SIEGEL DISKS AND RELATED TOPICS

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Abstract. We survey some results on Siegel disks.

Discrete dynamical systems:

$$f : X \to X$$
$$x \mapsto f(x) \mapsto f(f(x)) \mapsto \cdots \mapsto f^n(x) = f(f^{n-1}(x)) \mapsto \cdots .$$

Complex dynamical systems: X complex manifold, f holomorphic

- rational: $z^2 + c$
- transcendental entire: λe^z, $\lambda \sin(z)$
- transcendental meromorphic: $\lambda \tan(z)$

History (1-dim):

- Newton (1670s): $N_f(z) = z - \dfrac{f(z)}{f'(z)}$ (examples vs theory)
- Fatou, Julia (1920s): based on Montel theorem, Kleinian groups.

– Poincaré (1880s). Nonlinear holomorphic germ $f : U \to \mathbb{C}$:

$$f(z) = \lambda z + \mathcal{O}(z^2), \quad \lambda \neq 0.$$

Question 1. Whether f is locally linearizable? (Applications in celestial mechanics).

$$\begin{array}{ccc} D & \xrightarrow{\ f\ } & f(D) \\ \big\downarrow {\scriptstyle ?\,\varphi} & & \big\downarrow {\scriptstyle \varphi} \\ \mathbb{D} & \xrightarrow{z \mapsto \lambda z} & \lambda\mathbb{D} \end{array}$$

- (Kœnigs, 1884 [Koe84]) If $|\lambda| \neq 1$, yes.
- (Leau-Fatou, 1897 [Lea97], [Fat19]) If $\lambda = e^{2\pi i\alpha}, \alpha \in \mathbb{Q}$, no, except $f^n = \mathrm{id}$ for some $n \geqslant 1$.

1. Irrationally indifferent fixed points

Set $\lambda = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Question 2. Does the dynamics of $f(z) = \lambda z + \mathcal{O}(z^2)$ behave as $R_\alpha(z) = \lambda z = e^{2\pi i\alpha} z$?

- Kasner (1912): Always yes?
- Pfeiffer (1917): Sometimes no?
- Julia (1919): Always no for nonlinear rational maps?

Theorem 1.1 (Cremer, 1928, Math. Ann. [Cre28]). *$\exists\, \alpha \in \mathbb{R} \backslash \mathbb{Q}$, s.t. φ does not exist for any non-linear polynomial.*

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Proof. Let $f(z) = \lambda z + \cdots + z^d$, where $\lambda = e^{2\pi \mathrm{i}\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $d \geqslant 2$. Suppose the conjugacy φ is defined in $\mathbb{D}(0, \delta)$ with $0 < \delta < 1$, i.e.,

$$\begin{array}{ccc} \mathbb{D}(0,\delta) & \xrightarrow{\ f\ } & \cdot \\ \Big\downarrow \varphi & & \Big\downarrow \varphi \\ \cdot & \xrightarrow{\ R_\alpha\ } & \cdot \end{array}$$

Consider the fixed points of f^q, where $q \geqslant 1$, they satisfy

$$f^q(z) - z = z^{d^q} + \cdots + (\lambda^q - 1)z = 0.$$

Denote them by $0, z_1, \cdots, z_{d^q-1}$. Note that $f^q(z) = \varphi^{-1}(\lambda^q \varphi(z))$ has no fixed point in $\mathbb{D}(0, \delta)$ except 0. Hence

$$\delta^{d^q} \leqslant \delta^{d^q-1} \leqslant \prod_{j=1}^{d^q-1} |z_j| = |\lambda^q - 1|, \quad \forall\, q \geqslant 1. \tag{1.1}$$

Now we construct an $\alpha \in (0, 1) \setminus \mathbb{Q}$ such that (1.1) does not hold.

Take a sequence of positive integers:

$$2 \leqslant q_1 < q_2 < \cdots < q_k < \cdots, \qquad \text{with } q_{k+1} > q_k + 1.$$

and denote

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{2^{q_k}}. \tag{1.2}$$

Note that if $0 \leqslant \alpha \leqslant \frac{1}{2}$,

$$4\alpha \leqslant \left| e^{2\pi \mathrm{i}\alpha} - 1 \right| = 2 \left| \frac{e^{\pi \mathrm{i}\alpha} - e^{-\pi \mathrm{i}\alpha}}{2} \right| = 2\,|\sin(\pi\alpha)| \leqslant 2\pi\alpha.$$

Hence

$$\left| \lambda^{2^{q_k}} - 1 \right| = \left| e^{2\pi \mathrm{i} \cdot 2^{q_k} \sum\limits_{m=k+1}^{\infty} \frac{1}{2^{q_m}}} - 1 \right| \asymp 2^{q_k - q_{k+1}}.$$

Taking logarithm of (1.1), and setting $q = 2^{q_k}$, we have

$$d^{2^{q_k}} \log \delta \leqslant (q_k - q_{k+1}) \log 2 + C, \quad \text{where } C > 1 \text{ is universal.}$$

This implies

$$q_{k+1} \leqslant q_k + \frac{C}{\log 2} + \frac{\log \frac{1}{\delta}}{\log 2} d^{2^{q_k}}, \quad \forall\, k \geqslant 1. \tag{1.3}$$

Suppose $q_k \to +\infty$ very fast, e.g., $\log q_{k+1} \geqslant k \cdot 2^{q_k}$. Then, for α defined in (1.2), (1.3) does not hold for any $d \geqslant 2$ and any $0 < \delta < 1$. □

Remark (Cremer)**.** If α is very "close" to a rational number, e.g., if

$$\limsup_{q \to +\infty} \left(\frac{1}{|\lambda^q - 1|} \right)^{\frac{1}{d^q}} = +\infty$$

for an integer $d \geqslant 2$, then any rational map $f(z) = \lambda z + \mathcal{O}(z^2)$ of degree d is not locally linearizable at 0.

Theorem 1.2 (Siegel, 1942, Ann. Math. [Sie42])**.** *If*

$$\frac{1}{|\lambda^q - 1|} < c\, q^{\kappa-1}$$

for some $c > 0$, $\kappa \geqslant 2$ and all integers $q \geqslant 1$, then any holomorphic germ $f(z) = \lambda z + \mathcal{O}(z^2)$ is locally linearizable at 0.

Idea of the proof: Based on $\varphi(f(z)) = \lambda\varphi(z)$, prove that the convergence radius of $\varphi(z) = z + \sum_{n\geqslant 1} b_n z^n$ is positive.

For any $q \geqslant 1$, let $p \in \mathbb{Z}$, s.t. $|q\alpha - p| \leqslant \frac{1}{2}$. Then

$$4|q\alpha - p| \leqslant |\lambda^q - 1| = \left|e^{2\pi \mathrm{i}(q\alpha - p)} - 1\right| = 2|\sin(\pi(q\alpha - p))| \leqslant 2\pi|q\alpha - p|.$$

Hence

$$\frac{1}{|\lambda^q - 1|} < cq^{\kappa-1} \Leftrightarrow |\lambda^q - 1| > \frac{\varepsilon'}{q^{\kappa-1}} \Leftrightarrow |q\alpha - p| > \frac{\varepsilon}{q^{\kappa-1}}$$
$$\Leftrightarrow \left|\alpha - \frac{p}{q}\right| > \frac{\varepsilon}{q^{\kappa}}.$$

Definition (Diophantine of order $\leqslant \kappa$)**.**

$$\mathcal{D}(\kappa) := \left\{\alpha \in \mathbb{R}\setminus\mathbb{Q} \;\middle|\; \exists\, \varepsilon > 0, \kappa \geqslant 2 \text{ s.t. } \left|\alpha - \frac{p}{q}\right| > \frac{\varepsilon}{q^{\kappa}},\ \forall \frac{p}{q} \in \mathbb{Q}\right\}.$$

Obviously, $\mathcal{D}(\kappa) \subset \mathcal{D}(\kappa')$ if $\kappa < \kappa'$.

Remark. (1) Every algebraic number is Diophantine (by Liouville).

Indeed, let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ be a root of the polynomial g of degree $d \geqslant 1$ with integer coefficients. Without loss of generality, assume that $g(p/q) \neq 0$, $\forall\, p/q \in \mathbb{Q}$. Then

$$\frac{1}{q^d} \leqslant \left|g\left(\frac{p}{q}\right)\right| = \left|g(\alpha) - g\left(\frac{p}{q}\right)\right| \leqslant M\left|\alpha - \frac{p}{q}\right|.$$

Hence $\sqrt{2} \in \mathcal{D}(2)$ and $\sqrt[3]{2} \in \mathcal{D}(3)$.

(2) $\mathcal{D}(2+) := \bigcap_{\kappa>2} \mathcal{D}(\kappa)$ has full measure in $\mathbb{R}$.

(3) $\mathcal{D}(2) \neq \emptyset$ has zero measure.

(4) $\mathcal{D}(\kappa) = \emptyset$, $\forall\, 0 < \kappa < 2$.

- Continued fraction expansion

$$(0,1)\setminus\mathbb{Q} \ni \alpha = [a_1, a_2, a_3, \cdots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}},$$

where $a_n \in \mathbb{Z}_+$. The n-th convergent to α:

$$\frac{p_n}{q_n} = [a_1, a_2, \cdots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}},$$

where p_n, $q_n \in \mathbb{Z}_+$ are coprime integers.

Note

$$\frac{p_1}{q_1} = \frac{1}{a_1}, \quad \frac{p_2}{q_2} = \frac{1}{a_1 + \frac{1}{a_2}} = \frac{a_2}{a_1 a_2 + 1}, \quad \cdots.$$

Inductively, for $n \geqslant 2$,

$$q_n = a_n q_{n-1} + q_{n-2}, \quad q_0 = 1, \quad q_1 = a_1,$$
$$p_n = a_n p_{n-1} + p_{n-2}, \quad p_0 = 0, \quad p_1 = 1.$$

By studing the closest returns of $R_\alpha(z) = e^{2\pi \mathrm{i}\alpha} z$, one has

(1) $\left|\lambda^k - 1\right| > |\lambda^{q_n} - 1|, \quad \forall\, k = 1, 2, \cdots, q_{n+1} - 1$ and $k \neq q_n$.

(2) $\frac{2}{q_{n+1}} \leqslant |\lambda^{q_n} - 1| \leqslant \frac{2\pi}{q_{n+1}}, \quad \forall n \geqslant 1.$

Hence

$$\alpha \in \mathcal{D}(\kappa) \Leftrightarrow q_{n+1} \leqslant C q_n^{\kappa-1},$$

where $C > 0$ is a constant. In particular,

$$\alpha \in \mathcal{D}(2) \Leftrightarrow q_{n+1} = a_{n+1} q_n + q_{n-1} \leqslant C q_n \Leftrightarrow \sup_n \{a_n\} < +\infty.$$

Every $\alpha \in \mathcal{D}(2)$ is called *bounded type* (or *constant type*). For example,

$$\alpha = \frac{\sqrt{5} - 1}{2} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}} \in \mathcal{D}(2).$$

Remark. Quadratic irrational $\Leftrightarrow$ $\{a_n\}_{n \geqslant 1}$ is eventually periodic $\Leftrightarrow$ α is the root of a quadratic polynomial with integer coefficients. ($\sqrt{2} = 1 + [2, 2, 2, \cdots]$)

Theorem 1.3 (Brjuno, 1965 [Brj65], [Brj71]; Rüssman, 1967 [Rüs67])**.** *If*

$$\alpha \in \mathcal{B} := \left\{ \alpha \in \mathbb{R} \setminus \mathbb{Q} \;\middle|\; \sum_{n=1}^{\infty} \frac{\log(q_{n+1})}{q_n} < +\infty \right\},$$

then any holomorphic germ $f(z) = e^{2\pi \mathrm{i}\alpha} z + \mathcal{O}(z^2)$ is locally linearizable.

Remark. (1) We have

$$\mathcal{D}(2) \subsetneq \mathcal{D}(2+) \subsetneq \underset{\forall\, \kappa > 2}{\mathcal{D}(\kappa)} \subsetneq \mathcal{D} := \bigcup_{\kappa \geqslant 2} \mathcal{D}(\kappa) \subsetneq \mathcal{B}.$$

(2) If $a_{n+1} = e^{a_n}$, then $\alpha \in \mathcal{B} \setminus \mathcal{D}$.

Theorem 1.4 (Yoccoz, 1988 [Yoc88], [Yoc95])**.** *If $\alpha \notin \mathcal{B}$, then $f(z) = e^{2\pi \mathrm{i}\alpha} z + z^2$ is not locally linearizable at 0. Moreover, f has the small cycles property: every neighborhood of 0 contains infinitely many periodic orbits.*

For $f(z) = e^{2\pi \mathrm{i}\alpha} z + \mathcal{O}(z^2)$, the origin is called a

- *Siegel point*, if f is locally linearizable at 0;
- *Cremer point*, otherwise.

Theorem 1.5 (Pérez-Marco, 1990 [Pér90])**.** *We have*

(1) *If*

$$\sum_{n=1}^{\infty} \frac{\log \log(q_{n+1})}{q_n} < \infty,$$

then every germ $f(z) = e^{2\pi \mathrm{i}\alpha} z + \mathcal{O}(z^2)$ which has a Cremer point at 0 has the small cycles property.

(2) *If the sum above diverges, then $\exists\, f(z) = e^{2\pi \mathrm{i}\alpha} + \mathcal{O}(z^2)$ which has a Cremer point at 0 but has no small cycles.*

Conjecture 1 (Douady, 1980s)**.** Suppose $f(z) = e^{2\pi i\alpha}z + \mathcal{O}(z^2)$ is a non-Möbius rational map / transcendental entire function having a Siegel point at 0. Then $\alpha \in \mathcal{B}$.

Known results:

- Pérez-Marco (1993 [Pér93]): Structurally stable polynomials.
- Geyer (2001, 2019 [Gey01, Gey19]) and Okuyama (2001, [Oku01]): Julia-saturated polynomials, including
$$f(z) = \lambda z\left(1 + \frac{z}{d}\right)^d \quad \text{and} \quad g(z) = \lambda\frac{(1+z)^d - 1}{d}.$$
- Geyer (2001, [Gey01]) and Okuyama (2005 [Oku05]): Structurally finite transcendental entire functions, including $f(z) = \lambda z e^z$.
- Manlove (2015): Julia-saturated rational maps.

The proofs are based on Yoccoz's result.

Remark. Douady's conjecture is still open for cubic polynomials $\lambda z + a_2 z^2 + z^3$, the sine family $\lambda \sin z$, and the exponential family $\lambda(e^z - 1)$.

Question 3. Does there exist $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, s.t. 0 is a Cremer point for any nonlinear transcendental function $f(z) = e^{2\pi i\alpha}z + \mathcal{O}(z^2)$?

2. Siegel disks

Definition. For
$$f(z) = e^{2\pi i\alpha}z + \mathcal{O}(z^2) : 0 \in U \to \mathbb{C},$$
The *Siegel disk* Δ_f of f at 0 is the biggest open subset of U containing 0 on which f is analytically conjugate to $R_\alpha(z) = e^{2\pi i\alpha}z$ (i.e., locally linearizable).

Each Δ_f is simply connected, cannot contain any periodic or critical points.

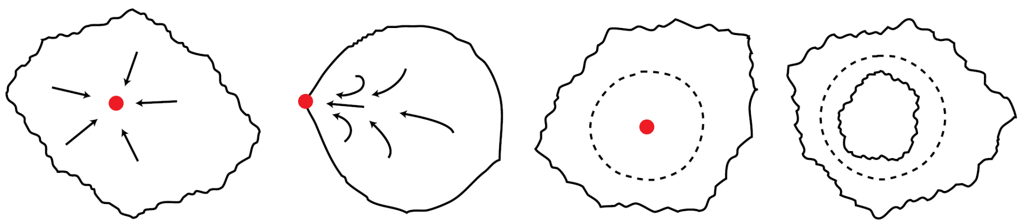
Conjecture 2 (Douady-Sullivan, 1986 [Dou87])**.** The Siegel disks of rational maps (deg $\geqslant 2$) are Jordan domains.

– Motivation

For a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, the Fatou set and Julia set are defined by (other definitions including equicontinuous, closure of repelling periodic points etc):

$$F(f) := \{z \in \widehat{\mathbb{C}} : \{f^n\}_{n\in\mathbb{N}} \text{ is a normal family in a neighborhood of } z\},$$
$$J(f) := \widehat{\mathbb{C}} \setminus F(f).$$

Classification of periodic Fatou components of rational maps (by Sullivan [Sul85]):



attracting basin — parabolic basin — Siegel disk — Herman ring

- The dynamics in $F(f)$ has been completely understood (no Baker and wandering domains);
- The dynamics on $J(f)$ is difficult.

Let $\mathrm{Crit}(f)$ be the set of critical points and the *postcritical set* of f is

$$\mathcal{P}(f) := \overline{\bigcup_{n\geqslant 1} f^n(\mathrm{Crit}(f))}.$$

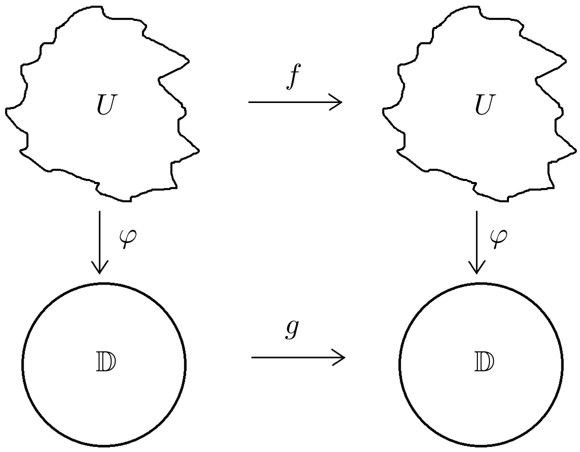
They play an essential role in the dynamics of f (contraction vs expanding). By Fatou (1920s):

- $U \cap \mathrm{Crit}(f) \neq \emptyset$ if U is an attracting or parabolic basin (i.e., critical periodic Fatou components).
- $\partial U \subset \mathcal{P}(f)$ if U is a Siegel disk or Herman ring.

By qc surgery (Shishikura [Shi87]), Siegel disks and Herman rings can be transformed into each other.

Theorem 2.1 (McMullen, 1988 [McM88]). *Let U be a simply connected fixed Fatou component of a rational map f with $\deg(f) \geqslant 2$. Then $f : U \to U$ is conformally conjugate to a Blaschke product $g : \mathbb{D} \to \mathbb{D}$, where*

$$g(z) = e^{\mathrm{i}\theta} \prod_{i=1}^{n} \frac{z - a_i}{1 - \overline{a}_i z}, \quad \textit{where } a_i \in \mathbb{D}.$$



By Carathéodory, the dynamics of $g|_{\partial\mathbb{D}}$ can be transferred to $f|_{\partial U}$ continuously (resp. homeomorphically) if ∂U is locally connected (resp. a Jordan curve).

LC of ∂U implies Jordan in the following cases:

- All bounded Fatou components (attracting, parabolic, Siegel) of polynomials;
- Siegel disks of rational maps.

Theorem 2.2 (Roesch-Yin, 2008 [RY08, RY22]). *For polynomials, all bounded critical Fatou components are Jordan domains.*

– Quadratic Siegel disks (Topology of $\partial\Delta_f$ and the position of critical points)

An orientation-preserving homeomorphism $h : \mathbb{R} \to \mathbb{R}$ is called *quasisymmetric* if $\exists\, k > 1$ s.t.

$$\frac{1}{k} \leqslant \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leqslant k, \quad \forall\, x \in \mathbb{R},\ \forall\, t > 0.$$

One can define quasisymmetric for a homeomorphism $g : \mathbb{T} \to \mathbb{T}$ similarly via the following diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\;h\;} & \mathbb{R} \\ \Big\downarrow{\scriptstyle e^{2\pi i x}} & & \Big\downarrow{\scriptstyle e^{2\pi i x}} \\ \mathbb{T} & \xrightarrow{\;g\;} & \mathbb{T}. \end{array}$$

The *rotation number* of $g : \mathbb{T} \to \mathbb{T}$ is

$$\mathrm{rot}(g) := \lim_{n\to+\infty} \frac{h^n(0)}{n} \mod 1.$$

Theorem 2.3 (Herman-Świątek, 1986 [Her86], [Świ88])**.** *Let $g : \mathbb{T} \to \mathbb{T}$ be a real-analytic critical circle homeomorphism of rotation number α. Then g is quasisymmetrically conjugate to $R_\alpha(z) = e^{2\pi i\alpha} z$ if and only if α is of bounded type.*

Now we show how to use qc surgery transforming cubic Blaschke products to a quadratic polynomials.

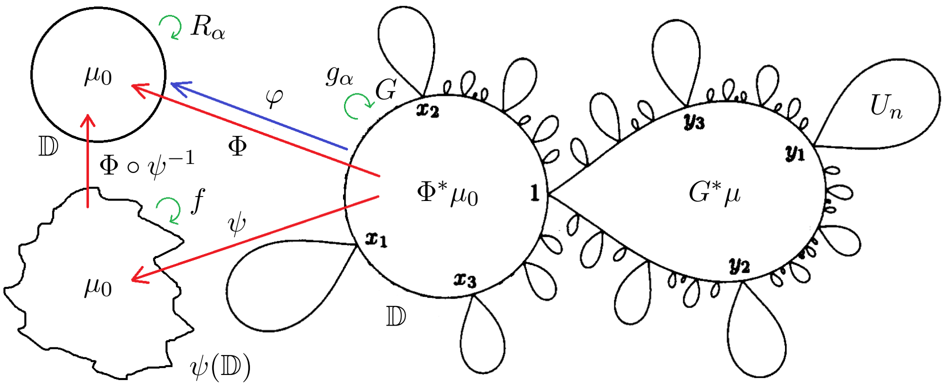
It is not hard to check that

$$z \mapsto z^2 \frac{z-3}{1-3z} : \mathbb{T} \to \mathbb{T}$$

is a real-analytic critical circle homeomorphism. $\forall\, \alpha \in (0,1) \setminus \mathbb{Q}$, $\exists 1\ \tau(\alpha) \in (0,1)$ s.t.

$$g_\alpha(z) = e^{2\pi i \tau(\alpha)} z^2 \frac{z-3}{1-3z},$$

has rotation number α.



Let $\alpha \in \mathcal{D}(2)$. By Herman-Świątek, $\exists$ quasisymmetric map $\varphi : \mathbb{T} \to \mathbb{T}$ s.t.

$$\varphi \circ g_\alpha \circ \varphi^{-1}(z) = R_\alpha(z) = e^{2\pi i\alpha} z.$$

Then $\exists$ a homeomorphism $\Phi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ s.t.

$$\Phi : \mathbb{D} \to \mathbb{D} \text{ is qc}, \quad \Phi|_{\mathbb{T}} = \varphi, \quad \Phi(0) = 0.$$

Define

$$G(z) := \begin{cases} g_\alpha(z) & z \in \widehat{\mathbb{C}} \setminus \mathbb{D} \\ \Phi^{-1} \circ R_\alpha \circ \Phi(z), & z \in \mathbb{D}. \end{cases}$$

Then $G : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasi-regular map of degree 2. Let μ_0 be the standard ellipse field. Define $\mu = \Phi^*\mu_0$ in $\mathbb{D}$. Then μ is invariant in $\mathbb{D}$ under G:

$$G^*\mu = (\Phi^{-1} \circ R_\alpha \circ \Phi)^*\mu = \Phi^*\Big(R_\alpha^*\big((\Phi \circ \Phi^{-1})^*\mu_0\big)\Big) = \Phi^*\mu_0 = \mu.$$

$\forall\, z$ in the drop $U_n \subset \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ which is mapped onto $\mathbb{D}$ by G^n, where $n \geqslant 1$ is minimal, define μ at z as $\mu = (G^n)^*\mu = (g_\alpha^n)^*\mu$. In the rest place, define $\mu = \mu_0$. Then $G^*\mu = \mu$ on $\widehat{\mathbb{C}}$. Note that $\|\mu\|_\infty < 1$. By MRMT (measurable Riemann mapping theorem),

$$\exists\, \text{qc}\ \psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}, \quad \text{s.t.} \quad \psi^*\mu_0 = \mu \text{ and } \psi \text{ fixes } 0, 1, \infty.$$

Note that

$$\begin{array}{ccc} (\widehat{\mathbb{C}}, \mu) & \xrightarrow{G} & (\widehat{\mathbb{C}}, \mu) \\ \downarrow \psi & & \downarrow \psi \\ (\widehat{\mathbb{C}}, \mu_0) & \xrightarrow{f} & (\widehat{\mathbb{C}}, \mu_0). \end{array}$$

Then $f = \psi \circ G \circ \psi^{-1} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quadratic rational map. By

- $f^{-1}(\infty) = \infty$, $f(0) = 0$,
- $f : \psi(\mathbb{D}) \to \psi(\mathbb{D})$ is conjugate to $R_\alpha : \mathbb{D} \to \mathbb{D}$ by a conformal map $\Phi \circ \psi^{-1} : \psi(\mathbb{D}) \to \mathbb{D}$, and
- 1 is a critical point of f,

we have $f(z) = e^{2\pi i \alpha}(z - \frac{z^2}{2})$.

Theorem 2.4 (Douady-Herman, 1986 [Dou87]). *$\forall\, \alpha \in \mathcal{D}(2)$, the boundary of the Siegel disk Δ_α of $P_\alpha(z) = e^{2\pi i \alpha} z + z^2$ is a quasi-circle passing through the critical point $\omega_\alpha = -\frac{e^{2\pi i \alpha}}{2}$.*

By a similar method but considering

$$g(z) = e^{2\pi i \tau} z^2 \frac{z - a}{1 - az}, \quad a > 3,$$

we have

Theorem 2.5 (Herman, 1986). *$\exists\, \alpha \in \mathcal{B} \setminus \mathcal{D}(2)$, s.t. $\partial\Delta_\alpha$ of P_α is a quasi-circle not passing through ω_α.*

By studying more general Blaschke products, Douady-Herman's result has been extended to

- Zakeri (1999) [Zak99]: cubic polynomials,
- Shishikura (2001) [Shi01]: all polynomials.

Theorem 2.6 (Zhang, Invent. Math. 2011 [Zha11]). *Every bounded type Siegel disk of a rational map* (deg $\geqslant 2$) *is bounded by a quasi-circle passing through at least one critical point.*

The proof: invariant curves in Δ_f are uniform quasi-circles.
Difficulty: The quasisymmetric constants have no uniform bound.

Remark. (1) Douady-Sullivan's conjecture holds for bounded type α.
(2) Petersen (2004) [Pet04]: The inverse of Zhang's result is also true.

Theorem 2.7 (Zakeri, Duke Math. J. 2010 [Zak10]). *Every bounded type Siegel disk centered at 0 of $f(z) = P(z)e^{Q(z)}$, where P, Q are polynomials, is bounded by a quasi-circle in $\mathbb{C}$ which contains at least one critical point of f.*

Remark. (1) Constructing transcendental meromorphic Blaschke models.
(2) This generalizes the results of

- Geyer (2001) [Gey01]: $\lambda z e^z$; and
- Keen-Zhang (2009) [KZ09]: $(\lambda z + az^2)e^z$.

Some other families on this topic (for bounded type α): the sine family $\lambda \sin z$ (Zhang, 2005 [Zha05]) and transcendental entire functions with 3 singular values (Chéritat-Epstein, 2018 [CE18]).

Theorem 2.8 (G. David, 1988 [Dav88], Solutions de l'equation de Beltrami avec $\|\mu\|_\infty = 1$)**.** *Suppose $\mu \in L^\infty(\mathbb{C})$, $\exists\, \varepsilon_0, C, \alpha > 0$, s.t. $\forall\, 0 < \varepsilon \leqslant \varepsilon_0$,*

$$\text{area}(\{z \in \mathbb{C} : |\mu(z)| > 1 - \varepsilon\}) \leqslant C \cdot e^{-\frac{\alpha}{\varepsilon}}. \tag{2.1}$$

Then $\exists$ homeomorphism $h : \mathbb{C} \to \mathbb{C}$, s.t. $\frac{\partial h}{\partial \bar{z}} = \mu \frac{\partial h}{\partial z}$ and h is unique if requiring $h(0) = 0$ and $h(1) = 1$.

Recall the qc surgery process of transforming g_α into P_α with $\alpha = [a_1, a_2, \cdots] \in (0,1) \setminus \mathbb{Q}$, especially about the definition of μ.

Theorem 2.9 (Petersen-Zakeri, Ann. Math. 2004 [PZ04])**.** *If $\log a_n \leqslant \mathcal{O}(\sqrt{n})$ as $n \to \infty$, then the globally defined μ satisfies* (2.1)*, and moreover, $\partial\Delta_\alpha$ of P_α is a David-circle passing through the critical point ω_α.*

Remark. (1) PZ has full measure in $(0,1)$, where

$$\text{PZ} := \{\alpha \in (0,1) \setminus \mathbb{Q} : \log a_n \leqslant \mathcal{O}(\sqrt{n}) \text{ as } n \to \infty\}.$$

(2) $\mathcal{D}(2) \subsetneq \text{PZ} \subsetneq \mathcal{D}(2+) = \bigcap_{\kappa>2} \mathcal{D}(\kappa)$.

Petersen-Zakeri's result has been generalized to

- All polynomials and $\lambda \sin z$ by Zhang (2014, 2016) [Zha14], [Zha16];
- Special α satisfying

$$\log^2 a_n \leqslant n \log n \cdot \log\log n \cdots \underbrace{\log\log \cdots \log n}_{k \text{ times}},$$

  by L. Shen (2018) [She18], based on studying degenerated Beltrami equations due to Astala-Iwaniec-Martin.

Theorem 2.10 (Avila-Buff-Chéritat, Acta Math. 2004 [ABC04])**.** *$\exists\, \alpha \in \mathcal{B} \setminus \text{PZ}$, s.t. $\partial\Delta_\alpha$ of P_α is C^∞-smooth.*

- Buff-Chéritat (2007) [BC07]: $\exists\, \alpha$, s.t. $\partial\Delta_\alpha$ is C^γ but not $C^{\gamma+1}$.

Question 4. Does there exist $\alpha \in \mathcal{B}$, s.t. H-dim$(\partial\Delta_f) = 2$?

High type irrational numbers:

$$\text{HT}_N := \{\alpha = [a_1, a_2, \cdots] \in (0,1) \setminus \mathbb{Q} \mid a_n \geqslant N, \forall\, n \geqslant 1\}.$$

Theorem 2.11 (Shishikura-Y., Cheraghi; arXiv, 2021 [SY21], [Che22])**.** *For any sufficiently high type α, if $P_\alpha(z) = e^{2\pi i\alpha} z + z^2$ has a Siegel disk Δ_α, then*

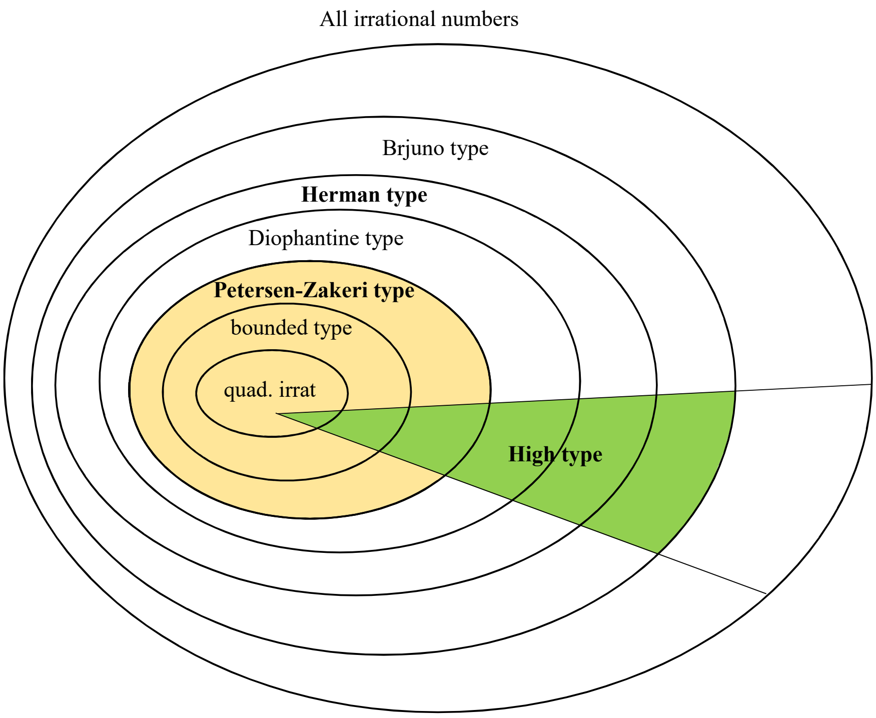
- *$\partial\Delta_\alpha$ is a Jordan curve; and*
- *$\partial\Delta_\alpha$ contains a critical point if and only if α is of Herman type.*

The proof: Constructing continuous curves converging to the boundary, by *parabolic renormalization* (Inou-Shishikura [IS08]).

High type numbers has non-empty intersection with usual types of irrationals.

Theorem 2.12 (Chéritat, Math. Ann., 2011 [Ché11])**.** *$\exists\, f(z) = e^{2\pi i\alpha} z + \mathcal{O}(z^2)$ whose Siegel disk Δ_f is compactly contained in Dom(f) but $\partial\Delta_f$ is a pseudo-circle, which is not locally connected.*

The proof: Range's approximate theorem.



3. Local connectivity of Julia sets with Siegel disks

Motivation:

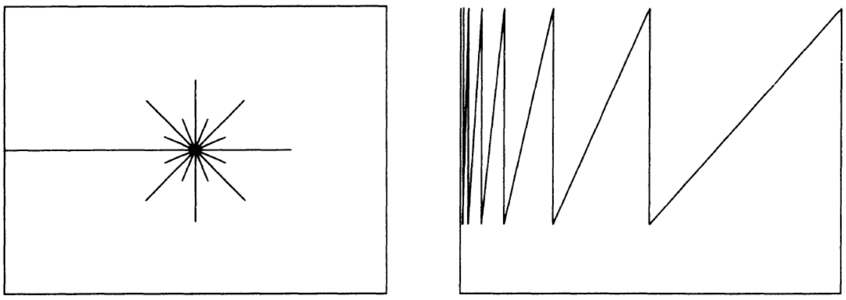
LC of $J(f)$ implies:

- well understanding of global dynamics (dynamics from $F(f)$ to $J(f)$ continuously);
- combinatoric model for $J(f)$ of polynomials.

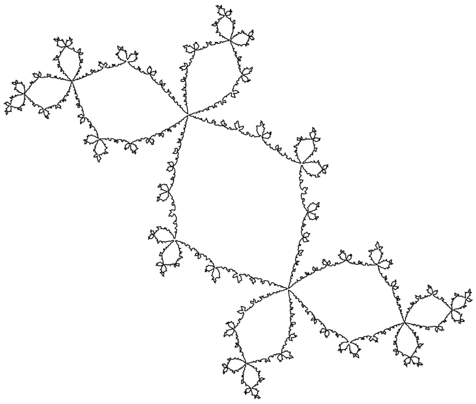
Two ways to prove local connectivity: The first is proving the local connectivity point by point. The second is:

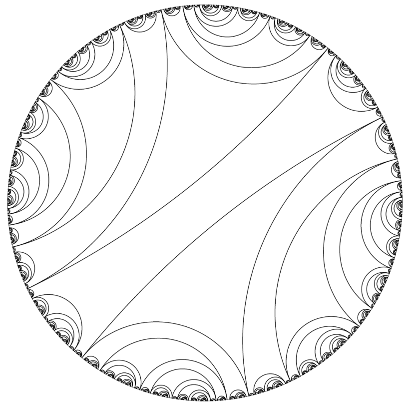
Theorem 3.1 (Whyburn, 1942). *A compact subset X in $\widehat{\mathbb{C}}$ is locally connected if and only if the following two conditions hold:*

(a) *The boundary of every component of $\widehat{\mathbb{C}}\setminus X$ is locally connected; and*
(b) $\forall\,\varepsilon>0$, *there are only finitely many components of $\widehat{\mathbb{C}}\setminus X$ whose spherical diameters $>\varepsilon$.*



Locally connected　　　　not locally connected



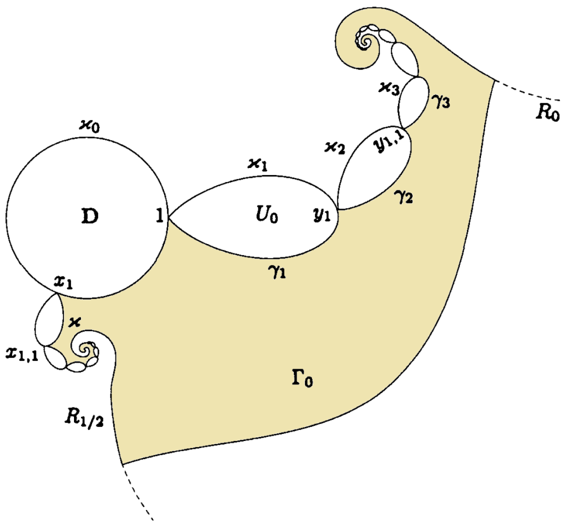


Hyperbolic ⇝ subhyperbolic ⇝ geometrically finite ⇝

Tools in the proof:

(1) Expanding metrics;
(2) Puzzles developed by Yoccoz [Hub93], Branner-Hubbard [BH92], Lyubich [Lyu97], Kozlovski-Shen-van Strien [KSS07], Roesch [Roe08], $\cdots$.
(3) Grötzsch modulus inequality, Kahn-Lyubich covering lemma [KL09].

Theorem 3.2 (Petersen, Acta Math. 1996 [Pet96])**.** *For any bounded type α, the Julia set of $P_\alpha(z) = e^{2\pi \mathrm{i}\alpha}z + z^2$ is locally connected.*



The proof: Cubic Blaschke model

$$g_\alpha(z) = e^{2\pi \mathrm{i}\tau(\alpha)} z^2 \frac{z-3}{1-3z}$$

and Petersen's puzzle. The principle:

$$\operatorname{diam}(\Gamma_n) \leqslant C \cdot \operatorname{length}(I_n)$$

and length $(I_n) \to 0$ as $n \to \infty$, where $C > 1$ is a constant.

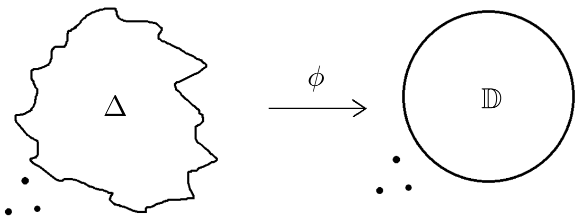
Remark. Petersen-Zakeri (2004) [PZ04] has generalized the result to almost all α.

Theorem 3.3 (Wang-Y.-Zhang-Zhang, arXiv, 2022 [WYZZ22])**.** *Suppose f is a rational map with Siegel disks such that the Julia set $J(f)$ is connected, and moreover, the forward orbit of every critical point of f satisfies one of the following:*

(a) *It is finite; or*
(b) *It lies in an attracting basin; or*
(c) *It intersects the closure of a bounded type Siegel disk.*

Then $J(f)$ is locally connected.

Difficulties: no expanding metric, no puzzles, nor analytic Blaschke models.
Idea of the proof: Constructing *quasi-Blaschke* models.

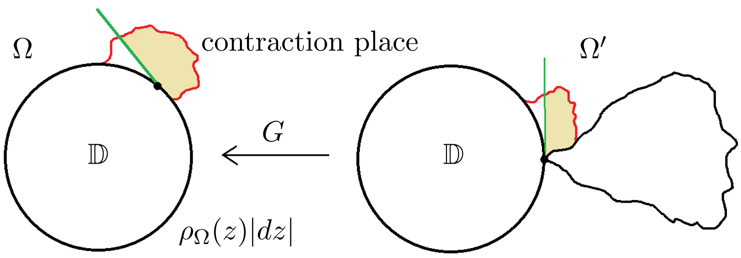


$\phi : \widehat{\mathbb{C}} \setminus \overline{\Delta} \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ conformal, and $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ quasiconformal.
Denote $z^* = 1/\overline{z}$. Define

$$G(z) := \begin{cases} \phi \circ f \circ \phi^{-1}(z) & \text{if } z \in \widehat{\mathbb{C}} \setminus \mathbb{D}, \\ \big(\phi \circ f \circ \phi^{-1}(z^*)\big)^* & \text{if } z \in \mathbb{D}. \end{cases}$$

Let $\mathcal{P}(G) = \phi(\mathcal{P}(f) \setminus \Delta)$.

Main Lemma. *$\forall\, \varepsilon > 0$, $\forall$ Jordan disk $V_0 \subset \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ with $\emptyset \neq \overline{V}_0 \cap \mathcal{P}(G) \subset \mathbb{T}$, $\exists\, N \geqslant 1$, s.t. $\forall\, n \geqslant N$, $\mathrm{diam}_{\widehat{\mathbb{C}}}(V_n) < \varepsilon$, where $\{V_n\}_{n \geqslant 0}$ is any pullback sequence of V_0 in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.*



Main Lemma ⇒ control the size of Fatou components.
Main Lemma + argument of homotopy class of the curves in immediate attracting basins ⇒ boundaries of attracting basins are locally connected.
Combining Zhang's result on Siegel disks [Zha11] ⇒ above theorem (WYZZ).

Developments: Parabolic basins are allowed (Fu-Y., arxiv, 2023 [FY23]).

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