1/2}\left|\int_0^z\varphi''(\zeta)\mathrm{d}\zeta\right||\mathrm{d}z| \leq 24\,\|\mu\|_\infty.\end{aligned} \tag{5.1}$$

We consider a function

$$\widehat{\varphi}(z) := \frac{\varphi(z)}{2\varphi(1/2)} : \mathbb{C}\to\mathbb{C}.$$

[20]In some literatures the Poincaré metric $\eta(z)|\mathrm{d}z| = 2(1-|z|^2)^{-1}|\mathrm{d}z|$ in $\mathbb{D}$ was also used.

Then φ is a quasiconformal mapping fixing 0, 1/2 and ∞ whose complex dilatation is μ. According to Teichmüller's distortion theorem, for any $z \neq 0, 1/2$, the hyperbolic distance of z and $\widehat{\varphi}(z)$ satisfies

$$\eta_{0,1/2}(z, \widehat{\varphi}(z)) \leq \log \frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty},$$

where $\eta_{0,1/2}$ is the Poincaré metric in $\mathbb{C} \setminus \{0, 1/2\}$ induced by $\eta(z) = (1-|z|^2)^{-1}|\mathrm{d}z|$ in $\mathbb{D}$.

For any give compact subset K in $\mathbb{C}$, the Euclidean metric is less than the Poincaré distant up to a constant depending only on K. Therefore, there exists a constant $C_1 > 0$ depending only on K such that

$$|z - \widehat{\varphi}(z)| \leq C_1 \log \frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty} \leq \frac{2C_1\|\mu\|_\infty}{1-\|\mu\|_\infty}. \tag{5.2}$$

On the other hand, by the Hölder continuity of the quasiconformal mappings (see [LV73, p. 71]), there exists a constant $C_2 > 0$ depending only on the compact set K and k_0 (note that $\|\mu\|_\infty \leq k_0$) such that

$$|\varphi(z)| \leq C_2. \tag{5.3}$$

Again by Koebe's distortion theorem, we have $|\varphi(1/2)| \geq 2/9$. Combining (5.1), (5.2) and (5.3), for any $z \in K$, we have

$$\begin{aligned} |z - \varphi(z)| &\leq |z - \widehat{\varphi}(z)| + \frac{|\varphi(z)|}{|\varphi(1/2)|}|\varphi(1/2) - 1/2| \\ &\leq \frac{2C_1\|\mu\|_\infty}{1-\|\mu\|_\infty} + \frac{9}{2}C_2 \cdot 24\,\|\mu\|_\infty \end{aligned}$$

The proof is finished if we set

$$C := \frac{2C_1}{1-\|\mu\|_\infty} + 108C_2. \qquad \square$$

Lemma 5.1 can be seen as a parallel version of Teichmuller's distortion theorem on the planar quasiconformal mappings. They both give the distortion between the quasiconformal mapping and the identity via the complex dilatation.

5.2. **Some limiting behaviors.** Recall that ε_0 and N_1 are constants introduced in Theorem 2.6 and Corollary 4.2 respectively. Let

$$\varepsilon_3 := \min\{\varepsilon_0, 1/N_1\}. \tag{5.4}$$

In the following, we assume that

$$\alpha = [0; a_1, a_2, \cdots] \in \mathrm{HT}_N \text{ with } N \geq 1/\varepsilon_3 \text{ and } a_{n+s} = a_n \text{ for } n \geq n_0, \tag{5.5}$$

where $n_0 \geq 1$ and $s \geq 1$ are integers.

Let $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$, where α is defined in (5.5). Inductively, we define $f_n = \mathcal{R} f_{n-1}$ for $n \geq 1$, where $f_0 := f$. As before, we use U_n to denote the domain of definition of f_n, $\mathcal{C}_n^{-1}$ to denote the preimage of $\mathcal{C}_n := \mathcal{C}_{f_n}$ under f_n with $\mathrm{cp}_n := \mathrm{cp}_{f_n} \in \mathcal{C}_n^{-1}$, $k_n := k_{f_n}$ is the positive integer defined in Proposition 2.5, and $\Delta_n := \Delta_{f_n}$ is the Siegel disk of f_n, where $n \geq 0$.

By Theorem 2.7(b), there exists a map $f_* \in \mathcal{IS}_\beta$ such that $\mathcal{R}^{\circ s} f_* = f_*$, where[21]

$$\beta = [0; a_{n_0 s+1}, \cdots, a_{(n_0+1)s}, a_{n_0 s+1}, \cdots, a_{(n_0+1)s}, \cdots] \in (0, \varepsilon_3]$$

is quadratic irrational. As introduced in §4.2, we use U_* to denote the domain of definition of f_*, $\mathcal{C}_*^{-1}$ to denote the preimage of $\mathcal{C}_* := \mathcal{C}_{f_*}$ under f_* with $\mathrm{cp}_* := \mathrm{cp}_{f_*} \in \mathcal{C}_*^{-1}$, $k_* := k_{f_*}$ is the positive integer defined in Proposition 2.5, and $\Delta_* := \Delta_{f_*}$ is the Siegel disk of f_*.

In the following, we use $d_H(\cdot,\cdot)$ to denote the Hausdorff metric on compact subsets of $\mathbb{C}$.

Lemma 5.2. *We have the following two limits:*

$$\lim_{n\to\infty} \mathrm{cp}_{ns} = \mathrm{cp}_* \text{ and } \lim_{n\to\infty} d_H(\mathcal{C}_{ns}^{-1}, \mathcal{C}_*^{-1}) = 0.$$

Proof. By Theorem 2.7(b), the sequence (f_{ns}) converges to f_* exponentially fast[22] as $n \to \infty$. Therefore, for any compact subset K of U_*, K is contained in U_{ns} if n is large enough and f_{ns} converges uniformly to f_* on K. Note that $f_{ns}(\mathrm{cp}_{ns}) = f_*(\mathrm{cp}_*) = \mathrm{cv}$ and cv has exactly only one preimage in U_{ns} or U_* under f_{ns} or f_*. This means that $\mathrm{cp}_{ns} \to \mathrm{cp}_*$ as $n \to \infty$.

According to [Shi00, Proposition 2.5.2(iii)], the correspondence $f \mapsto \Phi_f$ with $f \in \mathcal{IS}_\beta$ is continuous with respect to the compact-open topology, where Φ_f is the Fatou coordinate defined on the petal $\mathcal{P}_f$. Therefore, Φ_{ns} converges locally uniformly to Φ_* in $\mathcal{P}_*$ as $n \to \infty$. Note that

$$\Phi_{ns}(\mathcal{C}_{ns}) = \Phi_*(\mathcal{C}_*) = \{\zeta \in \mathbb{C} : 1/2 \le \mathrm{Re}\,\zeta \le 3/2 \text{ and } -2 < \mathrm{Im}\,\zeta \le 2\}.$$

and Φ_f is univalent on $\mathcal{P}_f$. This means that $d_H(\mathcal{C}_{ns}, \mathcal{C}_*) \to 0$ as $n \to \infty$. On the other hand, $f_{ns} : \mathcal{C}_{ns}^{-1} \to \mathcal{C}_{ns}$ and $f_* : \mathcal{C}_*^{-1} \to \mathcal{C}_*$ are both branched covering maps with degree two and branched over cv. Since $\mathrm{cp}_{ns} \to \mathrm{cp}_*$ as $n \to \infty$, it follows that $d_H(\mathcal{C}_{ns}^{-1}, \mathcal{C}_*^{-1}) \to 0$ as $n \to \infty$. □

Recall that $0 < \lambda < 1$ is a constant appeared in Proposition 4.4. Let $A > 1$ be a large number such that

$$\tau := \frac{1+\lambda}{2} \cdot \left(1 - \frac{1}{A}\right)^{-2} < 1. \tag{5.6}$$

By the continuity stated in Lemma 5.2, we have the following corollary.

Corollary 5.3. *There exist a constant $\delta > 0$ depending only on β and A (and hence on λ), and an integer n_1 such that if $n \ge n_1$, then*

$$\overline{\mathbb{D}}(\mathrm{cp}_{ns}, A\delta) \cup \overline{\mathbb{D}}(\mathrm{cp}_*, A\delta) \Subset \mathrm{int}(\mathcal{C}_{ns}^{-1} \cap \mathcal{C}_*^{-1}).$$

Definition. Let J_* be the closed subarc of $\partial\Delta_*$ containing cp_* whose two ends are $f_*^{\circ q_M}(\mathrm{cp}_*)$ and $f_*^{\circ q_{M+1}}(\mathrm{cp}_*)$, where $M \ge 1$ is the minimal integer such that $\mathrm{diam}\, J_* \le \delta/2$. Similarly, for $n \ge 0$, let J_{ns} be the closed subarc of $\partial\Delta_{ns}$ containing cp_{ns} whose two ends are $f_{ns}^{\circ q_M}(\mathrm{cp}_{ns})$ and $f_{ns}^{\circ q_{M+1}}(\mathrm{cp}_{ns})$.

[21] There are s choices of β and hence of f_*. In order to guarantee that $f_{ns} \to f_*$ as $n \to \infty$, we did not choose $\beta = [0; a_{n_0}, \cdots, a_{n_0+s-1}, a_{n_0}, \cdots, a_{n_0+s-1}, \cdots]$.

[22] Note that the space $\mathcal{IS}_\alpha$ is equipped with two topologies: the first one is induced by the Teichmüller metric and the second one is the compact-open topology that we defined in §2.2. In particular, the convergence in the Teichmüller metric implies the uniform convergence on compact sets.

Lemma 5.4. *We have the following limit:*

$$\lim_{n\to\infty} d_H(J_{ns}, J_*) = 0.$$

Proof. For each $f \in \mathcal{IS}_\beta$, the dynamics of f on $\partial\Delta_f$ is conjugate to the rigid irrational rotation on the unit circle. By Lemma 3.2 and Proposition 3.3, as in the proof of Theorem 3.5, one can construct a sequence of continuous maps $(\gamma_m^f)_{m\in\mathbb{N}}$ defined from the unit circle $\mathbb{S}^1$ to a set which contains the first $q_m + 1$ points of the critical orbit of $f \in \mathcal{IS}_\beta$. According to the estimation in (3.2), the sequence (γ_m^f) converges uniformly to $\gamma^f = \partial\Delta_f$ which satisfies

$$|\gamma_m^f(z) - \gamma^f(z)| < \widetilde{K} \cdot \rho^m \text{ with } z \in \mathbb{S}^1,$$

where $\widetilde{K} > 1$ and $0 < \rho < 1$ are both constants independent on f.

Let $\varepsilon > 0$ be any given small number. There is a positive integer m_1 such that if $m \geq m_1$, then $\widetilde{K} \cdot \rho^m < \varepsilon/9$. For each $m \geq 0$ and $n \geq 0$, we denote

$$\begin{aligned} J_*^m &:= \{f_*^{\circ k}(\mathrm{cp}_*) : 0 \leq k \leq q_m\} \cap J_* \text{ and} \\ J_{ns}^m &:= \{f_{ns}^{\circ k}(\mathrm{cp}_{ns}) : 0 \leq k \leq q_m\} \cap J_{ns}. \end{aligned}$$

By the definition of J_*, we have $J_*^m = \{\mathrm{cp}_*\}$ if $m < M$. For any $m \geq M+1$, the set J_*^m (consisting of finite points) divides the closed Jordan arc J_* into several closed subarcs. Note that there is a correspondence between these subarcs and the subintervals in the unit circle. According to the classification of the subintervals as stated in Lemma 3.2 and the estimation in Proposition 3.3, there exists an integer m_2 such that if $m \geq m_2$, then the distance between the two ends of each of these subarcs is at most $\varepsilon/9$. Note that the estimation in Proposition 3.3 holds for all the maps in the Inou-Shishikura class. This means that the same results hold for J_{ns}, where $n \geq 1$.

Set $m_0 := \max\{m_1, m_2\}$. Then if $m \geq m_0$, we have

$$d_H(J_*^m, J_*) < \varepsilon/3 \text{ and } d_H(J_{ns}^m, J_{ns}) < \varepsilon/3 \text{ for all } n \geq 0.$$

Note that f_{ns} converges locally uniformly to f_* and cp_{ns} converges to cp_* as $n \to \infty$. There exists a positive integer n' such that if $n \geq n'$, then

$$d_H(J_{ns}^{m_0}, J_*^{m_0}) < \varepsilon/3.$$

Therefore, if $n \geq n'$, we have

$$d_H(J_{ns}, J_*) \leq d_H(J_{ns}, J_{ns}^{m_0}) + d_H(J_{ns}^{m_0}, J_*^{m_0}) + d_H(J_*^{m_0}, J_*) < \varepsilon.$$

By the arbitrary of ε, the proof is finished. □

Recall that $\mathcal{IS}_\alpha$ is equipped with a complete Teichmüller metric for any $\alpha \in [0, 1)$. Any convergent sequence under this metric implies the uniform convergence of the sequence on the compact set. The following result is an immediate corollary of Lemma 5.4. It has its own interest although it will be not used in this article.

Corollary 5.5. *Let $\alpha \in \mathrm{HT}_N$ be an irrational of bounded type. The map $f \mapsto \partial\Delta_f$ defined from $\mathcal{IS}_\alpha$ to $\mathbb{C}$ is continuous with respect to the Teichmüller metric and the Hausdorff topology respectively.*

5.3. **A sequence of contracting functions.** By the definition of near-parabolic renormalization, for each $n \geq 1$, the map $\mathcal{R}f_{n-1}$ is induced by the map

$$f_{n-1}^{\circ k_{n-1}} : \mathcal{C}_{n-1}^{-k_{n-1}} \cup (\mathcal{C}_{n-1}^{\sharp})^{-k_{n-1}} \to \mathcal{C}_{n-1} \cup \mathcal{C}_{n-1}^{\sharp}$$

under the map $\mathbb{E}\mathrm{xp} \circ \Phi_{n-1}$, i.e. we have a holomorphic map[23]

$$f_n : \mathbb{E}\mathrm{xp} \circ \Phi_{n-1}(\mathcal{C}_{n-1}^{-k_{n-1}} \cup (\mathcal{C}_{n-1}^{\sharp})^{-k_{n-1}}) \to \mathbb{E}\mathrm{xp} \circ \Phi_{n-1}(\mathcal{C}_{n-1} \cup \mathcal{C}_{n-1}^{\sharp}).$$

Considering the interior of $\mathcal{C}_{n-1}$ and noting that $\Omega = \mathbb{E}\mathrm{xp} \circ \Phi_{n-1}(\mathrm{int}\, \mathcal{C}_{n-1})$ is defined in (4.1), we have the restriction

$$f_n : \mathbb{E}\mathrm{xp} \circ \Phi_{n-1}(\mathrm{int}\, \mathcal{C}_{n-1}^{-k_{n-1}}) \to \Omega. \tag{5.7}$$

By Corollary 4.2, we have $\overline{\mathcal{C}}_n \Subset \Omega$. Since the map in (5.7) is proper, we have

$$\overline{\mathcal{C}}_n^{-1} \Subset \mathbb{E}\mathrm{xp} \circ \Phi_{n-1}(\mathrm{int}\, \mathcal{C}_{n-1}^{-k_{n-1}}). \tag{5.8}$$

Therefore, $\mathbb{E}\mathrm{xp}$ has an inverse branch $\mathbb{L}\mathrm{og}_n$ defined on $\overline{\mathcal{C}}_n^{-1}$ (since $\overline{\mathcal{C}}_n^{-1}$ is simply connected and avoids the origin, see Proposition 2.5(a), the assumption in (2.10) and Figure 4) such that

$$\mathbb{L}\mathrm{og}_n(\overline{\mathcal{C}}_n^{-1}) \Subset \Phi_{n-1}(\mathrm{int}\, \mathcal{C}_{n-1}^{-k_{n-1}}).$$

Hence as in the proof of Theorem 4.3, we can define an anti-holomorphic map

$$\Upsilon_n(z) := f_{n-1}^{\circ(k_{n-1}-1)} \circ \Phi_{n-1}^{-1} \circ \mathbb{L}\mathrm{og}_n(z) : \overline{\mathcal{C}}_n^{-1} \to \mathbb{C}, \tag{5.9}$$

which satisfies
(i) $\Upsilon_n(\overline{\mathcal{C}}_n^{-1}) \Subset \mathrm{int}\, \mathcal{C}_{n-1}^{-1}$;
(ii) $\Upsilon_n(\mathrm{cp}_n) = \mathrm{cp}_{n-1}$; and
(iii) $\Upsilon_n(\Gamma_n) \subset \Gamma_{n-1}$, where Γ_n is the component of $\partial\Delta_n \cap \mathcal{C}_n^{-1}$ containing cp_n (see the latter part of the proof of Theorem 4.3 for a similar proof). Therefore, the following composition is well-defined:

$$\widehat{\Psi}_n := \Upsilon_1 \circ \cdots \circ \Upsilon_n : \overline{\mathcal{C}}_n^{-1} \to \mathbb{C}. \tag{5.10}$$

As an immediate corollary, we have

Lemma 5.6. *For all $n \geq 1$, the univalent (if n is even) or anti-univalent (if n is odd) map $\widehat{\Psi}_n$ satisfies* (i) $\widehat{\Psi}_n(\overline{\mathcal{C}}_n^{-1}) \Subset \mathrm{int}\, \mathcal{C}_0^{-1}$*;* (ii) $\widehat{\Psi}_n(\mathrm{cp}_n) = \mathrm{cp}_0$*; and* (iii) $\widehat{\Psi}_n(\Gamma_n) \subset \Gamma_0 \subset \partial\Delta_0$.

5.4. **The self-similarity when one zooms.** Recall that Υ is the contracting map defined in (4.6). In the proof of Theorem 4.3, we assumed that $s = 1$ for simplicity. As clarified before, the contracting map Υ can be constructed such that it is anti-holomorphic if s is odd and it is holomorphic if s is even (Indeed, Υ can be written as the composition of s anti-holomorphic maps as in (5.9)). Recall that $A > 1$ is a large number satisfying (5.6).

Lemma 5.7. *The sequence of (anti-)holomorphic maps $\Upsilon_{(n-1)s+1} \circ \cdots \circ \Upsilon_{ns}$ converges to Υ uniformly on $\overline{\mathbb{D}}(\mathrm{cp}_*, A\delta)$ as $n \to \infty$.*

[23] As remarked before, the map f_n defined in this form is not contained in the Inou-Shishikura's class. However, the restriction of f_n on some subdomain is.

Proof. Without loss of generality, we assume that $s=1$. The case for $s>1$ is similar. Recall that k_* and k_n are the positive integers associated to f_* and f_n respectively defined in Proposition 2.5(b), and $\boldsymbol{k}''$ is a constant introduced in Proposition 2.5 which satisfies $k_f \leq \boldsymbol{k}''$ for all $f \in \mathcal{IS}_\beta$. Define $\hat{k} := \boldsymbol{k}''$. We claim that one can rewrite the maps Υ in (4.6) and Υ_n in (5.9) as

$$(5.11)\qquad \begin{aligned} \Upsilon(z) &= f_*^{\circ(\hat{k}-1)} \circ \Phi_*^{-1} \circ \widetilde{\mathbb{L}\mathrm{og}}(z) : \overline{\mathcal{C}}_*^{-1} \to \mathbb{C}, \text{ and} \\ \Upsilon_n(z) &= f_{n-1}^{\circ(\hat{k}-1)} \circ \Phi_{n-1}^{-1} \circ \widehat{\mathbb{L}\mathrm{og}}_n(z) : \overline{\mathcal{C}}_n^{-1} \to \mathbb{C}, \end{aligned}$$

where $\widetilde{\mathbb{L}\mathrm{og}} = \mathbb{L}\mathrm{og} - (\hat{k} - k_*)$ and $\widehat{\mathbb{L}\mathrm{og}}_n = \mathbb{L}\mathrm{og}_n - (\hat{k} - k_{n-1})$ with $n \geq 1$. Indeed, by (5.8), the map Υ_n in (5.11) is well defined since[24]

$$\overline{\mathcal{C}}_n^{-1} \Subset \mathbb{E}\mathrm{xp} \circ \Phi_{n-1}(\mathrm{int}\,\mathcal{C}_{n-1}^{-k_{n-1}}) = \mathbb{E}\mathrm{xp} \circ \Phi_{n-1}(\mathrm{int}\,\mathcal{C}_{n-1}^{-\hat{k}}).$$

In particular, we have

$$(5.12)\qquad \widehat{\mathbb{L}\mathrm{og}}_n(\overline{\mathcal{C}}_n^{-1}) \Subset \Phi_{n-1}(\mathrm{int}\,\mathcal{C}_{n-1}^{-\hat{k}}).$$

It is easy to check that the new Υ_n defined in (5.11) is equal to the old one in (5.9) since $f_{n-1} \circ \Phi_{n-1}^{-1}(\zeta - 1) = \Phi_{n-1}^{-1}(\zeta)$ when the both sides are well defined. The similar arguments can be applied to the map Υ. Hence the claim in (5.11) is proved. In particular, we have

$$(5.13)\qquad \widetilde{\mathbb{L}\mathrm{og}}(\overline{\mathcal{C}}_*^{-1}) \Subset \Phi_*(\mathrm{int}\,\mathcal{C}_*^{-\hat{k}}).$$

According to [Shi00, Proposition 2.5.2(iii)], the Fatou coordinate Φ_f defined on the petal $\mathcal{P}_f$ depends continuously on $f \in \mathcal{IS}_\beta$ with respect to the compact-open topology. Note that f_n tends to f_* locally uniformly in U_*. Therefore, Φ_n converges locally uniformly to Φ_* in $\mathcal{P}_*$ as $n \to \infty$. Since Φ_f is injective on $\mathcal{P}_f$, it follows that Φ_n^{-1} converges locally uniformly to Φ_*^{-1} in

$$\Phi_*(\mathcal{P}_*) = \{\zeta \in \mathbb{C} : 0 < \mathrm{Re}\,\zeta < \lfloor \tfrac{1}{\beta} \rfloor - \boldsymbol{k}\}$$

as $n \to \infty$. By a similar argument as in Lemma 5.2, the Hausdorff distance $d_H(\mathcal{C}_n^{-k}, \mathcal{C}_*^{-k})$ tends to 0 as $n \to \infty$, where $k \geq 0$. By (5.12) and (5.13), since both of the widths of $\Phi_{n-1}(\mathrm{int}\,\mathcal{C}_{n-1}^{-\hat{k}})$ and $\Phi_*(\mathrm{int}\,\mathcal{C}_*^{-\hat{k}})$ are at most 1, it follows that $\widehat{\mathbb{L}\mathrm{og}}_n = \widetilde{\mathbb{L}\mathrm{og}}$ and $\Upsilon_n = f_{n-1}^{\circ(\hat{k}-1)} \circ \Phi_{n-1}^{-1} \circ \widehat{\mathbb{L}\mathrm{og}}_n$ converges to $\Upsilon = f_*^{\circ(\hat{k}-1)} \circ \Phi_*^{-1} \circ \widetilde{\mathbb{L}\mathrm{og}}$ locally uniformly in $\mathrm{int}\,\mathcal{C}_*^{-1}$ as $n \to \infty$. In particular, the convergence is uniform on $\overline{\mathbb{D}}(\mathrm{cp}_*, A\delta)$ since $\overline{\mathbb{D}}(\mathrm{cp}_*, A\delta) \Subset \mathrm{int}\,\mathcal{C}_*^{-1}$ by Corollary 5.3. □

Note that the sequence (f_{ns}) converges to f_* exponentially fast in the Teichmüller metric as $n \to \infty$. Recall that U_* is the domain of definition of f_*. The following lemma shows that (f_{ns}) converges locally uniformly to f_* "exponentially fast" in U_*.

Lemma 5.8. *There exist constants $D_0 > 1$ and $0 < \varsigma < 1$ independent on n such that*

$$|\mathrm{cp}_{ns} - \mathrm{cp}_*| < D_0 \cdot \varsigma^n, \text{ where } n \geq 1.$$

[24]We can assume that N is large enough such that $\Phi_f(\mathcal{C}_f^{-\hat{k}})$ is defined for all $f \in \mathcal{IS}_\beta \cup \{Q_\beta\}$.

Moreover, for any compact subset K_1 of U_* and any compact subset K_2 of $\Phi_*(\mathcal{P}_*)$, there exist constants $D_1 > 1$, $0 < u < 1$ depending only on K_1, and $D_2 > 1$, $0 < v < 1$ depending only on K_2 such that[25] *for all $n \geq 1$,*

$$\begin{aligned} &|f_{ns}(z) - f_*(z)| < D_1 \cdot u^n, \text{ where } z \in K_1, \text{ and} \\ &|\Phi_{ns}^{-1}(\zeta) - \Phi_*^{-1}(\zeta)| < D_2 \cdot v^n, \text{ where } \zeta \in K_2. \end{aligned} \tag{5.14}$$

Proof. The idea of the proof is to use the exponential convergence of f_{ns} with respective to the Teichmüller metric. Without loss of generality, we assume that $s = 1$. Let $f_n = P \circ \varphi_n^{-1}$ and $f_* = P \circ \varphi_*^{-1}$ be the maps in the Inou-Shishikura class, where $P(z) = z(1+z)^2$. Note that φ_n and φ_* are both univalent maps defined on a fixed Jordan domain U which has a quasiconformal extension to $\mathbb{C}$ with normalized condition $\varphi_n(0) = \varphi_*(0) = 0$ and $\varphi_n'(0) = \varphi_*'(0) = 1$.

Recall that the Teichmüller distance between f_n and f_* (or between φ_n and φ_*) is defined in (2.12). Since f_n converges to f_* exponentially fast in the Teichmüller metric, it follows that there exist two constants $C_0 \geq 1$ and $0 < \hat{u} < 1$, and two quasiconformal extentions $\hat{\varphi}_n$ and $\hat{\varphi}_*$ of φ_n and φ_* respectively, such that the complex dilatation of $\hat{\varphi}_n \circ \hat{\varphi}_*^{-1}$ satisfies

$$\|\mu_{\hat{\varphi}_n \circ \hat{\varphi}_*^{-1}}\|_\infty \leq C_0 \cdot \hat{u}^n, \text{ where } n \geq 1.$$

According to Lemma 5.1, for any given compact subset K_1 of $\mathbb{C}$, there exists a constant $C_1 > 0$ depending only on K_1 such that

$$|\hat{\varphi}_n \circ \hat{\varphi}_*^{-1}(z) - z| \leq C_1 \cdot \hat{u}^n, \text{ where } z \in K_1.$$

By the Hölder continuity of the quasiconformal mappings (see [LV73, p. 71]), there exist two constants $C_2 \geq 1$ and $0 < u < 1$ such that

$$|\hat{\varphi}_n^{-1}(z) - \hat{\varphi}_*^{-1}(z)| \leq C_2 \cdot u^n, \text{ where } z \in K_1.$$

Since $f_n = P \circ \varphi_n^{-1}$ and $f_* = P \circ \varphi_*^{-1}$, it follows that for $z \in K_1$, one has

$$|f_n(z) - f_*(z)| \leq D_1 \cdot u^n \text{ for some } D_1 > 0. \tag{5.15}$$

Similarly, if we consider the complex dilatation of $\hat{\varphi}_n^{-1} \circ \hat{\varphi}_*$, then there exists two constant $D_0 > 0$ and $0 < \varsigma < 1$ such that

$$|\hat{\varphi}_n(z) - \hat{\varphi}_*(z)| \leq D_0 \cdot \varsigma^n, \text{ where } z \in \mathbb{D}.$$

In particular, we have

$$|\mathrm{cp}_n - \mathrm{cp}_*| = |\hat{\varphi}_n(-1/3) - \hat{\varphi}_*(-1/3)| \leq D_0 \cdot \varsigma^n.$$

According to the proof of [Shi00, Proposition 2.5.2(ii) and (iii)], the Fatou coordinate Φ_f defined on the petal $\mathcal{P}_f$ is the composition of two quasiconformal mappings which depend continuously on $f \in \mathcal{IS}_\beta$ with respect to the compact-open topology. In particular, for any compact subset $\hat{K}$ of $\mathcal{P}_*$, there exist two constants $\hat{D}_2$ and $0 < \hat{v} < 1$ depending on $\hat{K}$ such that $|\Phi_n(z) - \Phi_*(z)| < \hat{D}_2 \cdot \hat{v}^n$ for all $z \in K'$ by (5.15) and Lemma 5.1. Similarly, if we consider the inverse of the Fatou coordinate, for any compact subset K_2 of $\Phi_*(\mathcal{P}_*)$, there exist two constants D_2 and $0 < v < 1$ depending on K such that $|\Phi_n^{-1}(\zeta) - \Phi_*^{-1}(\zeta)| < D_2 \cdot v^n$, where $\zeta \in K_2$. □

[25]Without loss of generality, one can assume that f_{ns} has definition on K_1 for all $n \geq 1$.

For the simplicity of notations, we denote

$$\varrho := \begin{cases} \frac{\partial \Upsilon}{\partial \overline{z}}(\mathrm{cp}_*) & \text{if } s \text{ is odd,} \\ \frac{\partial \Upsilon}{\partial z}(\mathrm{cp}_*) & \text{if } s \text{ is even.} \end{cases}$$

For each $n \geq 1$, we denote

$$\varrho_n := \begin{cases} \frac{\partial \Upsilon_{(n-1)s+1} \circ \cdots \circ \Upsilon_{ns}}{\partial \overline{z}}(\mathrm{cp}_{ns}) & \text{if } s \text{ is odd,} \\ \frac{\partial \Upsilon_{(n-1)s+1} \circ \cdots \circ \Upsilon_{ns}}{\partial z}(\mathrm{cp}_{ns}) & \text{if } s \text{ is even.} \end{cases}$$

The explicit analysis on the convergence of f_n and Φ_n stated in Lemma 5.8 can help us to prove that the sequence (ϱ_n) converges to ϱ exponentially fast as $n \to \infty$.

Lemma 5.9. *The sequence (ϱ_n) converges to ϱ exponentially fast as $n \to \infty$.*

Proof. For simplify the notations, we still assume that $s = 1$. According to Lemma 5.2 and Corollary 5.3, we assume that W is a rounded open disk centered at cp_* with radius δ such that $\mathrm{cp}_n \in W$ and $\overline{W} \subset \mathrm{int}(\mathcal{C}_n^{-1} \cap \mathcal{C}_*^{-1})$ if n is large enough. In particular, there exists a large integer n_1 such that if $n \geq n_1$, then

$$\min_{z \in \partial W} |\mathrm{cp}_n - z| \geq \delta/2 \text{ and } |\mathrm{cp}_* - z| = \delta \text{ if } z \in \partial W. \tag{5.16}$$

Recall that $\widetilde{\mathbb{L}}\mathrm{og}$ and $\widehat{\mathbb{L}}\mathrm{og}_n$ are two inverse branches of the modified exponential map $\mathbb{E}\mathrm{xp}$ defined in (5.11). By Lemma 5.10, there exists an integer $n_2 \geq n_1$ such that if $n \geq n_2$, then these two inverse are equal to each other. Let us denote both of them by $\mathbb{L}\mathrm{og}$. Note that $0 \notin \overline{W}$. For $n \geq n_2$, we denote

$$K_2 := \mathbb{L}\mathrm{og}(\overline{W}) \text{ and } K_1 := \Phi_*^{-1}(K_2).$$

Then K_2 is a compact subset of $\Phi_*(\mathcal{P}_*) = \Phi_n(\mathcal{P}_n)$ and K_1 is a compact subset of U_*.

Note that f_{n-1} tends to f_* uniformly on K_1 as $n \to \infty$. By Lemma 5.8, for $\zeta \in K_2$, there exist constants D_1, D_2, $0 < u$, $v < 1$ and M depending on K_1 and K_2 such that

$$\begin{aligned} & |f_{n-1} \circ \Phi_{n-1}^{-1}(\zeta) - f_* \circ \Phi_*^{-1}(\zeta)| \\ \leq\ & |f_{n-1} \circ \Phi_{n-1}^{-1}(\zeta) - f_{n-1} \circ \Phi_*^{-1}(\zeta)| + |f_{n-1} \circ \Phi_*^{-1}(\zeta) - f_* \circ \Phi_*^{-1}(\zeta)| \\ \leq\ & M \cdot D_2 \cdot v^n + D_1 \cdot u^n \leq M' \cdot \nu^n \end{aligned}$$

for some $M' \geq 1$ and $\nu := \max\{u, v\} \in (0,1)$. Similarly, by using interpolation as above, there exist two constants $\hat{M}$ and $\nu \leq \hat{\nu} < 1$ depending only on W and $\hat{k}$ such that if $z \in \overline{W}$, then

$$\begin{aligned} & |\Upsilon_n(z) - \Upsilon(z)| \\ =\ & |f_*^{\circ(\hat{k}-1)} \circ \Phi_*^{-1} \circ \mathbb{L}\mathrm{og}(z) - f_{n-1}^{\circ(\hat{k}-1)} \circ \Phi_{n-1}^{-1} \circ \mathbb{L}\mathrm{og}(z)| \leq \hat{M} \cdot \hat{\nu}^n. \end{aligned} \tag{5.17}$$

By Cauchy's integral formula and (5.16), we have

$$\begin{aligned} |\overline{\varrho}_n - \overline{\varrho}| =\ & \left| \frac{1}{2\pi \mathrm{i}} \int_{\partial W} \frac{\overline{\Upsilon}_n(z)}{(z - \mathrm{cp}_n)^2} \mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \int_{\partial W} \frac{\overline{\Upsilon}(z)}{(z - \mathrm{cp}_*)^2} \mathrm{d}z \right| \\ \leq\ & \frac{4}{\delta^3} \max_{z \in \partial W} \left| \overline{\Upsilon}_n(z)(z - \mathrm{cp}_*)^2 - \overline{\Upsilon}(z)(z - \mathrm{cp}_n)^2 \right|. \end{aligned}$$

According to Lemmas 5.7, 5.8 and (5.17), there exists a constant $\tilde{M} > 0$ such that if $z \in \partial W$, we have

$$\begin{aligned} & \left|\overline{\Upsilon}_n(z)(z-\mathrm{cp}_*)^2 - \overline{\Upsilon}(z)(z-\mathrm{cp}_n)^2\right| \\ \leq & \left|\overline{\Upsilon}_n(z)\right| \cdot |(2z - \mathrm{cp}_n - \mathrm{cp}_*)| \cdot |\mathrm{cp}_n - \mathrm{cp}_*| + |z-\mathrm{cp}_n|^2 \cdot \left|\overline{\Upsilon}_n(z) - \overline{\Upsilon}(z)\right| \\ \leq & \tilde{M} \cdot 4\delta \cdot D_0 \cdot \varsigma^n + (2\delta)^2 \cdot \hat{M} \cdot \hat{\nu}^n \leq 4\delta(D_0\tilde{M} + \delta\hat{M}) \cdot \hat{\varsigma}^n, \end{aligned}$$

where $\hat{\varsigma} = \max\{\varsigma, \hat{\nu}\} \in (0,1)$. Therefore, if $n \geq n_2$, we have

$$|\overline{\varrho}_n - \overline{\varrho}| \leq 16\,\delta^{-2}(D_0\tilde{M} + \delta\hat{M}) \cdot \hat{\varsigma}^n.$$

This means that (ϱ_n) converges to ϱ exponentially fast as $n \to \infty$. $\square$

For the map $f_0 = f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ fixed at the beginning of §5.2, we define

$$L(z) = \begin{cases} \mathrm{cp}_f + \varrho(\overline{z - \mathrm{cp}_f}) & \text{if } s \text{ is odd}, \\ \mathrm{cp}_f + \varrho(z - \mathrm{cp}_f) & \text{if } s \text{ is even}. \end{cases} \tag{5.18}$$

Recall that $\tau < 1$ is a positive number defined in (5.6), $\delta > 0$ is a constant introduced in Corollary 5.3 and $\widehat{\Psi}_n$ is a map defined in (5.10), where $n \geq 1$.

Lemma 5.10. *The sequence of* holomorphic *maps* $\Theta_n := L^{-n} \circ \widehat{\Psi}_{ns}$ *converges uniformly to a holomorphic map* Θ *in* $\mathbb{D}(\mathrm{cp}_*, \frac{3}{4}\delta)$.

Proof. If s is odd, then Θ_n is the composition of $n(s+1)$ anti-holomorphic maps and hence Θ_n is holomorphic. If s is even, then Θ_n is evidently holomorphic. Since $\Theta_n(\mathrm{cp}_{ns}) = \mathrm{cp}_f$ and $\mathrm{cp}_{ns} \to \mathrm{cp}_*$ as $n \to \infty$, it is sufficient to prove that $\frac{\partial \Theta_n}{\partial z}$ converges uniformly on $\mathbb{D}(\mathrm{cp}_*, \frac{3}{4}\delta)$. For simplifying the notations (replacing ns by n), we assume that $s = 1$.

By Corollary 5.3, the anti-univalent map Υ_n is defined on $\mathbb{D}(\mathrm{cp}_n, A\delta)$ if $n \geq n_1$. Since $|\varrho_n| \to |\varrho| = \lambda$ as $n \to \infty$ by Lemma 5.7, there exists $n_2 \geq n_1$ such that $|\varrho_n| < (1+\lambda)/2$ if $n \geq n_2$. By the choice of A in (5.6), if $n \geq n_2$, according to Koebe's distortion theorem, we have

$$|\Upsilon_n(z) - \Upsilon_n(\mathrm{cp}_n)| \leq |\varrho_n| \left(1 - \tfrac{1}{A}\right)^{-2} \delta < \tau\delta, \text{ where } z \in \mathbb{D}(\mathrm{cp}_n, \delta).$$

Inductively, for any $n > n_2$ and $n_2 < k \leq n$, we have

$$|\Upsilon_k(z) - \Upsilon_k(\mathrm{cp}_k)| < \tau^{n-k+1}\delta, \text{ where } z \in \mathbb{D}(\mathrm{cp}_k, \tau^{n-k}\delta). \tag{5.19}$$

Still by Koebe's distortion theorem, if $n_2 < k \leq n$ and for $z \in \mathbb{D}(\mathrm{cp}_k, \tau^{n-k}\delta)$, we have

$$\begin{aligned} |\varrho_k|\Big(1 - \frac{\tau^{n-k}}{A}\Big)\Big(1 + \frac{\tau^{n-k}}{A}\Big)^{-3} & \leq \Big|\frac{\partial \Upsilon_k}{\partial \overline{z}}(z)\Big| \\ & \leq |\varrho_k|\Big(1 + \frac{\tau^{n-k}}{A}\Big)\Big(1 - \frac{\tau^{n-k}}{A}\Big)^{-3}. \end{aligned} \tag{5.20}$$

In the following argument, we suppose that n is an even number. The completely similar argument can be applied to odd n. Let $m \geq 1$ be an even integer (similar arguments can be applied to odd m also). For a positive even integer $n_3 < n$ and $z \in \mathbb{D}(\mathrm{cp}_n, \delta)$, we define[26]

$$\begin{aligned} w_1 &= \frac{1}{|\varrho|^{n_3}} \cdot \frac{\partial \Upsilon_{n-n_3+1} \circ \cdots \circ \Upsilon_n}{\partial z}(z) \text{ and} \\ w_1' &= \frac{1}{|\varrho|^{n_3+m}} \cdot \frac{\partial \Upsilon_{n-n_3+1} \circ \cdots \circ \Upsilon_n \circ \Upsilon_{n+1} \circ \cdots \circ \Upsilon_{n+m}}{\partial z}(z). \end{aligned}$$

Let $z_1 = \Upsilon_{n-n_3+1} \circ \cdots \circ \Upsilon_n(z)$ and $z_1' = \Upsilon_{n-n_3+1} \circ \cdots \circ \Upsilon_{n+m}(z)$. If $n > n_2 + n_3 + 1$, by (5.19) we have $z_1, z_1' \in \mathbb{D}(\mathrm{cp}_{n-n_3}, \tau^{n_3}\delta)$. Without loss of generality, we assume that n_2 is also even. Denote

$$\begin{aligned} w_2 &= \frac{1}{|\varrho|^{n-n_2-n_3}} \cdot \frac{\partial \Upsilon_{n_2+1} \circ \cdots \circ \Upsilon_{n-n_3}}{\partial z}(z_1) \text{ and} \\ w_2' &= \frac{1}{|\varrho|^{n-n_2-n_3}} \cdot \frac{\partial \Upsilon_{n_2+1} \circ \cdots \circ \Upsilon_{n-n_3}}{\partial z}(z_1'). \end{aligned}$$

Let $\varepsilon > 0$ be a given number. By (5.20), there exists an integer $n_4 \geq n_2 + n_3 + 2$ such that if $n \geq n_4$ and if the integer n_3 is chosen large enough, then

$$|w_2|, |w_2'| \in \mathbb{D}(Z_2, \varepsilon), \text{ where } Z_2 = \frac{|\varrho_{n_2+1} \cdots \varrho_{n-n_3}|}{|\varrho|^{n-n_2-n_3}}.$$

Let $z_2 = \Upsilon_{n_2+1} \circ \cdots \circ \Upsilon_{n-n_3}(z_1)$ and $z_2' = \Upsilon_{n_2+1} \circ \cdots \circ \Upsilon_{n-n_3}(z_1')$. Still by (5.19), we have $z_2, z_2' \in \mathbb{D}(\mathrm{cp}_{n_2}, \tau^{n-n_2}\delta)$. Denote

$$w_3 = \frac{1}{|\varrho|^{n_2}} \cdot \frac{\partial \Upsilon_1 \circ \cdots \circ \Upsilon_{n_2}}{\partial z}(z_2) \text{ and } w_3' = \frac{1}{|\varrho|^{n_2}} \cdot \frac{\partial \Upsilon_1 \circ \cdots \circ \Upsilon_{n_2}}{\partial z}(z_2').$$

There exists an integer $n_5 \geq n_4$ such that if $n \geq n_5$, then by the continuity, we have

$$w_3, w_3' \in \mathbb{D}(Z_3, \varepsilon), \text{ where } Z_3 = \frac{\varrho_1 \overline{\varrho}_2 \cdots \overline{\varrho}_{n_2-1} \varrho_{n_2}}{|\varrho|^{n_2}}.$$

According to Lemma 5.7, there exists an integer $n_6 \geq n_5$ such that if $n \geq n_6$, then

$$w_1, w_1' \in \mathbb{D}(Z_1, \varepsilon), \text{ where } Z_1 = \frac{1}{|\varrho|^{n_3}} \cdot \frac{\partial \Upsilon^{\circ n_3}}{\partial z}(z).$$

By Lemma 5.2, there exists $n_7 \geq n_6$ such that if $n \geq n_7$, then $\mathbb{D}(\mathrm{cp}_*, \frac{3}{4}\delta) \subset \mathbb{D}(\mathrm{cp}_n, \delta)$. Therefore, for every even $n \geq n_7$ and $z \in \mathbb{D}(\mathrm{cp}_*, \frac{3}{4}\delta)$, we have

$$\begin{aligned} &|\tfrac{\partial \Theta_n}{\partial z}(z) - \tfrac{\partial \Theta_{n+m}}{\partial z}(z)| = |w_1 w_2 w_3 - w_1' w_2' w_3'| \\ \leq\; & |w_1 - w_1'| \cdot |w_2 w_3| + |w_1'| \cdot (|w_2 - w_2'| \cdot |w_3| + |w_2'| \cdot |w_3 - w_3'|) \\ \leq\; & 2\varepsilon \cdot [(|Z_2| + \varepsilon)(|Z_3| + \varepsilon) + (|Z_1| + \varepsilon)(|Z_2| + |Z_3| + 2\varepsilon)]. \end{aligned}$$

By Lemma 5.9, there exists a finite number $\widehat{\varrho} > 0$ such that

$$\lim_{n\to\infty} \left|\tfrac{\partial \Theta_n}{\partial z}(\mathrm{cp}_n)\right| = \lim_{n\to\infty} \frac{|\varrho_1 \cdots \varrho_n|}{|\varrho|^n} = \widehat{\varrho}. \tag{5.21}$$

Note that there is a constant $C_1 > 0$ and $C_2 > 0$ independent on n such that $|Z_1| \leq C_1$, $C_2 \leq |Z_2| \leq C_1$ and $|Z_2| \cdot |Z_3| \to \widehat{\varrho}$ as $n \to \infty$. This means that

[26] We require that the integers m, n and n_3 are even numbers. This is convenient for us to judge the holmorphicity or anti-holomorphicity of a given map. For example, when n is even, one can check directly that $\frac{\partial L^{\circ n}}{\partial z}(\mathrm{cp}_f) = |\varrho|^n$ and $\frac{\partial \Upsilon_1 \circ \cdots \circ \Upsilon_n}{\partial z}(\mathrm{cp}_n) = \varrho_1 \overline{\varrho}_2 \cdots \varrho_{n-1} \overline{\varrho}_n$ by chain rule.

the sequence $(\frac{\partial\Theta_n}{\partial z})_{n\in\mathbb{N}}$ forms a Cauchy sequence in $\mathbb{D}(\mathrm{cp}_*,\frac{3}{4}\delta)$. Therefore, it converges uniformly in $\mathbb{D}(\mathrm{cp}_*,\frac{3}{4}\delta)$. □

Proposition 5.11 (Self-similarity of the boundary of the Siegel disk when one zooms)**.** *The successive blowups $L^{-n}(\partial\Delta_f)$ of the boundary of the Siegel disk of f converge to a L-invariant Jordan curve through ∞, in the Hausdorff metric on compact subsets of the Riemann sphere.*

Proof. By Corollary 5.3 and Lemma 5.4, there exists an integer $n'\geq n_1$ such that if $n\geq n'$, then $J_{ns}\subset\mathbb{D}(\mathrm{cp}_{ns},\frac{2}{3}\delta)\subset\mathbb{D}(\mathrm{cp}_*,\frac{3}{4}\delta)\subset\mathcal{C}_{ns}^{-1}$, where J_{ns} is a closed Jordan arc of the Siegel disk of Δ_{ns} defined just above Lemma 5.4. Recall that $\widehat{\Psi}_{ns}$ is defined in $\mathcal{C}_{ns}^{-1}$. For $n\geq 1$, we define

$$I_n:=\widehat{\Psi}_{ns}(J_{ns}).$$

By Lemma 5.6, I_n is a subset of $\partial\Delta_f$ which is a closed Jordan arc containing the critical point cp_f. If $n\geq n'$, the Hausdorff distance between $L^{-n}(I_n)$ and $L^{-(n+1)}(I_{n+1})$ satisfies

$$\begin{aligned}&d_H(L^{-n}(I_n),L^{-(n+1)}(I_{n+1}))\\ =\;& d_H(L^{-n}\circ\widehat{\Psi}_{ns}\circ\widehat{\Psi}_{ns}^{-1}(I_n),L^{-(n+1)}\circ\widehat{\Psi}_{(n+1)s}\circ\widehat{\Psi}_{(n+1)s}^{-1}(I_{n+1}))\\ =\;& d_H(\Theta_n(J_{ns}),\Theta_{n+1}(J_{(n+1)s}))\\ \leq\;& d_H(\Theta_n(J_{ns}),\Theta_n(J_{(n+1)s}))+d_H(\Theta_n(J_{(n+1)s}),\Theta_{n+1}(J_{(n+1)s})).\end{aligned}$$

By Lemmas 5.4 and 5.10, we have $d_H(L^{-n}(I_n),L^{-(n+1)}(I_{n+1}))\to 0$ as $n\to\infty$. This means that the boundary of the Siegel disk of f is self-similar when one zooms by a fixed rate $1/\varrho$ at each time.

By (5.21), the diameter of $L^{-n}(I_n)=\Theta_n(J_{ns})$ is not less than a universal constant for all $n\geq 1$. So the convergence of $L^{-n}(\partial\Delta_f)$ in the Hausdorff metric on the compact subsects of the Riemann sphere is straightforward. This completes the proof of this proposition and Theorem 1.1 modulo the self-similarity in the case of non-quadratic irrational. □

Let us sum up the idea of the proof of the self-similarity: We want to obtain the self-similarity of the boundary of the Siegel disk of f_0 via zooming near the critical point cp_0 by a fixed rate $1/\varrho$. By means of near-parabolic renormalization, one can construct a sequence of expanding maps (the inverse of Υ_n) defined from a neighborhood of cp_{n-1} to cp_n such that the expanding factor is nearly $1/\varrho_n$ for each n (suppose that $s=1$). Since the sequence of the boundaries of the Siegel disks of (f_n) is convergent in the Hausdorff topology under near-parabolic renormalization, one just needs to compare the expanding factors ϱ and ϱ_n. Fortunately, the exponentially fast convergence of (ϱ_n) was guaranteed by Lemma 5.9.

If we do not know Lemma 5.9 in prior, one can still obtain the self-similarity by zooming with the expanding factor $1/\varrho_n$ at each time, i.e. zooming with different rates at each time. We will use this idea to hand with the self-similarity of non-quadratic irrational case in the next section.

5.5. **The self-similarity is dynamical.** The following result shows that the dynamical self-similarity in in Proposition 4.4 can be extended to the quadratic polynomials and all the maps in the Inou-Shishikura class.

Proposition 5.12. *Let α be the quadratic irrational defined in* (5.5)*. For any $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$, we have the following limit*

$$\lambda = \lim_{n\to\infty} \left| \frac{f^{\circ q_{n+s}}(\mathrm{cp}_f) - \mathrm{cp}_f}{f^{\circ q_n}(\mathrm{cp}_f) - \mathrm{cp}_f} \right|,$$

where λ is the constant introduced in Proposition 4.4.

Proof. We still assume that $n_0 = 1$ and $s = 1$, i.e. there exists a renormalization fixed point $f_* \in \mathcal{IS}_\alpha$ such that $\mathcal{R}f_* = f_*$. The argument for $n_0 > 1$ or $s > 1$ is similar. Set $f_0 := f$. For $n \geq 1$, let $f_n := \mathcal{R}f_{n-1}$ be the sequence defined in (2.14) and cp_n its unique critical point.

Note that the holomorphic map f_* maps a neighborhood of cp_* to that of $\mathrm{cv} = -4/27$ with local degree two. For each $\varepsilon > 0$, by Propositions 3.3 and 4.4, there exists an integer $k_1 \geq 1$ such that if $k \geq k_1$, then

$$\left| \left| \frac{f_*^{\circ q_{k+1}}(\mathrm{cv}) - \mathrm{cv}}{f_*^{\circ q_k}(\mathrm{cv}) - \mathrm{cv}} \right| - \lambda^2 \right| < \varepsilon. \tag{5.22}$$

Recall that $\Psi_n : \widetilde{\mathcal{P}}_n \to \mathcal{P}_0$ is a holomorphic (actually univalent) or an anti-holomorphic (actually anti-univalent) map (if n is even or odd, respectively) defined in (2.19). By Proposition 3.3 and the pre-compactness of the class $\cup_{\alpha\in(0,\varepsilon_3]}\mathcal{IS}_\alpha$, there exists an integer $k_2 \geq 1$, such that if $k \geq k_2$, then $f_m^{\circ q_k}(\mathrm{cv})$ and $f_m^{\circ q_{k+1}}(\mathrm{cv})$ are contained in $\mathcal{P}_m \subset \widetilde{\mathcal{P}}_m$ for all $m \geq 1$. Note that $\mathbb{E}\mathrm{xp}(\mathrm{cv}) = \mathrm{cv}$. By (2.17) and Lemma 2.8(a), there exists an integer $l \in [0, \boldsymbol{k}']$ such that Ψ_m maps the triple $(\mathrm{cv}, f_m^{\circ q_k}(\mathrm{cv}), f_m^{\circ q_{k+1}}(\mathrm{cv}))$ to the triple $f_0^{\circ l}(\mathrm{cv}, f_0^{\circ q_{k+m}}(\mathrm{cv}), f_0^{\circ q_{k+m+1}}(\mathrm{cv}))$. By Koebe's distortion theorem and Proposition 3.3, there exists an integer $k_3 \geq k_2$ such that if $k \geq k_3$, then for all $m \geq 1$, one has

$$\left| \left| \frac{f_0^{\circ q_{k+m+1}} \circ f_0^{\circ l}(\mathrm{cv}) - f_0^{\circ l}(\mathrm{cv})}{f_0^{\circ q_{k+m}} \circ f_0^{\circ l}(\mathrm{cv}) - f_0^{\circ l}(\mathrm{cv})} \right| - \left| \frac{f_m^{\circ q_{k+1}}(\mathrm{cv}) - \mathrm{cv}}{f_m^{\circ q_k}(\mathrm{cv}) - \mathrm{cv}} \right| \right| < \varepsilon.$$

Since $0 \leq l \leq \boldsymbol{k}'$, there exists a constant $C_0 \geq 1$ depending only on f_0 such that

$$\left| \left| \frac{f_0^{\circ q_{k+m+1}}(\mathrm{cv}) - \mathrm{cv}}{f_0^{\circ q_{k+m}}(\mathrm{cv}) - \mathrm{cv}} \right| - \left| \frac{f_m^{\circ q_{k+1}}(\mathrm{cv}) - \mathrm{cv}}{f_m^{\circ q_k}(\mathrm{cv}) - \mathrm{cv}} \right| \right| < C_0 \cdot \varepsilon. \tag{5.23}$$

Set $k_0 = \max\{k_1, k_3\}$ and $\delta := |f_*^{\circ q_{k_0}}(\mathrm{cv}) - \mathrm{cv}| > 0$. Note that (f_n) converges to f_* locally uniformly as $n \to \infty$. There exists an integer $m_0 := m_0(k_0) > 0$ such that if $m \geq m_0$, then

$$\begin{aligned} |f_m^{\circ q_{k_0+1}}(\mathrm{cv}) - f_*^{\circ q_{k_0+1}}(\mathrm{cv})| < \delta^2\varepsilon,\ \ |f_m^{\circ q_{k_0}}(\mathrm{cv}) - f_*^{\circ q_{k_0}}(\mathrm{cv})| < \delta^2\varepsilon \\ \text{and}\ \ |f_m^{\circ q_{k_0}}(\mathrm{cv}) - \mathrm{cv}| \geq \delta/2. \end{aligned} \tag{5.24}$$

By Proposition 3.3, there exists a constant $C_1 > 0$ depending only on k_0 such that

$$|f_*^{\circ q_{k_0}}(\mathrm{cv})| < C_1 \quad \text{and} \quad |f_*^{\circ q_{k_0+1}}(\mathrm{cv})| < C_1. \tag{5.25}$$

By (5.24) and (5.25), there exists a constant $C_2 > 0$ depending only on k_0 such that if $m \geq m_0$, one has

$$\left| \frac{f_m^{\circ q_{k_0+1}}(\mathrm{cv}) - \mathrm{cv}}{f_m^{\circ q_{k_0}}(\mathrm{cv}) - \mathrm{cv}} - \frac{f_*^{\circ q_{k_0+1}}(\mathrm{cv}) - \mathrm{cv}}{f_*^{\circ q_{k_0}}(\mathrm{cv}) - \mathrm{cv}} \right| < C_2 \cdot \varepsilon. \tag{5.26}$$

By (5.22), (5.23) and (5.26), if $n \geq k_0 + m_0$, we have

$$\begin{aligned} &\left| \left| \frac{f_0^{\circ q_{n+1}}(\mathrm{cv}) - \mathrm{cv}}{f_0^{\circ q_n}(\mathrm{cv}) - \mathrm{cv}} \right| - \lambda^2 \right| \\ \leq & \left| \left| \frac{f_0^{\circ q_{n+1}}(\mathrm{cv}) - \mathrm{cv}}{f_0^{\circ q_n}(\mathrm{cv}) - \mathrm{cv}} \right| - \left| \frac{f_{n-k_0}^{\circ q_{k_0+1}}(\mathrm{cv}) - \mathrm{cv}}{f_{n-k_0}^{\circ q_{k_0}}(\mathrm{cv}) - \mathrm{cv}} \right| \right| \\ &+ \left| \frac{f_{n-k_0}^{\circ q_{k_0+1}}(\mathrm{cv}) - \mathrm{cv}}{f_{n-k_0}^{\circ q_{k_0}}(\mathrm{cv}) - \mathrm{cv}} - \frac{f_*^{\circ q_{k_0+1}}(\mathrm{cv}) - \mathrm{cv}}{f_*^{\circ q_{k_0}}(\mathrm{cv}) - \mathrm{cv}} \right| + \left| \left| \frac{f_*^{\circ q_{k_0+1}}(\mathrm{cv}) - \mathrm{cv}}{f_*^{\circ q_{k_0}}(\mathrm{cv}) - \mathrm{cv}} \right| - \lambda^2 \right| \\ \leq & \, C_0 \cdot \varepsilon + C_2 \cdot \varepsilon + \varepsilon = (C_0 + C_2 + 1)\varepsilon. \end{aligned}$$

This means that

$$\lambda^2 = \lim_{n\to\infty} \left| \frac{f_0^{\circ q_{n+1}}(\mathrm{cv}) - \mathrm{cv}}{f_0^{\circ q_n}(\mathrm{cv}) - \mathrm{cv}} \right|.$$

Note that the holomorphic map $f = f_0$ maps a neighborhood of cp_f to that of cv with local degree two. This completes the proof of this proposition and Theorem 1.3 modulo the statement for the three special functions. □

6. The case for the non-quadratic irrationals

In this section, we study the self-similarity of the boundaries of the Siegel disks when the rotation numbers are not necessarily quadratic irrational. Let $\alpha = [0; a_1, a_2, \cdots] \in \mathrm{HT}_N$ be an irrational of high type, where $N \geq 1/\varepsilon_3$ and ε_3 is defined in (5.4). For $f_0 := f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$, let $f_n = \mathcal{R}^{\circ n} f_0$ be the n-th near-parabolic renormalization. By definition, for each $n \geq 0$, we denote

$$h_n = f_n(e^{-2\pi \mathrm{i} \alpha_n} z) \in \mathcal{IS}_0 \cup \{Q_0\}, \tag{6.1}$$

where $\alpha_n = [0; a_{n+1}, a_{n+2}, \cdots] \in \mathrm{HT}_N$. Recall that $\mathcal{S}_N$ is the bounded type irrationals of the subset of HT_N. If $\alpha \in \mathcal{S}_N$, then there exists a subsequence of (α_n) whose limit exists. The following result is a general statement of Theorem 2.7(b).

Theorem 6.1 ([IS08, Main Theorem 3]). *Let $f_0 \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in \mathcal{S}_N$, where $N \geq 1/\varepsilon_3$. Then there exist a subsequence $(n_k)_{k\in\mathbb{N}}$ of positive integers and a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ of $(h_n)_{n\in\mathbb{N}}$ defined in* (6.1) *such that α_{n_k} converges to an irrational α_* of bounded type and h_{n_k} converges to a limit $h_* \in \mathcal{IS}_0$ exponentially fast under the metric $\mathrm{d}_{\mathrm{Teich}}$ as $k \to \infty$.*

We still use $f_*(z) := h_*(e^{2\pi \mathrm{i} \alpha_*} z)$ to denote the limit of the sequence $(f_{n_k})_{k\in\mathbb{N}}$ although f_* is not necessarily invariant under the near-parabolic renormalization. Let us continue using the notations introduced in §5.2. Recall that $d_H(\cdot,\cdot)$ denotes the Hausdorff metric on compact subsets of $\mathbb{C}$. Similar to the proof of Lemma 5.2, we have

Lemma 6.2. $\lim_{k\to\infty} \mathrm{cp}_{n_k} = \mathrm{cp}_*$ *and* $\lim_{k\to\infty} d_H(\mathcal{C}_{n_k}^{-1}, \mathcal{C}_*^{-1}) = 0$.

Recall that Υ_n is defined in (5.9) whose properties are studied §5.3. For each $k \geq 1$, we denote

$$\varrho_k := \begin{cases} \frac{\partial \Upsilon_{n_{k-1}+1}\circ\cdots\circ\Upsilon_{n_k}}{\partial \bar z}(\mathrm{cp}_{n_k}) & \text{if } n_k - n_{k-1} \text{ is odd,} \\ \frac{\partial \Upsilon_{n_{k-1}+1}\circ\cdots\circ\Upsilon_{n_k}}{\partial z}(\mathrm{cp}_{n_k}) & \text{if } n_k - n_{k-1} \text{ is even.} \end{cases}$$

Lemma 6.3. *There exists a constant* $0 < \lambda' < 1$ *such that* $0 < |\varrho_k| \leq \lambda' < 1$ *for all* $k \geq 1$.

Proof. By choosing a deeper subsequence of $(n_k)_{k\in\mathbb{N}}$ if necessary, we suppose that $n_k - n_{k-1}$ tends to infinity as $k \to \infty$. Consider the univalent (or anti-univalent) map $\Upsilon_{n_{k-1}+1} \circ \cdots \circ \Upsilon_{n_k} : \mathcal{C}_{n_k}^{-1} \to \mathcal{C}_{n_{k-1}}^{-1}$, then $\mathrm{int}\,\mathcal{C}_{n_{k-1}}^{-1} \setminus \Upsilon_{n_{k-1}+1}\circ\cdots\circ\Upsilon_{n_k}(\overline{\mathcal{C}}_{n_k}^{-1})$ is annulus whose conformal modulus tends to infinity as $k \to \infty$. Indeed, by Corollary 4.2 and the estimation in the proof of Proposition 4.1, the conformal modulus of $\mathrm{int}\,\mathcal{C}_{n-1}^{-1} \setminus \Upsilon_n(\overline{\mathcal{C}}_n^{-1})$ has a uniform lower bound for all $n \geq 1$. On the other hand, if we notice Lemma 6.2, it follows by Schwarz's lemma (for holomorphic or anti-holomorphic maps) that ϱ_k tends to 0 as $k \to \infty$. The proof is complete. □

By Lemma 6.3, there exists a large constant $A' > 1$ such that

$$\frac{1+\lambda'}{2} \cdot \left(1 - \frac{1}{A'}\right)^{-2} < 1. \tag{6.2}$$

By the continuity stated in Lemma 6.2, we have the following corollary.

Corollary 6.4. *There exist a constant* $\delta' > 0$ *depending only on* α *and* A' *(and hence on* λ'*), and an integer* k_1 *such that if* $k \geq k_1$*, then*

$$\overline{\mathbb{D}}(\mathrm{cp}_{n_k}, A'\delta') \cup \overline{\mathbb{D}}(\mathrm{cp}_*, A'\delta') \Subset \mathrm{int}(\mathcal{C}_{n_k}^{-1} \cap \mathcal{C}_*^{-1}).$$

We define the subarcs J_* and J_{n_k} in the boundaries of the Siegel disks Δ_* and Δ_{n_k} respectively as in §5.2.

Definition. Let J_* be the closed subarc of $\partial\Delta_*$ containing cp_* whose two ends are $f_*^{\circ q_M}(\mathrm{cp}_*)$ and $f_*^{\circ q_{M+1}}(\mathrm{cp}_*)$, where $M \geq 1$ is the minimal integer such that $\mathrm{diam}\, J_* \leq \delta'/2$. Similarly, for $k \geq 0$, let J_{n_k} be the closed subarc of $\partial\Delta_{n_k}$ containing cp_{n_k} whose two ends are $f_{n_k}^{\circ q_M}(\mathrm{cp}_{n_k})$ and $f_{n_k}^{\circ q_{M+1}}(\mathrm{cp}_{n_k})$.

Note that the estimation in the proof of Lemma 5.4 relies on the proof of Theorem 3.5. However, the proof of Theorem 3.5 is valid for the bounded type irrationals, not only for the quadratic irrationals. Therefore, the proof of the following statement is completely similar to that of Lemma 5.4.

Lemma 6.5. $\lim_{k\to\infty} d_H(J_{n_k}, J_*) = 0$.

For each $k \geq 1$, define

$$L_k(z) = \begin{cases} \mathrm{cp}_f + \varrho_k(\overline{z - \mathrm{cp}_f}) & \text{if } n_k - n_{k-1} \text{ is odd,} \\ \mathrm{cp}_f + \varrho_k(z - \mathrm{cp}_f) & \text{if } n_k - n_{k-1} \text{ is even.} \end{cases}$$

Proposition 6.6 (Self-similarity of the boundary of the bounded type Siegel disks)**.** *The successive blowups* $L_k^{-1}(\partial\Delta_f)$ *of the boundary of the Siegel disk of* f *converge to a Jordan curve through* ∞*, in the Hausdorff metric on compact subsets of the Riemann sphere.*

Proof. We only give a sketch of the proof since the proof is similar to that of Proposition 5.11.

Recall that $\delta' > 0$ is a constant introduced in Corollary 6.4 and $\widehat{\Psi}_n$ is a map defined in (5.10), where $n \geq 1$. By using (6.2) and following the idea of Lemma 5.10, one can prove that the sequence of *holomorphic* maps $\Theta_k := L_k^{-1} \circ \widehat{\Psi}_{n_k}$ converges uniformly to a holomorphic map Θ on $\mathbb{D}(\mathrm{cp}_*, \frac{3}{4}\delta')$. Note that we do not need the result that ϱ_k tends to a limit exponentially fast since the partial derivative of $\Upsilon_{n_{k-1}+1} \circ \cdots \circ \Upsilon_{n_k}$ in z or $\overline{z}$ at cp_{n_k} is absorbed by the definition of L_k.

By Corollary 6.4 and Lemma 6.5, there exists an integer $k' \geq k_1$ such that if $k \geq k'$, then $J_{n_k} \subset \mathbb{D}(\mathrm{cp}_{n_k}, \frac{2}{3}\delta') \subset \mathbb{D}(\mathrm{cp}_*, \frac{3}{4}\delta') \subset \mathcal{C}_{n_k}^{-1}$. Recall that $\widehat{\Psi}_{n_k}$ is defined in $\mathcal{C}_{n_k}^{-1}$. For $n \geq 1$, we define

$$I_k := \widehat{\Psi}_{n_k}(J_{n_k}).$$

By Lemma 5.6, I_k is a subset of $\partial\Delta_f$ which is a closed Jordan arc containing the critical point cp_f. If $k \geq k'$, the Hausdorff distance between $L^{-k}(I_k)$ and $L^{-(k+1)}(I_{k+1})$ tends to zero as $k \to \infty$. This means that the boundary of the Siegel disk of f is self-similar when one zooms at the critical point cp_f by a varied rate $1/\varrho_k$ at k-th time. This ends the proof of this proposition and Theorem 1.1. □

See Figure 7 for the Siegel disk of a quadratic polynomial. The rotation number is bounded but not quadratic irrational. The self-similarity at the critical point can be seen clearly.

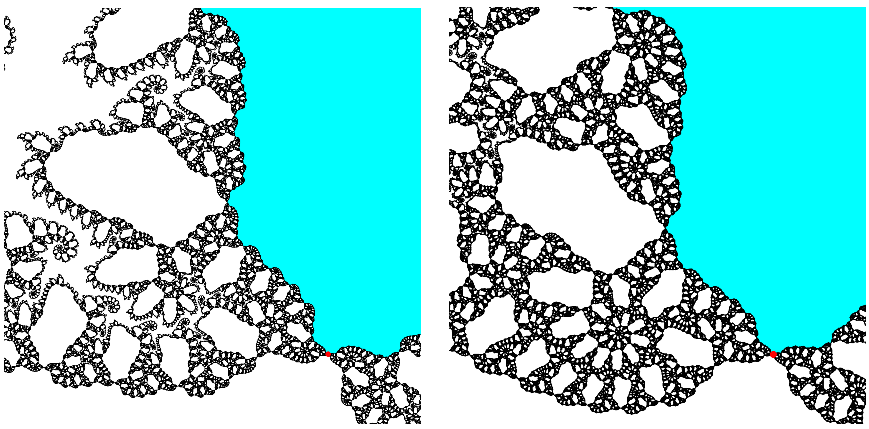

Figure 7: The Julia set (black part) and the Siegel disk (cyan part) of $P_\alpha(z) = e^{2\pi i\alpha}z + z^2$, where $\alpha = [0; 5, 8, 5, 5, 8, 5, 5, 5, 8, 5, 5, 5, 5, 8, \cdots]$ is of bounded type but not quadratic irrational. The picture on the right is the zoom of the left one near the critical point (more and more deeper and the widths of this two pictures are 0.18 and 0.027 respectively). One can obtain the clues of the self-similarity of the boundary of the Siegel disk at the critical point (the red point).

Let us emphasize again that if the rotation number is quadratic irrational, then the self-similarity of the boundaries of the Siegel disks can be obtained by zooming at the critical point with a fixed rate at each time. If the rotation number is not quadratic irrational (but is of bounded type), one can obtain the self-similarity by zooming with a more and more deeper way and with varied rates.

One may wonder if one can obtain the similar result of the dynamical similarity as in Theorem 1.3 for the case of non-quadratic irrationals. Unfortunately, the following limit is not exist in general:

$$\lim_{k\to\infty}\left|\frac{f^{\circ q_{n_{k+1}}}(\mathrm{cp}_f)-\mathrm{cp}_f}{f^{\circ q_{n_k}}(\mathrm{cp}_f)-\mathrm{cp}_f}\right|.$$

This is because $n_{k+1}-n_k$ is not a constant in general and can even tend to infinity as k tends to infinity.

7. Some applications

We will give some applications of Theorem 1.1 in this section and thence prove Corollary 1.2. For this, we introduce an invariant class $\mathcal{F}_0$ as in [IS08]

$$\mathcal{F}_0 := \left\{ f: U_f \to \mathbb{C} \;\middle|\; \begin{array}{l} 0\in U_f \text{ open} \subset \mathbb{C},\ f \text{ is holomorphic in } U_f, f(0) \\ = 0, f'(0)=1,\ f: U_f\setminus\{0\}\to\mathbb{C}^* \text{ is a branched} \\ \text{covering map with a unique critical value } cv_f, \\ \text{all critical points are of local degree 2} \end{array} \right\}.$$

Proposition 7.1 ([IS08, Proposition 5.3])**.** *There exists a natural injection*

$$\big((\mathcal{F}_0\setminus\{\textit{quadratic polynomials}\})/\underset{linear}{\sim}\big)\hookrightarrow \mathcal{IS}_0.$$

In particular, if the critical value cv_f of $f\in\mathcal{F}_0$ is normalized to $-\frac{4}{27}$, then the restriction of f on a domain is contained in $\mathcal{IS}_0$ provided f is not a quadratic polynomial.

One can refer [IS08, §3] for more information on $\mathcal{F}_0$.

Proposition 7.2. *The map $E_0(z)=ze^z:\mathbb{C}\to\mathbb{C}$ is contained in $\mathcal{F}_0$. For $\alpha\in[0,1)$, there exists a domain $U_{\widetilde{E}_\alpha}\subset\mathbb{C}$ such that the restriction of*

$$\widetilde{E}_\alpha(z) := e^{2\pi \mathrm{i}\alpha} z\, e^{\frac{27}{4e}e^{2\pi\mathrm{i}\alpha}z}$$

on $U_{\widetilde{E}_\alpha}\subset\mathbb{C}$ is contained in $\mathcal{IS}_\alpha$.

Proof. The map E_0 is holomorphic on $\mathbb{C}$, $E_0(0)=0$, $E_0'(0)=1$ and $E_0:\mathbb{C}^*\to\mathbb{C}^*$ is a branched covering map with a unique critical value $-1/e$ and the unique critical point -1 is of local degree 2. According to the definition of $\mathcal{F}_0$, we know that $E_0:\mathbb{C}\to\mathbb{C}$ is in $\mathcal{F}_0$.

Note that the unique critical value of the conjugacy $\widetilde{E}_0(z)=z\,e^{\frac{27}{4e}z}$ of E_0 is normalized to $-\frac{4}{27}$. By Proposition 7.1, there exists a domain $U_E\subset\mathbb{C}$ such that the restriction of $\widetilde{E}_0$ on U_E is contained in $\mathcal{IS}_0$ since $\widetilde{E}_0$ is not a quadratic polynomial. Note that $\widetilde{E}_\alpha(z)=\widetilde{E}_0(e^{2\pi\mathrm{i}\alpha}z)$. Hence for all $\alpha\in[0,1)$, the restriction of $\widetilde{E}_\alpha$ on $e^{-2\pi\mathrm{i}\alpha}\cdot U_E$ is contained in $\mathcal{IS}_\alpha$ by (2.8). □

Let $S_\alpha(z)=e^{2\pi\mathrm{i}\alpha}\sin(z)$ be the sine family defined on $\mathbb{C}$, where $\alpha\in[0,1)$. Obviously, the restriction of $S_0(z)=\sin(z)$ on any domain (containing 0) in $\mathbb{C}$ is not contained in $\mathcal{IS}_0$ since every map in $\mathcal{IS}_0$ has exactly one attracting axis at the origin but S_0 has two. This means that the restriction of S_α on any domain in $\mathbb{C}$ is not contained in $\mathcal{IS}_\alpha$ for any α. Note that S_α is an odd function, the set of critical points of S_α is $\{\pi/2 + k\pi : k \in \mathbb{Z}\}$ and S_α has exactly two critical values $\pm e^{2\pi i\alpha}$. So we can consider the semi-conjugacy of $S_{\alpha/2}$ as following: there exists an entire function $\widetilde{S}_\alpha$ such that $\widetilde{S}_\alpha \circ h = h \circ S_{\alpha/2}(z)$, where $h(z) = -4z^2/(27e^{2\pi i\alpha})$. Thus we have

$$\widetilde{S}_\alpha(z) = -\frac{4}{27}\sin^2\sqrt{-\frac{27e^{2\pi i\alpha}}{4}z}. \tag{7.1}$$

Proposition 7.3. *The restriction of $\widetilde{S}_\alpha$ on a domain is contained in $\mathcal{IS}_\alpha$ for $\alpha \in [0, 1)$.*

Proof. Note that $\widetilde{S}_0(z) = -\frac{4}{27}\sin^2\sqrt{-\frac{27z}{4}}$ is an entire function. We first prove that the restriction of $\widetilde{S}_0$ on some domain is contained in $\mathcal{F}_0$. Define

$$Dom(\widetilde{S}_0) := \mathbb{C} \setminus \{-\tfrac{4}{27}k^2\pi^2 : k \in \mathbb{N}^+\}.$$

Then $0 \in Dom(\widetilde{S}_0)$, $\widetilde{S}_0(0) = 0$ and $\widetilde{S}_0'(0) = 1$. Moreover, $\widetilde{S}_0 : Dom(\widetilde{S}_0) \setminus \{0\} \to \mathbb{C}^*$ is a branched covering map with a unique critical value $-\frac{4}{27}$ and all critical points are of local degree 2. Therefore, the restriction of $\widetilde{S}_0$ on $Dom(\widetilde{S}_0)$ is contained in $\mathcal{F}_0$. By Proposition 7.1, there exists a domain $U_S \subset \mathbb{C}$ such that the restriction of $\widetilde{S}_0$ on U_S is contained in $\mathcal{IS}_0$ since $\widetilde{S}_0$ is not a quadratic polynomial. Note that $\widetilde{S}_\alpha(z) = \widetilde{S}_0(e^{2\pi i\alpha}z)$. Hence for all $\alpha \in [0, 1)$, the restriction of $\widetilde{S}_\alpha$ on $e^{-2\pi i\alpha} \cdot U_S$ is contained in $\mathcal{IS}_\alpha$ by (2.8). □

Proof of Corollary 1.2. Although the quadratic polynomial P_α with $\alpha \in \mathrm{HT}_N$ does not belong to the Inou-Shishikura class, its near-parabolic renormalization does: $\mathcal{R}P_\alpha \in \mathcal{IS}_{1/\alpha}$ (see [IS08, §3] or Theorem 2.6(a)). The statement about P_α follows from Theorem 1.1. The result on E_α is an immediate corollary of Proposition 7.2. The statement about $\widetilde{S}_\alpha$ is also true by Proposition 7.3. Note that $S_{\alpha/2}$ is semi-conjugated to $\widetilde{S}_\alpha$ by the map $h(z) = -4z^2/(27e^{2\pi i\alpha})$ and h is univalent in a small neighborhood of $\pi/2$ or $-\pi/2$. This means that the boundary of the Siegel disks of $S_{\alpha/2}$ is a Jordan curve and self-similar at the critical point $\pi/2$ or $-\pi/2$. □

Proof of the second part of Theorem 1.3. The first part of Theorem 1.3 has been proved in Proposition 5.12. For the second part, the statement of P_α and E_α is true since these two maps can be iterated infinitely times under $\mathcal{R}$. The statement for $\widetilde{S}_\alpha$ is also true by Proposition 7.3. The sine family $S_{\alpha/2}$ is semi-conjugated to $\widetilde{S}_\alpha$ by the holomorphic map h which is univalent in a small neighborhood of $\pi/2$ or $-\pi/2$. This means that the limit in this theorem exists for $S_{\alpha/2}$. □

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