Estimates of the Parabolic Renormalization for Local Degree Three

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About this file:

This file is the Maple worksheet for checking the numerical estimations in the proof of the Main Theorem in the paper:
Parabolic and near-parabolic renormalizations for local degree three.

Remark:
(1) The numbers are accurated to **30** decimal places and the results show the first **20**.
(2) We calculate the values of the given functions at given precise values (for examples, at 0.2, $\sqrt{6}$, π, e^3, etc).
(3) Although there are some **loop statements** in the calculations (such as **for *k* from 1 to 10**), **they have no relationship to the iterations**. We calculate the values $f(x_k)$ of the given function f at given precise values x_k (such as x_k=1 + 0.05 ·k).

Last updated: June 29, 2024

§ 2 - Proposition 2.2: the values of some constants

$$\mu := 11 - 4\sqrt{6.0};$$
$$cp\_P := 1 - \frac{2}{3}\sqrt{6.0};$$
$$cv\_P := -\frac{16}{3\left(8\sqrt{6.0}+3\right)};$$
$$v1P := 9\cdot\sqrt{6.0} - \frac{47}{2} - \frac{1}{2}\cdot\sqrt{996\sqrt{6.0}-2439};$$
$$v2P := 9\cdot\sqrt{6.0} - \frac{47}{2} + \frac{1}{2}\cdot\sqrt{996\sqrt{6.0}-2439};$$

$$\mu := 1.20204102886728760721$$
$$cp\_P := -0.63299316185545206546$$
$$cv\_P := -0.23603083295666381917$$

$$v1P := -1.87046002590206283420$$
$$v2P := -1.03872460400073139825$$

§ 6 - Lemma 6.1

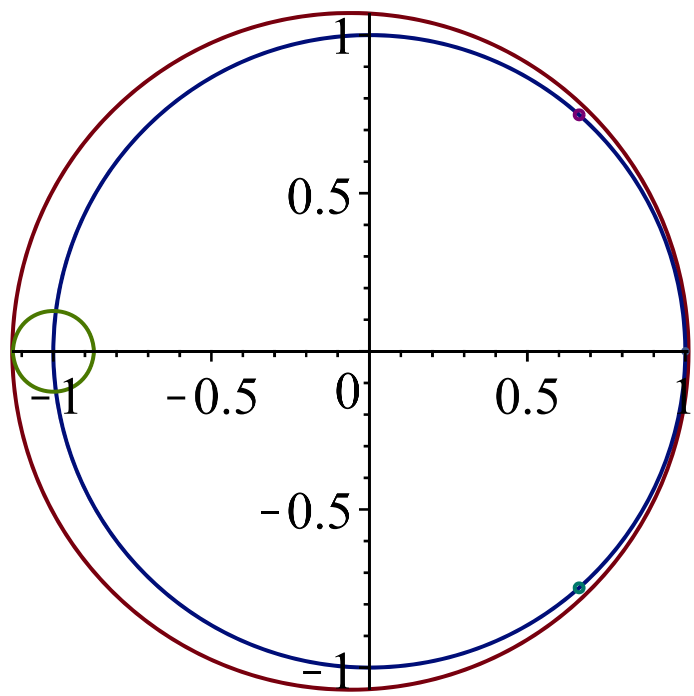
$$cp := \frac{1}{5}\cdot\left(1+4\cdot\sqrt{6.0}+2\sqrt{2\cdot\left(9+\sqrt{6.0}\right)}\right);$$
$$cpp := \frac{1}{5}\left(1+4\sqrt{6.0}-2\sqrt{2\left(9+\sqrt{6.0}\right)}\right);$$
$$\omega := \frac{8\sqrt{6.0}-3}{25}+\frac{6\sqrt{6.0}+4}{25}\cdot i;$$
$$cv := \frac{3\left(8\sqrt{6.0}+3\right)}{4};$$
$$v1 := \frac{2.0+v1P}{-v1P}+\frac{2\sqrt{-1.0-v1P}}{-v1P}\cdot i;$$
$$v2 := \frac{2.0+v2P}{-v2P}+\frac{2\sqrt{-1.0-v2P}}{-v2P}\cdot i;$$

$$cp := 4.07370692077735080938$$
$$cpp := 0.245476667675734147773$$
$$\omega := 0.663836717690616991142+0.747877538267962743357\ I$$
$$cv := 16.9469384566990685891\,8$$
$$v1 := 0.0692556763064017869\,9+0.997598943112583143 11\ I$$
$$v2 := 0.925438169363505071 93+0.378898660178581922 74\ I$$

§ 7 - Lemma 7.1: Figure 9 (left)

$a0 := -0.06 :$
$r0 := 1.07 :$
$\varepsilon 1 := 0.128 :$
$\varepsilon 2 := 0.007 :$
$\varepsilon 3 := 0.014 :$

$plot([[-0.06+r0\cdot\cos(t),\ r0\cdot\sin(t),\ t=0..2\ \pi],\ [\cos(t),\ \sin(t),\ t=0..2\ \pi],\ [-1+\varepsilon 1\cdot\cos(t),\ \varepsilon 1\cdot\sin(t),\ t=0..2\ \pi],\ [1+\varepsilon 2\cdot\cos(t),\ \varepsilon 2\cdot\sin(t),\ t=0..2\ \pi],\ [\mathrm{Re}(\omega)+\varepsilon 3\cdot\cos(t),\ \mathrm{Im}(\omega)+\varepsilon 3\cdot\sin(t),\ t=0..2\ \pi],\ [\mathrm{Re}(\omega)+\varepsilon 3\cdot\cos(t),\ -\mathrm{Im}(\omega)+\varepsilon 3\cdot\sin(t),\ t=0..2\ \pi]],\ scaling=constrained)$



Lemma 7.1: (7.2*) - (7.5*)

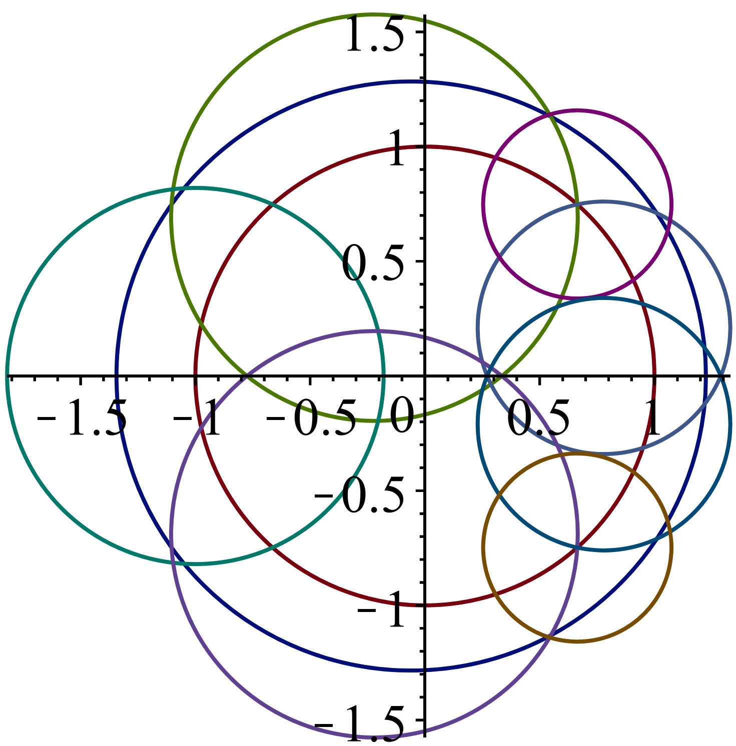
$$\eta := 3 :$$
$$\frac{\varepsilon 1^{6} \cdot (2 - \varepsilon 1)^{4}}{(1 + \varepsilon 1) \cdot (2 + \varepsilon 1)^{8}};$$
$$evalf\left(cv \cdot e^{-2\pi\eta}\right);$$
$$\frac{(2 - \varepsilon 2)^{6} \varepsilon 2^{4}}{(1 + \varepsilon 2) \cdot (1 + \varepsilon 2)^{8}};$$
$$\frac{(2 + \varepsilon 3)^{6} \cdot (1 + \varepsilon 3)^{4}}{(1 - \varepsilon 3) \cdot \varepsilon 3^{4} \cdot (1 - \varepsilon 3)^{4}};$$
$$evalf\left(cv \cdot e^{2\pi\eta}\right);$$
$$evalf\left((\mathrm{Re}(\omega) + 0.06)^{2} + \mathrm{Im}(\omega)^{2} - (r0 - \varepsilon 3)^{2}\right);$$

$$1.13867846403811237469 \; 10^{-7}$$
$$1.10365447674806197504 \; 10^{-7}$$
$$1.41309290964804813806 \; 10^{-7}$$
$$1.97066090029753385368 \; 10^{9}$$
$$2.60225214599212317845 \; 10^{9}$$
$$-0.0318755938771259610\,3$$

Lemma 7.3: Figure 9 (right)

$$\omega := \frac{8\sqrt{6.0}-3}{25} + \frac{6\sqrt{6.0}+4}{25}\mathrm{i}:$$
$$a4 := -0.22 + 0.69\mathrm{i}:$$
$$\varepsilon 4 := |a4 - \omega|:$$
$$a5 := 0.78 + 0.21\mathrm{i}:$$
$$\varepsilon 5 := |a5 - \omega|:$$
$$\varepsilon 6 := 0.41:$$
$$\varepsilon 7 := 0.82:$$
$$r1 := 1.2:$$

$plot([[\cos(t), \sin(t), t = 0..2\pi], [a0 + r0\cdot r1\cdot\cos(t), r0\cdot r1\cdot\sin(t), t = 0..2\pi], [\mathrm{Re}(a4) + \varepsilon 4\cdot\cos(t), \mathrm{Im}(a4) + \varepsilon 4\cdot\sin(t), t = 0..2\pi], [\mathrm{Re}(a5) + \varepsilon 5\cdot\cos(t), \mathrm{Im}(a5) + \varepsilon 5\cdot\sin(t), t = 0..2\pi], [\mathrm{Re}(\omega) + \varepsilon 6\cdot\cos(t), \mathrm{Im}(\omega) + \varepsilon 6\cdot\sin(t), t = 0..2\pi], [-1 + \varepsilon 7\cdot\cos(t), \varepsilon 7\cdot\sin(t), t = 0..2\pi], [\mathrm{Re}(a4) + \varepsilon 4\cdot\cos(t), -\mathrm{Im}(a4) + \varepsilon 4\cdot\sin(t), t = 0..2\pi], [\mathrm{Re}(a5) + \varepsilon 5\cdot\cos(t), -\mathrm{Im}(a5) + \varepsilon 5\cdot\sin(t), t = 0..2\pi], [\mathrm{Re}(\omega) + \varepsilon 6\cdot\cos(t), -\mathrm{Im}(\omega) + \varepsilon 6\cdot\sin(t), t = 0..2\pi]], scaling = constrained)$



Lemma 7.3: (7.8*) - (7.11*)

$$y4p := x \to \mathrm{Im}(a4) + \sqrt{\varepsilon 4^2 - (x - \mathrm{Re}(a4))^2}:$$
$$y4m := x \to \mathrm{Im}(a4) - \sqrt{\varepsilon 4^2 - (x - \mathrm{Re}(a4))^2}:$$

$y5p := x \to \mathrm{Im}(a5) + \sqrt{\varepsilon 5^2 - (x - \mathrm{Re}(a5))^2}$:
$y5m := x \to \mathrm{Im}(a5) - \sqrt{\varepsilon 5^2 - (x - \mathrm{Re}(a5))^2}$:
$y6p := x \to \mathrm{Im}(\omega) + \sqrt{\varepsilon 6^2 - (x - \mathrm{Re}(\omega))^2}$:
$y6m := x \to \mathrm{Im}(\omega) - \sqrt{\varepsilon 6^2 - (x - \mathrm{Re}(\omega))^2}$:
$y7p := x \to \sqrt{\varepsilon 7^2 - (x+1)^2}$:
$y7m := x \to -\sqrt{\varepsilon 7^2 - (x+1)^2}$:

$x60 := 0.54 : x61 := \mathrm{Re}(\omega) : x62p := 1.07 : x62m := 1.067 :$
$x63 := \mathrm{Re}(\omega) + \varepsilon 6 :$
$(x60 - \mathrm{Re}(\omega))^2 + (y4p(x60) - \mathrm{Im}(\omega))^2 - \varepsilon 6^2;$
$(x60 - \mathrm{Re}(a4))^2 + (y6p(x60) - \mathrm{Im}(a4))^2 - \varepsilon 4^2;$
$(x62p - \mathrm{Re}(\omega))^2 + (y5p(x62p) - \mathrm{Im}(\omega))^2 - \varepsilon 6^2;$
$(x62m - \mathrm{Re}(a5))^2 + (y6m(x62m) - \mathrm{Im}(a5))^2 - \varepsilon 5^2;$

0.004847655527974287 98
−0.005559894224053677 34
0.001799072726483041 22
−0.005775095520950215 71

Lemma 7.3: (7.12*) - (7.13*)

> $y4p(x60) - \frac{23}{26} \cdot (x60 + 1);$

$\xi 41p := x \to \left((x+1)^2 + (y4p(x))^2\right)^3 : \xi 42p := x \to \left((x-1)^2 + (y4p(x))^2\right)^2 : \xi 43p := x \to \left(x^2 + (y4p(x))^2\right)^{\frac{1}{2}}$:

$\xi 44p := x \to \left((x - \mathrm{Re}(\omega))^2 + (y4p(x) - \mathrm{Im}(\omega))^2\right)^2 : \xi 45p := x \to \left((x - \mathrm{Re}(\omega))^2 + (y4p(x) + \mathrm{Im}(\omega))^2\right)^2$:

$\Xi 4p := (x, xp) \to \frac{\xi 41p(xp) \cdot \xi 42p(xp)}{\xi 43p(x) \cdot \xi 44p(x) \cdot \xi 45p(x)}$:
$x601 := 0.6$:

$\Xi 4p(x60, x601);$
$\Xi 4p(x601, x61);$

−0.217426220961493276 67
140.640535177720308262 40
217.073752473583006139 20

Lemma 7.3: (7.14*) - (7.17*)

$$> s61 := \frac{\mathrm{Im}(a5+\omega)}{\mathrm{Re}(a5-\omega)} : \mathrm{Re}(a5) + \frac{\varepsilon 5}{\sqrt{1+s61^2}};$$

$$\mathrm{Im}(a5) - \frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)-1}\cdot(\mathrm{Re}(a5)-1);$$

$$\xi 51p := x \rightarrow \left((x+1)^2 + (y5p(x))^2\right)^3 : \xi 52p := x \rightarrow \left((x-1)^2 + (y5p(x))^2\right)^2 : \xi 53p := x \rightarrow \left(x^2 + (y5p(x))^2\right)^{\frac{1}{2}} :$$

$$\xi 54p := x \rightarrow \left((x-\mathrm{Re}(\omega))^2 + (y5p(x)-\mathrm{Im}(\omega))^2\right)^2 : \xi 55p := x \rightarrow \left((x-\mathrm{Re}(\omega))^2 + (y5p(x)+\mathrm{Im}(\omega))^2\right)^2 :$$

$$\xi 55pmax := \left(\sqrt{(\mathrm{Re}(a5)-\mathrm{Re}(\omega))^2 + (\mathrm{Im}(a5)+\mathrm{Im}(\omega))^2} + \varepsilon 5\right)^4 :$$

$$x611 := 0.99 :$$

$$\frac{\xi 51p(x61)\cdot \xi 52p(x611)}{\xi 53p(x611)\cdot \xi 54p(x611)\cdot \xi 55pmax};$$

$$Hat\Xi 5p := (x,\ xp) \rightarrow \frac{\xi 51p(x)\cdot \xi 52p(xp)}{\xi 53p(xp)\cdot \xi 54p(xp)\cdot \xi 55p(x)} :$$

$$x612 := 1.05 : x613 := 1.06 :$$

$$Hat\Xi 5p(x611,\ x612);$$
$$Hat\Xi 5p(x612,\ x613);$$
$$Hat\Xi 5p(x613,\ x62p);$$

0.846247722430934546 37
−0.279443871706149590 80
132.594990847302565012 15
137.608951410458748943 85
141.476217224046328744 87
126.547650628304715926 05

Lemma 7.3: (7.18*) - (7.23*)

$$> s62 := \frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)+1} : \mathrm{Re}(\omega) + \frac{\varepsilon 6}{\sqrt{1+s62^2}};$$

$$s63 := \frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)-1} : \mathrm{Re}(\omega) - \frac{\varepsilon 6}{\sqrt{1+s63^2}};$$

$$s64 := \frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)} : \mathrm{Re}(\omega) + \frac{\varepsilon 6}{\sqrt{1+s64^2}};$$

$$\xi 61p := x \to \left((x+1)^2 + (y6p(x))^2\right)^3 : \xi 62p := x \to \left((x-1)^2 + (y6p(x))^2\right)^2 : \xi 63p := x \to \left(x^2 + (y6p(x))^2\right)^{\frac{1}{2}} :$$

$$\xi 64p := x \to \left((x - \mathrm{Re}(\omega))^2 + (y6p(x) - \mathrm{Im}(\omega))^2\right)^2 : \xi 65p := x \to \left((x - \mathrm{Re}(\omega))^2 + (y6p(x) + \mathrm{Im}(\omega))^2\right)^2 :$$

$$x621 := \mathrm{Re}(\omega) : x622 := 0.99 : x623 := 1.07 : x63 := \mathrm{Re}(\omega) + \varepsilon 6 :$$

$$\frac{\xi 61p(x60) \cdot \xi 62p(x621)}{\xi 63p(x621) \cdot \varepsilon 6^4 \cdot \xi 65p(x621)}; \frac{\xi 61p(x621) \cdot \xi 62p(x622)}{(1 + \varepsilon 6) \cdot \varepsilon 6^4 \cdot \xi 65p(x621)}; \frac{\xi 61p(x622) \cdot \xi 62p(x623)}{\xi 63p(x622) \cdot \varepsilon 6^4 \cdot \xi 65p(x622)}; \frac{\xi 61p(x623) \cdot \xi 62p(x623)}{\xi 63p(x622) \cdot \varepsilon 6^4 \cdot \xi 65p(x622)};$$

$$\frac{\left((x63+1)^2 + (\mathrm{Im}(\omega))^2\right)^3 \cdot \left((x63-1)^2 + (\mathrm{Im}(\omega))^2\right)^2}{\xi 63p(x623) \cdot \varepsilon 6^4 \cdot \xi 65p(x623)};$$

$$\xi 61m := x \to \left((x+1)^2 + (y6m(x))^2\right)^3 : \xi 62m := x \to \left((x-1)^2 + (y6m(x))^2\right)^2 : \xi 63m := x \to \left(x^2 + (y6m(x))^2\right)^{\frac{1}{2}} :$$

$$\xi 64m := x \to \left((x - \mathrm{Re}(\omega))^2 + (y6m(x) - \mathrm{Im}(\omega))^2\right)^2 : \xi 65m := x \to \left((x - \mathrm{Re}(\omega))^2 + (y6m(x) + \mathrm{Im}(\omega))^2\right)^2 :$$

$$\Xi 6m := (x, xp) \to \frac{\xi 61m(xp) \cdot \xi 62m(xp)}{\xi 63m(x) \cdot \xi 64m(x) \cdot \xi 65m(x)} :$$

$$x631 := 1.069 : x632 := 1.072 :$$

$$\Xi 6m(x631, x62m); \Xi 6m(x632, x631);$$

$$\frac{\xi 61m(x632) \cdot \xi 62m(x632)}{\left((x63)^2 + (\mathrm{Im}(\omega))^2\right)^{\frac{1}{2}} \cdot \varepsilon 6^4 \cdot \left((x63 - \mathrm{Re}(\omega))^2 + (2 \cdot \mathrm{Im}(\omega))^2\right)^2};$$

1.03779590814045251750

0.49574589736391490929

0.93600977194376995791

209.68008810657422391242

130.09402372058928046278

130.87957982914634149710

129.16867319287319578361
146.28228102792950757915
125.39980120633022022925
126.55060098307776497184
133.07634800778924572818

Lemma 7.3: (7.24*) - (7.31*)

$x70 := -1 - \varepsilon7 : x71 := -1.095 : x72 := \mathrm{Re}(a4) - \varepsilon4; x73 := -0.77 :$
$(x71 - \mathrm{Re}(a4))^2 + (y7p(x71) - \mathrm{Im}(a4))^2 - \varepsilon4^2;$
$(x71 + 1)^2 + (y4p(x71))^2 - \varepsilon7^2;$
$y4m(x73);$

$\xi71p := x \to \varepsilon7^6 : \xi72p := x \to ((x-1)^2 + (y7p(x))^2)^2 : \xi73p := x \to (x^2 + (y7p(x))^2)^{\frac{1}{2}} :$
$\xi74p := x \to ((x - \mathrm{Re}(\omega))^2 + (y7p(x) - \mathrm{Im}(\omega))^2)^2 : \xi75p := x \to ((x - \mathrm{Re}(\omega))^2 + (y7p(x) + \mathrm{Im}(\omega))^2)^2 :$
$\Xi7p := (x, xp) \to \frac{\xi71p(xp) \cdot \xi72p(xp)}{\xi73p(x) \cdot \xi74p(x) \cdot \xi75p(x)} :$

$s71 := -\frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega) + 1} : -1 - \frac{\varepsilon7}{\sqrt{1 + s71^2}}; x701 := -1.3 :$
$\frac{\xi71p(x70) \cdot \xi72p(x70)}{\xi73p(x701) \cdot \xi74p(x701) \cdot \xi75p(x70)};$
$\Xi7p(x701, x70);$
$\Xi7p(x71, x701);$

$s72 := -\frac{\mathrm{Im}(a4)}{\mathrm{Re}(a4) - 1} : \mathrm{Re}(a4) - \frac{\varepsilon4}{\sqrt{1 + s72^2}};$
$\frac{\xi41p(x71) \cdot \xi42p(x71)}{\xi43p(x72) \cdot \xi44p(x71) \cdot \xi45p(x72)};$

$s73 := \frac{\mathrm{Im}(a4)}{\mathrm{Re}(a4) + 1} : \mathrm{Re}(a4) - \frac{\varepsilon4}{\sqrt{1 + s73^2}};$
$s74 := \frac{\mathrm{Im}(\omega - a4)}{\mathrm{Re}(\omega - a4)} : \mathrm{Re}(a4) - \frac{\varepsilon4}{\sqrt{1 + s74^2}};$

$\xi41m := x \to ((x + 1)^2 + (y4m(x))^2)^3 : \xi42m := x \to ((x-1)^2 + (y4m(x))^2)^2 : \xi43m := x$

$\rightarrow \left(x^2 + (y4m(x))^2\right)^{\frac{1}{2}}$:

$\xi 44m := x \rightarrow \left((x - \mathrm{Re}(\omega))^2 + (y4m(x) - \mathrm{Im}(\omega))^2\right)^2$: $\xi 45m := x \rightarrow \left((x - \mathrm{Re}(\omega))^2 + (y4m(x) + \mathrm{Im}(\omega))^2\right)^2$:

$\Xi 4m := (x, xp) \rightarrow \frac{\xi 41m(xp) \cdot \xi 42m(xp)}{\xi 43m(x) \cdot \xi 44m(x) \cdot \xi 45m(x)}$:

$x721 := -1$:

$\frac{\xi 41m(x72) \cdot \xi 42m(x72)}{\xi 43m(x721) \cdot \xi 44m(x72) \cdot \xi 45m(x721)}$; $\Xi 4m(x721, x72)$;

$\Xi 4m(x73, x721)$; $\frac{\xi 41m(x73) \cdot \xi 42m(x721)}{\xi 43m(x73) \cdot \xi 44m(x73) \cdot \xi 45m(x73)}$;

$x72 := -1.10572972907884428202$

-0.00339729069894500147

0.02129639834332052276

-0.00427455158178085860

-1.74791838089967105215

0.01893617718805266832

0.02274289545324064641

0.02613246307297592611

-0.99096566278102725571

0.02538634716629377051

-0.88340852237559048660

-1.10383671769061699142

0.01892703739895571484

0.02071476008895760615

0.00017978282899865685

0.00006825486293344913

Lemma 7.3: (7.32*) - (7.41*)

$y := x \rightarrow \sqrt{(r0 \cdot r1)^2 - (x - a0)^2}$:

$x4 := a0 + r0 \cdot r1 : x1 := -1.1 : x2 := 0.54 : x3 := 1.03$:

$(x1 + 1)^2 + (y(x1))^2 - \varepsilon 7^2$;

$(x1 - \mathrm{Re}(a4))^2 + (y(x1) - \mathrm{Im}(a4))^2 - \varepsilon 4^2$; $(x2 - \mathrm{Re}(a4))^2 + (y(x2) - \mathrm{Im}(a4))^2 - \varepsilon 4^2$;

$(x2 - \mathrm{Re}(\omega))^2 + (y(x2) - \mathrm{Im}(\omega))^2 - \varepsilon 6^2$; $(x3 - \mathrm{Re}(\omega))^2 + (y(x3) - \mathrm{Im}(\omega))^2 - \varepsilon 6^2$;

$(x3 - \mathrm{Re}(a5))^2 + (y(x3) - \mathrm{Im}(a5))^2 - \varepsilon 5^2$; $(x4 - \mathrm{Re}(a5))^2 + (\mathrm{Im}(a5))^2 - \varepsilon 5^2$;

$x1p := -0.5 : x2p := 0.3 :$
$(x1p - \mathrm{Re}(a4))^2 + (\mathrm{Im}(a4))^2 - \varepsilon 4^2; (x2p - \mathrm{Re}(a4))^2 + (\mathrm{Im}(a4))^2 - \varepsilon 4^2; (x2p - \mathrm{Re}(a5))^2 + (\mathrm{Im}(a5))^2 - \varepsilon 5^2;$

$$-0.09534400000000000000$$
$$-0.006144219810720678 09$$
$$-0.0087231486831579160 5$$
$$-0.0027536390117356913 2$$
$$-0.0292309966052462433 3$$
$$-0.0206801148605608851 0$$
$$-0.0615701543300931410 8$$
$$-0.2300171529740828901 0$$
$$-0.0380171529740828901 0$$
$$-0.0283061543300931410 8$$

§ 8 - Lemma 8.1: The definition of Q_2max

$> b0 := \frac{2\cdot(13 + 32\cdot\sqrt{6.0})}{25}; b1 := \frac{2029 + 256\cdot\sqrt{6.0}}{125};$
$a11 := 2\cdot(617 + 688\cdot\sqrt{6.0}) : a01 := 25\cdot(119 + 16\cdot\sqrt{6.0}) :$
$a12 := 3889250 + 837000\cdot\sqrt{6.0} : a02 := 2755539 + 487396\cdot\sqrt{6.0} :$
$a13 := 31356325 + 8965425\cdot\sqrt{6.0} : a03 := 66811702 + 23697378\cdot\sqrt{6.0} :$
$a14 := 102142212 + 38104768\cdot\sqrt{6.0} : a04 := 240990025 + 94826600\cdot\sqrt{6.0} :$

$$Q2max := r \to \frac{2^4}{5^5}\cdot\frac{a11\cdot r + a01}{r\cdot(r-1)^2} + \frac{2^6}{5^{10}}\cdot\frac{a12\cdot r + a02}{(r-1)^4} + \frac{2^{11}}{5^{14}}\cdot\frac{a13\cdot r + a03}{(r-1)^6} + \frac{2^{12}}{5^{16}}\cdot\frac{a14\cdot r + a04}{(r-1)^8} :$$

$$b0 := 7.3106937415249359313 9$$
$$b1 := 21.248554993219948745 11$$

Lemma 8.2 (8.1*) and Lemma 8.3 (8.2*)

$> Q2max(11); \frac{64\cdot(617 + 688\cdot\sqrt{6.0})}{3125};$

$$LogDQmax := r \to \frac{b1}{r^2} + \frac{50}{r^3} + \frac{cp^4}{2\cdot r^3\cdot(r - cp)} :$$

$LogDQmax(5.6);$

$$0.29980316960524462599$$
$$47.150058353353247366 34$$
$$1.4760015715743009182 9$$

§ 9 - Lemma 9.1: some functions

> $r0 := 1.07 : a0 := -0.06 : c00 := 0.06 : c01max := 2.14 :$
$\varphi 1max := r \rightarrow r0 \cdot \sqrt{-\log\left(1 - \left(\frac{r0}{r - |a0|}\right)^2\right)} :$
$LogDvarphimax := r \rightarrow -\log\left(1 - \left(\frac{r0}{r - |a0|}\right)^2\right) :$

Lemma 9.2 and the value of h0

> $\log(11.0) + \frac{0.06\ \pi}{3};$
$h0 := \frac{14 \cdot \sqrt{6.0}}{25} + 0.45;$

$$2.46072712587016640883$$
$$h0 := 1.82171425595857973499$$

Lemma 9.3: preparations

> $\omega := \frac{8\sqrt{6.0} - 3}{25} + \frac{6\sqrt{6.0} + 4}{25} \mathrm{i} :$
$\alpha 1 := \arctan\left(\frac{\mathrm{Im}(a5)}{\mathrm{Re}(a5) - a0}\right); \ \alpha 2m := 0.54 : \alpha 2p := 0.55 :$
$\alpha 3 := \arctan\left(\frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega) - a0}\right); \ \alpha 4 := \arctan\left(\frac{\mathrm{Im}(\omega) + \varepsilon 6}{\mathrm{Re}(\omega) - a0}\right); \ \alpha 5 := \pi - \arctan\left(\frac{h0 + 1}{a0 + 1}\right);$
$r2 := \theta \rightarrow \frac{e^{\pi - \theta} + 1}{e^{\pi - \theta} - 1} :$

$$\alpha 1 := 0.244978663126864154 17$$
$$\alpha 3 := 0.801731960850496207 37$$
$$\alpha 4 := 1.012095627387603041 09$$
$$\alpha 5 := 1.892364613257717380 83$$

Lemma 9.3: (9.2*) - (9.7*)

> $B5 := \theta \rightarrow \mathrm{Im}(a5) \cdot \sin(\theta) + (\mathrm{Re}(a5) - a0) \cdot \cos(\theta) : C5 := \theta \rightarrow (\mathrm{Re}(a5) - a0)^2 + \mathrm{Im}(a5)^2 - \varepsilon 5^2 :$
$r5 := \theta \rightarrow \frac{1}{r0} \cdot \left(B5(\theta) + \sqrt{B5(\theta)^2 - C5(\theta)}\right) :$
$r5(0) - r2(\alpha 1);$

$(a0 + r0 \cdot r5(\alpha 2p) \cdot \cos(\alpha 2p) - \mathrm{Re}(\omega))^2 + (r0 \cdot r5(\alpha 2p) \cdot \sin(\alpha 2p) - \mathrm{Im}(\omega))^2 - \varepsilon 6^2;$

$r5(\alpha 2p) - r2(\alpha 2p)$;

$B6 := \theta \to \mathrm{Im}(\omega)\cdot\sin(\theta) + (\mathrm{Re}(\omega) - a0)\cdot\cos(\theta)$: $C6 := \theta \to (\mathrm{Re}(\omega) - a0)^2 + \mathrm{Im}(\omega)^2 - \varepsilon 6^2$:

$r6 := \theta \to \frac{1}{r0}\cdot\left(B6(\theta) + \sqrt{B6(\theta)^2 - C6(\theta)}\right)$:

$r6(\alpha 4) - r2(\alpha 4)$;

$(a0 + r0\cdot r6(\alpha 2m)\cdot\cos(\alpha 2m) - \mathrm{Re}(a5))^2 + (r0\cdot r6(\alpha 2m)\cdot\sin(\alpha 2m) - \mathrm{Im}(a5))^2 - \varepsilon 5^2$;

$r6(\alpha 2m) - r2(\alpha 3)$;

0.143531225055757757975

−0.009170714509615563 73

0.063164373072837506 97

0.006296010511506318 56

−0.003176019108756644 77

0.015279109943865995 22

Lemma 9.3: (9.8*)

$r7 := \theta \to \frac{h0 - a0}{\sqrt{2.0}\cdot r0\cdot\sin\left(\theta + \frac{\pi}{4}\right)}$:

```
tt4 := Vector[row](1..24, 1) :

tt4[1] := α4 :

for k from 2 to 10 do
  tt4[k] := 1 + 0.01·k :
end do:

for k from 11 to 15 do
  tt4[k] := 0.9 + 0.02·k :
end do:

for k from 16 to 23 do
  tt4[k] := 0.45 + 0.05·k :
end do:

tt4[24] := α5 :

for k from 1 to 23 do
 print(r7(tt4[k]) − r2(tt4[k+1]));
end do;
```

0.00386269642522696840
0.00311999972727045979
0.00309358830311987737
0.00317073994419218916
0.00335259261728716774
0.00364036227344779157
0.00403534509713779964
0.00453891989777661152
0.00515255065049962817
0.00238123779397653201
0.00408058968341865673
0.00624433982898773465
0.00888259408274461 14
0.01202984025661217296
0.00338905066505096618
0.01407774098439650695
0.02845769936545678804
0.04702287002515158514
0.07039891324057986741
0.09937977760593882453
0.13497774615090780213
0.17849286242455292721
0.00833083292375814570

Lemma 9.3: (9.9*)

$$\theta 51 := 2.38 : \theta 52 := 2.6 :$$

$$r8 := \theta \rightarrow \frac{h0 + 1}{r0 \cdot \sin(\theta)} :$$

$$r8(\alpha 5) - r2(\theta 51) ; r8(\theta 51) - r2(\theta 52) ;$$

0.02779813700724900011
0.03885264547486461139

§ 10 - Proof of Proposition 3.3: (10.1*)

$$c00 + c01max + \varphi 1max(11) ;$$

$$(cv - 11) \cdot \sin\left(\frac{\pi}{6}\right) ;$$

2.30490423435328582262
2.97346922834953429459

§ 11 - Lemma 11.1: some functions

$$\beta max := r \to c01max + \frac{b1}{2\cdot r} + Q2max(r) + \varphi 1max(r) :$$

$$\sigma 1 := (r, \theta) \to \frac{\frac{b1\cdot\sin(\theta)}{2\cdot r}}{b0 - c00 + \frac{b1\cdot\cos(\theta)}{2\cdot r}} : \sigma 2 := (r, \theta) \to \sqrt{(b0-c00)^2 + \left(\frac{b1}{2\cdot r}\right)^2 + 2\cdot(b0-c00)\cdot\left(\frac{b1}{2\cdot r}\right)\cdot\cos(\theta)} :$$

$$ArgDeltaFmax := (r, \theta) \to -\arctan(\sigma 1(r, \theta)) + \arcsin\left(\frac{\beta max(r)}{\sigma 2(r, \theta)}\right) :$$

$$ArgDeltaFmin := (r, \theta) \to -\arctan(\sigma 1(r, \theta)) - \arcsin\left(\frac{\beta max(r)}{\sigma 2(r, \theta)}\right) :$$

$$AbsDeltaFmax := (r, \theta) \to \sigma 2(r, \theta) + \beta max(r) :$$

$$AbsDeltaFmin := (r, \theta) \to \sigma 2(r, \theta) - \beta max(r) :$$

$$LogDQmax := r \to \frac{b1}{r^2} + \frac{50}{r^3} + \frac{cp^4}{2\cdot r^3\cdot(r-cp)} :$$

$$LogDFmax := r \to LogDQmax(r) + LogDvarphimax(r) :$$

$$b0 - c00 - \frac{b1}{2\cdot 6.1};$$

$$\beta max(6.1);$$

5.5090089060151040670З

5.50467057792382093189

Lemma 11.2: (11.4*) - (11.9*)

$$\phi 1max := r \to 4\cdot r0\cdot\sqrt{-\log\left(1 - \left(\frac{4\cdot r0}{r}\right)^2\right)} :$$

$$c00 + c01max + \phi 1max(cv - 2.35);$$

$$\frac{2.35}{\exp(-LogDvarphimax(cv - 2.35 - 3.5))};$$

$$2.4\cdot\exp(LogDQmax(cv - 2.4));$$

$$Q := \zeta \to \frac{(\zeta+1)^6\cdot(\zeta-1)^4}{\zeta\cdot\left(\zeta^2 + \frac{6 - 16\sqrt{6.0}}{25}\cdot\zeta + 1\right)^4} :$$

$$2.75 + Q(cv) - 25.5;$$

$(25.5-22)\sin\left(\frac{7\cdot\pi}{20.0}\right)$;

$\frac{b1}{20}+Q2max(20)$;

3.4832549178342241219O
2.3722966340674875442 7
2.7084987060624689772 9
2.8359998404511810163 5
3.1185228346592875182 6
1.1367178602811668551 3

§ 12 - (12.1*), (12.2*)

>

$\frac{\phi 1max(125)+\frac{b1}{122}+Q2max(122)}{5}$;

$LogDFmax(5\cdot 25)$;

0.064480496094180106 98
0.001459437215851019 96

§ 13 - Lemma 13.1: (13.1*) - (13.5*)

> $\theta 1:=\frac{3\cdot\pi}{20.0}:\theta 2:=\frac{\pi}{4.0}:$

$u1theta1:=8.5:u2theta1:=6.1:u3:=22\cdot\cos(\theta 1):u4:=17.3:$

$u1theta2:=9:u2theta2:=6.6:$

$u0theta1:=\frac{u1theta1}{\cos(\theta 1)};u0theta2:=\frac{u1theta2}{\cos(\theta 2)};$

$u2theta1+c00\cdot\cos(\theta 1)+c01max+\varphi 1max(u2theta1)$;

$u2theta2+c00\cdot\cos(\theta 2)+c01max+\varphi 1max(u2theta2)$;

$u4+c00\cdot\cos(\theta 1)+c01max+\varphi 1max(u4)$;

$u3$;

$ArgDeltaFmax(u2theta1,\ \theta 1);ArgDeltaFmax(u2theta1,-\theta 1)$;

$-ArgDeltaFmin(u2theta1,\ \theta 1);-ArgDeltaFmin(u2theta1,-\theta 1)$;

$ArgDeltaFmax(u2theta2,\ \theta 2);ArgDeltaFmax(u2theta2,-\theta 2)$;

$-ArgDeltaFmin(u2theta2,\ \theta 2);-ArgDeltaFmin(u2theta2,-\theta 2)$;

$\frac{\pi}{2.0}-\theta 2$;

$$u0theta1 := 9.53977301989206686083$$
$$u0theta2 := 12.72792206135785543922$$
$$8.48452638180902784498$$
$$8.95867636590075742178$$
$$19.55993399135540146085$$
$$19.60214353214409297191$$
$$0.58278335743794402436$$
$$0.76195694106910770477$$
$$0.76195694106910770477$$
$$0.58278335743794402436$$
$$0.50699611304471416608$$
$$0.77671952010329964444$$
$$0.77671952010329964444$$
$$0.50699611304471416608$$
$$0.78539816339744830962$$

Lemma 13.2: (13.8*) - (13.12*)

$$r4 := 0.34 : u5 := u3 - u1theta1 :$$
$$\left| b0 - c00 + \frac{b1 \cdot e^{-\theta 1 \cdot i}}{2 \cdot u4} - \frac{2 \cdot u5 \cdot r4^2 \cdot e^{\theta 1 \cdot i}}{1 - r4^2} \right| + \beta max(u4) - \frac{2 \cdot u5 \cdot r4}{1 - r4^2};$$
$$-ArgDeltaFmin(u4, \theta 1) + \frac{1}{2} \cdot LogDFmax(u4) - \frac{1}{2} \cdot \log\left(1 - r4^2\right);$$
$$\frac{\pi}{5.0};$$
$$-ArgDeltaFmax(u4, \theta 1) - \frac{1}{2} \cdot LogDFmax(u4) + \frac{1}{2} \cdot \log\left(1 - r4^2\right);$$
$$-\frac{3 \cdot \pi}{20.0};$$
$$\frac{\exp\left(\frac{1}{2} \cdot LogDFmax(u4)\right)}{AbsDeltaFmin(u4, \theta 1) \cdot \sqrt{1 - r4^2}};$$
$$\frac{\sqrt{1 - r4^2}}{AbsDeltaFmax(u4, \theta 1) \cdot \exp\left(\frac{1}{2} \cdot LogDFmax(u4)\right)};$$

$$-0.16168616388580024059$$
$$0.52448591012420947342$$
$$0.62831853071795864769$$
$$-0.45300852954272587266$$
$$-0.47123889803846898577$$
$$0.22756989531066101254$$

$$0.08396094459145346716$$

Lemmas 13.3 and 13.4: (13.14*) - (13.18*)

> $\tan(1.245);\ \frac{\sqrt{1+8.0^2}}{0.083};\ u6 := 10.7:$
> $\frac{b1}{u6} + Q2max(u6);$
> $(22 - b0)\cdot\cos(\theta 1) - u6;$
> $LogDQmax(u6);$
> $LogDvarphimax(u6);$

$$2.96002722022930978290$$
$$97.13563552166927292008$$
$$2.30676792952983824200$$
$$2.38826771210230559591$$
$$0.24337117780356480165$$
$$0.01016458475250778226$$

§ 14 - Proof of Lemma 14.1: (14.2*)

> $t0 := 6.5\cdot\sqrt{2.0} - cp:$
> $\vartheta 1 := t \to 3\cdot\arctan\left(\frac{t}{cp - cpp}\right) : \vartheta 2 := t \to \arctan\left(\frac{t - \mathrm{Im}(\nu 1)}{cp - \mathrm{Re}(\nu 1)}\right) : \vartheta 3 := t \to \arctan\left(\frac{t + \mathrm{Im}(\nu 1)}{cp - \mathrm{Re}(\nu 1)}\right) :$
> $\vartheta 4 := t \to \arctan\left(\frac{t - \mathrm{Im}(\nu 2)}{cp - \mathrm{Re}(\nu 2)}\right) : \vartheta 5 := t \to \arctan\left(\frac{t + \mathrm{Im}(\nu 2)}{cp - \mathrm{Re}(\nu 2)}\right) : \vartheta 6 := t \to \arctan\left(\frac{t}{cp}\right) :$
> $\vartheta 7 := t \to 4\cdot\arctan\left(\frac{t - \mathrm{Im}(\omega)}{cp - \mathrm{Re}(\omega)}\right) : \vartheta 8 := t \to 4\cdot\arctan\left(\frac{t + \mathrm{Im}(\omega)}{cp - \mathrm{Re}(\omega)}\right) :$
> $\vartheta := (tp, tpp) \to \vartheta 1(tp) + \vartheta 2(tp) + \vartheta 3(tp) + \vartheta 4(tp) + \vartheta 5(tp) - \vartheta 6(tpp) - \vartheta 7(tpp) - \vartheta 8(tpp) :$

```
tt0 := Vector[row](1..12, 1) :

for k from 1 to 8 do
  tt0[k] := 0.5·k :
end do:

for k from 9 to 10 do
  tt0[k] := 0.8 + 0.4·k :
end do:
```

```
for k from 11 to 12 do
  tt0[k] := 2.8 + 0.2·k :
end do:

-3·π/4.0;
ϑ(0, tt0[1]);
for k from 1 to 11 do
  print(ϑ(tt0[k], tt0[k+1]));
end do;

print(UpperBound);

π/2.0;
ϑ(tt0[1], 0);
for k from 1 to 11 do
  print(ϑ(tt0[k+1], tt0[k]));
end do;
```

−2.35619449019234492985

−1.23514999751183231790

−1.49606541821614649468

−1.71508257093148626925

−1.89431544511283241888

−2.03857421441118071473

−2.15388764171248501588

−2.24631547239307359132

−2.32118907111209880730

−2.28246752262110089376

−2.33491721226778090936

−2.21302433840138189861

−2.24272766053190201444

UpperBound

1.57079632679489661923

0.93455688536982032885

0.60229702096812457078

0.25233978854296200296

−0.09776367079249926725

−0.43125703520698235277

−0.73604218090914990876

−1.00606295846578549493

−1.24038286436402052055

−1.51909522077489457546

−1.65060363785030370120

−1.89754846118692243283

−1.94356760599035849081

(14.4*) - (14.6*)

> $\arctan\left(\frac{t0}{cp}\right)$; $0.25\cdot\pi$; $\arctan\left(\frac{t0-\mathrm{Im}(\omega)}{cp-\mathrm{Re}(\omega)}\right)$; $\arctan\left(\frac{t0+\mathrm{Im}(\omega)}{cp-\mathrm{Re}(\omega)}\right)$;

0.89859046656085943898
0.78539816339744830962
0.90827852884623904536
1.04428623924510790449

(14.7*)

> $Q3max := r \to \frac{2^4}{5^5}\cdot\frac{a01}{r\cdot(r-1)^2} + \frac{2^6}{5^{10}}\cdot\frac{a12\cdot r + a02}{(r-1)^4} + \frac{2^{11}}{5^{14}}\cdot\frac{a13\cdot r + a03}{(r-1)^6} + \frac{2^{12}}{5^{16}}\cdot\frac{a14\cdot r + a04}{(r-1)^8}$:

$(6.5 + b0)\cdot\cos\left(\frac{\pi}{4.0}\right) + \frac{2^4}{5^5}\cdot\frac{a11}{(6.5-1)^2}\cdot\cos\left(\frac{\pi}{4.0}\right) + Q3max(6.5)$;

$cv\cdot\cos\left(\frac{\pi}{4.0}\right)$;

10.73024502229520039083
11.98329510308299569107

(14.9*), (14.10*)

> $cp + c00 - c01max - \varphi 1max(cp)$;

$6.5 + c00\cdot\cos\left(\frac{\pi}{4.0}\right) - c01max - \varphi 1max(6.5)$;

1.70318641690299564382
4.22340128593619150452

(14.12*)

> $y2 := x \to h0 - x$:

$\theta 21 := x \to 6\cdot\arctan\left(\frac{y2(x)}{x+1}\right)$: $\theta 22 := x \to 4\cdot\arctan\left(\frac{y2(x)}{x-1}\right)$: $\theta 23 := x \to \arctan\left(\frac{y2(x)}{x}\right)$:

$\theta 24 := x \to 4\cdot\arctan\left(\frac{y2(x)-\mathrm{Im}(\omega)}{x-\mathrm{Re}(\omega)}\right)$: $\theta 25 := x \to 4\cdot\arctan\left(\frac{y2(x)+\mathrm{Im}(\omega)}{x-\mathrm{Re}(\omega)}\right)$:

$\Theta 2 := (x, xp) \to \theta 21(xp) + \theta 22(xp) - \theta 23(x) - \theta 24(x) - \theta 25(x)$:

```
xx1 := Vector[row](1..16, 1):

for k from 1 to 16 do
 xx1[k] := 0.04·k:
end do:
```

$$-\frac{5\cdot\pi}{4.0};$$

$$\theta 21(xx1[1]) + \theta 22(xx1[1]) - \frac{\pi}{2} - \theta 24(0) - \theta 25(0) - 4.0\cdot\pi;$$

```
for k from 1 to 15 do
 print(Θ2(xx1[k], xx1[k+1]) − 4.0·π);
end do;
```

$$\theta 21(\mathrm{Re}(\omega)) + \theta 22(\mathrm{Re}(\omega)) - \theta 23(xx1[16]) - \theta 24(xx1[16]) - \theta 25(xx1[16]) - 4.0\cdot\pi;$$

```
print(UpperBound);
```

$$\theta 21(0) + \theta 22(0) - \theta 23(xx1[1]) - \theta 24(xx1[1]) - \theta 25(xx1[1]) - 4.0\cdot\pi;$$

```
for k from 1 to 15 do
 print(Θ2(xx1[k+1], xx1[k]) − 4.0·π);
end do;
```

$$\theta 21(0.64) + \theta 22(0.64) - \theta 23(\mathrm{Re}(\omega)) + 2\cdot\pi + 2\cdot\pi - 4.0\cdot\pi;$$

$$-\frac{3\cdot\pi}{4.0};$$

−3.926990816987241548
−2.848504466967304784
−2.932097957396893715
−3.012806625334881565
−3.089858453396871623
−3.162360302433373039
−3.229272069578384560
−3.289372474145273745
−3.341213315054460984
−3.383057837365484334
−3.412797222601566294
−3.427837128934694829
−3.424943776092124234
−3.400036944426608346
−3.347917500715230801
−3.261925281801134046

−3.13355327301948037359
−2.84850331003137938974
UpperBound
−2.54968710012175310907
−2.62256715206856270458
−2.69188144691535598294
−2.75673642745085361712
−2.81608473685359207371
−2.86868973713923055450
−2.91307842679133218105
−2.94747839896365852270
−2.96973285591274674034
−2.97718560330164983527
−2.96652551365347680250
−2.93357782125673106952
−2.87302985614143576482
−2.77808705256081748660
−2.64008519460842041082
−2.44816840193514999540
−2.40422785175700498726
−2.35619449019234492885

(14.13*)

> $\Theta 2(-1, -0.975) - 5 \cdot \pi;$
$\Theta 2(-0.975, -0.95) - 5 \cdot \pi;$
$\Theta 2(-0.95, -0.925) - 5 \cdot \pi;$
$\Theta 2(-0.925, -0.9) - 5 \cdot \pi;$

$6 \cdot \frac{\pi}{2} + \theta 22(-1) - \theta 23(-0.975) - \theta 24(-0.975) - \theta 25(-0.975) - 5 \cdot \pi;$
$\Theta 2(-0.95, -0.975) - 5 \cdot \pi;$
$\Theta 2(-0.925, -0.95) - 5 \cdot \pi;$
$\Theta 2(-0.9, -0.925) - 5 \cdot \pi;$

−0.81321631031846283577
−0.85148437015116298236
−0.89039122686944389039
−0.92994285272022355353
−0.72923865212318048489
−0.76592178317749984831
−0.80320746989737533208
−0.84110092166076974627

(14.14*)

> $\xi 21 := x \rightarrow \left((x+1)^2 + (y2(x))^2 \right)^3 : \xi 22 := x \rightarrow \left((x-1)^2 + (y2(x))^2 \right)^2 : \xi 23 := x \rightarrow \left(x^2 \right.$

$+ (y2(x))^2)^{\frac{1}{2}}$:

$\xi 24 := x \to ((x - \mathrm{Re}(\omega))^2 + (y2(x) - \mathrm{Im}(\omega))^2)^2$: $\xi 25 := x \to ((x - \mathrm{Re}(\omega))^2 + (y2(x) + \mathrm{Im}(\omega))^2)^2$:

$\Xi 2 := (x, xp) \to \dfrac{\xi 21(xp) \cdot \xi 22(xp)}{\xi 23(x) \cdot \xi 24(x) \cdot \xi 25(x)}$:

$h2 := (x, xp) \to \Xi 2(xp, x) \cdot \cos(\Theta 2(xp, x) - 5 \cdot \pi)$:

$v2 := (x, xp) \to \Xi 2(xp, x) \cdot \sin(\Theta 2(xp, x) - 5 \cdot \pi) - (h2(x, xp) - 4.2 \cdot \sqrt{2.0})$:

$\Xi 2(-0.975, -1) \cdot \cos\left(6 \cdot \frac{\pi}{2} + \theta 22(-1) - \theta 23(-0.975) - \theta 24(-0.975) - \theta 25(-0.975) - 5 \cdot \pi\right)$;

$h2(-0.975, -0.95)$;
$h2(-0.95, -0.925)$;
$h2(-0.925, -0.9)$;

$-\Xi 2(-0.975, -1) - (1.7 - 4.2 \cdot \sqrt{2.0})$;
$-\Xi 2(-0.95, -0.975) - (1.7 - 4.2 \cdot \sqrt{2.0})$;
$-\Xi 2(-0.925, -0.95) - (1.7 - 4.2 \cdot \sqrt{2.0})$;
$-\Xi 2(-0.9, -0.925) - (1.7 - 4.2 \cdot \sqrt{2.0})$;

1.65018873952893605063
1.60327690654200639299
1.55375985879198771982
1.50149177242775673638
2.02670409461158207016
2.01522112351875750342
2.00214675187858270344
1.98737808845007345461

(14.15*)

```
> xx2 := Vector[row](1..22, 1) :

for k from 4 to 22 do
 xx2[k] :=-1.1 + 0.05·k :
end do:

for k from 4 to 21 do
 print(Θ2(xx2[k], xx2[k + 1]) − 5.0·π);
end do;
```

```
print(UpperBound);

for k from 4 to 20 do
 print(Θ2(xx2[k+1], xx2[k]) - 5.0·π);
end do;
```

$$\theta 21kmax := \theta 21(-0.05) + \theta 22(-0.05) - \left(-\frac{\pi}{2.0}\right) - \theta 24(0) - \theta 25(0) - 5.0 \cdot \pi;$$

−1.03643500329735476260
−1.12124355297374212177
−1.20878509868438550604
−1.29907363546058950715
−1.39210526544358567471
−1.48785462422673390678
−1.58627082352581793428
−1.68727286617836491712
−1.79074448635399792549
−1.89652836242108872906
−2.00441964023562535133
−2.11415868773825059958
−2.22542297323600878305
−2.33781791309275381727
−2.45086646028050545910
−2.56399708963120255772
−2.67652965877804602742
−2.78765835665315605552
UpperBound
−0.85362474912923930588
−0.93133719629987248799
−1.01145847952169215741
−1.09398973503132282460
−1.17891391140137736055
−1.26619208549127754667
−1.35575922302908485703
−1.44751931257860825225
−1.54133979052804040292
−1.63704515919201468430
−1.73440967757297431147
−1.83314897086095484223
−1.93291035447245458146
−2.03326159255828191985
−2.13367769641780649769
−2.23352519582510587091
−2.33204305718537648425

$\theta 21kmax := -2.42831903377872545385$

(14.16*)

```
> for k from 4 to 20 do
   print(h2(xx2[k], xx2[k+1]));
  end do;

  Ξ2(0, -0.05)·cos(θ21kmax);

  print(h2v2);

  for k from 4 to 20 do
   print(v2(xx2[k], xx2[k+1]));
  end do;

  Ξ2(0, -0.05)·sin(θ21kmax) − (Ξ2(0, -0.05)·cos(θ21kmax) − 4.2·√2.0);
```

1.65345995750206524151
1.53073671626430813838
1.39278744559759995430
1.23755131514382687909
1.06259011741632667613
0.86500021668173761637
0.64129926927427925383
0.38727907424340978220
0.09781244970308859895
−0.23340302966705358877
−0.61418908847196691587
−1.05435449478010571166
−1.56633949995058661194
−2.16611642273439250493
−2.87446731178321662310
−3.71882573205998902301
−4.73597673421534067227
−5.97608494757796017082

h2v2

2.39023277001321098158
2.35069805719461509577
2.32210716219911378082
2.30638408279717376777
2.30585008147696892575
2.32332137529106596811
2.36223510692250645785
2.42681307992016866407
2.52227636260345367989

2.65512903540619404089
2.83353680329158207611
3.06783701317049673809
3.37123245859774138135
3.76074471503207657593
4.25853737261698657705
4.89377094885187178907
5.70522734525430929774
6.74505279659485127395

(14.17*)

$y3 := x \to h0 + 1 :$

$\xi 31 := x \to \left((x+1)^2 + (y3(x))^2 \right)^3 : \xi 32 := x \to \left((x-1)^2 + (y3(x))^2 \right)^2 : \xi 33 := x \to \left(x^2 + (y3(x))^2 \right)^{\frac{1}{2}} :$

$\xi 34 := x \to \left((x - \mathrm{Re}(\omega))^2 + (y3(x) - \mathrm{Im}(\omega))^2 \right)^2 : \xi 35 := x \to \left((x - \mathrm{Re}(\omega))^2 + (y3(x) + \mathrm{Im}(\omega))^2 \right)^2 :$

$\Xi 3 := (x, xp) \to \frac{\xi 31(xp) \cdot \xi 32(xp)}{\xi 33(x) \cdot \xi 34(x) \cdot \xi 35(x)} :$

$\theta 31 := x \to 6 \cdot \arctan\left(\frac{y3(x)}{x+1} \right) : \theta 32 := x \to 4 \cdot \arctan\left(\frac{y3(x)}{x-1} \right) : \theta 33 := x \to \arctan\left(\frac{y3(x)}{x} \right) :$

$\theta 34 := x \to 4 \cdot \arctan\left(\frac{y3(x) - \mathrm{Im}(\omega)}{x - \mathrm{Re}(\omega)} \right) : \theta 35 := x \to 4 \cdot \arctan\left(\frac{y3(x) + \mathrm{Im}(\omega)}{x - \mathrm{Re}(\omega)} \right) :$

$\Theta 3 := (x, xp) \to \theta 31(xp) + \theta 32(xp) - \theta 33(x) - \theta 34(x) - \theta 35(x) + \pi :$

```
xx3 := Vector[row](1..40, 1) :

for k from 1 to 20 do
 xx3[k] :=-1 - 0.04·k :
end do:

for k from 21 to 29 do
 xx3[k] := 2.2 - 0.2·k :
end do:

for k from 30 to 40 do
 xx3[k] := 8 - 0.4·k :
end do:
```

$6 \cdot \left(-\frac{\pi}{2.0} \right) + \theta 32(-1) - \theta 33(-1.04) - \theta 34(-1.04) - \theta 35(-1.04) + \pi;$

```
for k from 1 to 7 do
 print(Θ3(xx3[k+1], xx3[k]));
end do;

print(UpperBound);

Θ3(-1, -1.04);

for k from 1 to 7 do
 print(Θ3(xx3[k], xx3[k+1]));
end do;
```

−0.84834442345354087491
−0.82020109973414391561
−0.79129758969614480813
−0.76167371111084496096
−0.73136962579681446905
−0.70042571039299935428
−0.66888243144224637881
−0.63678022517159434617
UpperBound
−0.63010482164982747504
−0.60380125453008264529
−0.57675908101590531447
−0.54901691635533747663
−0.52061372750858191358
−0.49158870114492542621
−0.46198111644394645044
−0.43183022309106223631

(14.18*)

> $4.2 \cdot \sqrt{2.0} - 1.7;$

```
Ξ3(-1,-1.04);

for k from 1 to 7 do
 print(Ξ3(xx3[k], xx3[k+1]));
end do;
```

4.23969696196699920497
2.06265693262301943370
2.00015852864596490375
1.94109148567548639756

1.88530037777486610581
1.83263470062488303124
1.78294911664356809548
1.73610362864888461151
1.69196369061739540188

(14.19*) - (14.20*)

```
> Ξ3(-1,-1.04)·cos(Θ3(-1,-1.04));

for k from 1 to 7 do
 print(Ξ3(xx3[k], xx3[k+1])·cos(Θ3(xx3[k], xx3[k+1])));
end do;

print(from8to28);

for k from 8 to 28 do
 print(Ξ3(xx3[k], xx3[k+1]));
end do;
```

1.66655615297622192298
1.64649711307032067802
1.62708956963522447342
1.60823277934600932508
1.58983638292389561194
1.57181952378065069870
1.55411001508006234969
1.53664355762832911009
from8to28
1.65040026355426503772
1.61128982391475286129
1.57451433142332162031
1.53996116253885245013
1.50752301522538455380
1.47709779011673637224
1.44858845261740139646
1.42190287996631253558
1.39695369680780048644
1.37365810236729417540
1.35193769191884395560
1.33171827485730230776
1.62545992961130154906

1.55756088592177390314
1.51751956353778832136
1.50038672415247041974
1.50235818483396278981
1.52049929292684018284
1.55252776179399957379
1.59664804201208671506
1.65142753022828316626

(14.21*)

```
for k from 29 to 39 do
 print(Θ3(xx3[k+1], xx3[k]));
end do;

print(UpperBound);

for k from 29 to 39 do
 print(Θ3(xx3[k], xx3[k+1]));
end do;
```

0.88414108885984615756
1.13885871076736819041
1.35650174669510685763
1.54232999766213305711
1.70127796365124448700
1.83769221201597725810
1.95527211903494642677
2.05710697032961793405
2.14575083366282412193
2.22330576937272879625
2.29150003468030069442
UpperBound
1.82130519334237769196
1.96028602283542276744
2.07922612437741096364
2.18103712867743864289
2.26840588675817779433
2.34367598685468543092
2.40883035776698007112
2.46551925652008386205
2.51510455507619152519

2.5587062673477424619
2.5972452226441159127

(14.22*)

```
> h3 := (x, xp) → Ξ3(x, xp)·cos(Θ3(xp, x)) :
  v3 := (x, xp) → Ξ3(x, xp)·sin(Θ3(xp, x)) − (4.2·√2.0 − h3(x, xp)) :

  for k from 29 to 39 do
   print(h3(xx3[k], xx3[k+1]));
  end do;

  print(h3v3);

  for k from 29 to 39 do
   print(v3(xx3[k], xx3[k+1]));
  end do;
```

1.5206965886545556721
1.0858072150049315127
0.5992554559850690649
0.0872688138889463447
−0.4338080054391941126
−0.9544019433845846226
−1.4692970411114626375
−1.9759575857001610572
−2.4734341632506444730
−2.9616774709182917426
−3.4411197723279022405
h3v3
−2.5638779277693239948
−2.4983974543899796154
−2.5869690995081962264
−2.7875707168967925211
−3.0677273090618491646
−3.4034800091874348213
−3.7776199478718271448
−4.1779717200389476210
−4.5959905719680090985
−5.0257040089333484241
−5.4629457489710523175

(14.23*)

```
> 2·Q2max(8) − 8 − 1.6 + b0;
```

−0.9396354866549818745

§ Appendix A - The position of two disks

Proof of Lemma 7.2(a): (A.1*) - (A.5*)

> $x42m := \dfrac{2\cdot \mathrm{Re}(a4)\cdot \mathrm{Im}(a4)\cdot \mathrm{Im}(\omega) + \left(\mathrm{Re}(a4)^2 - \mathrm{Im}(a4)^2\right)\cdot \mathrm{Re}(\omega)}{\mathrm{Re}(a4)^2 + \mathrm{Im}(a4)^2};$

$y4m(x42m);\ x40m := -0.22 - \varepsilon 4;\ x41m := -1 :$

$s41 := -\dfrac{(1 + \mathrm{Re}(a4))\cdot \mathrm{Im}(a4) + \varepsilon 4\cdot\sqrt{(1 + \mathrm{Re}(a4))^2 + \mathrm{Im}(a4)^2 - \varepsilon 4^2}}{\varepsilon 4^2 - (1 + \mathrm{Re}(a4))^2};$

$x41tilde := \dfrac{\mathrm{Re}(a4) + \mathrm{Im}(a4)\cdot s41 - s41^2}{1 + s41^2};$

$s42 := -\dfrac{(1 - \mathrm{Re}(a4))\cdot \mathrm{Im}(a4) + \varepsilon 4\cdot\sqrt{(1 - \mathrm{Re}(a4))^2 + \mathrm{Im}(a4)^2 - \varepsilon 4^2}}{(1 - \mathrm{Re}(a4))^2 - \varepsilon 4^2} :$

$x42tilde := \dfrac{\mathrm{Re}(a4) + \mathrm{Im}(a4)\cdot s42 + s42^2}{1 + s42^2};$

$x42m := -0.97422037135258412999$

0.22559846639911600920

$x40m := -1.10572972907884428202$

$s41 := -5.81045211005569968382$

$x41tilde := -1.09289661103671340916$

$x42tilde := 0.60514061962699594343$

(A.6*) - (A.11*)

> $\theta 41m := x \rightarrow 6\cdot \arctan\left(\dfrac{y4m(x)}{x+1}\right) :\ \theta 42m := x \rightarrow 4\cdot \arctan\left(\dfrac{y4m(x)}{x-1}\right) :\ \theta 43m := x \rightarrow \arctan\left(\dfrac{y4m(x)}{x}\right) :$

$\theta 44m := x \rightarrow 4\cdot \arctan\left(\dfrac{y4m(x) - \mathrm{Im}(\omega)}{x - \mathrm{Re}(\omega)}\right) :\ \theta 45m := x \rightarrow 4\cdot \arctan\left(\dfrac{y4m(x) + \mathrm{Im}(\omega)}{x - \mathrm{Re}(\omega)}\right) :$

$\Theta 4m := (x, xp) \rightarrow \theta 41m(xp) + \theta 42m(xp) - \theta 43m(x) - \theta 44m(x) - \theta 45m(x) + \pi :$

$x401m := -1.096 :$

$\Theta 4m(x40m, x401m) + 2\cdot\pi;$

$\Theta 4m(x401m, x40m) + 2\cdot\pi;$

$6\cdot \arctan(s41) + \theta 42m(-1) - \theta 43m(x401m) - \theta 44m(x401m) - \theta 45m(x401m) + \pi + 2\cdot\pi;$

$6\cdot\left(-\dfrac{\pi}{2.0}\right) + \theta 42m(x401m) - \theta 43m(-1) - \theta 44m(-1) - \theta 45m(-1) + \pi + 2\cdot\pi;$

$y41m := x \to s41 \cdot (x+1)$: $x411m := -1.04$:

$\Theta41m := (x, xp) \to 6 \cdot \arctan(s41) + 4 \cdot \arctan\left(\frac{y41m(xp)}{xp-1}\right) - \arctan\left(\frac{y41m(x)}{x}\right) - 4 \cdot \arctan\left(\frac{y41m(x) - \mathrm{Im}(\omega)}{x - \mathrm{Re}(\omega)}\right) - 4 \cdot \arctan\left(\frac{y41m(x) + \mathrm{Im}(\omega)}{x - \mathrm{Re}(\omega)}\right) + 3 \cdot \pi$:

$\Theta41m(x41tilde, x411m)$;
$\Theta41m(x411m, x41tilde)$;
$\Theta41m(x411m, -1)$;
$\Theta41m(-1, x411m)$;

3.13418076253032808990
2.24502382998534015678
3.08452471530709453411
0.30005875595709586067
3.08601065478715113288
1.14612314292602546612
2.15574703263482464122
0.56883884835602326325

(A.12*)

$x40p := -0.22 - \varepsilon4$: $x41p := -1$: $x42p := 0$: $x43p := \mathrm{Re}(\omega)$:

$\theta41p := x \to 6 \cdot \arctan\left(\frac{y4p(x)}{x+1}\right)$: $\theta42p := x \to 4 \cdot \arctan\left(\frac{y4p(x)}{x-1}\right)$: $\theta43p := x \to \arctan\left(\frac{y4p(x)}{x}\right)$:

$\theta44p := x \to 4 \cdot \arctan\left(\frac{y4p(x) - \mathrm{Im}(\omega)}{x - \mathrm{Re}(\omega)}\right)$: $\theta45p := x \to 4 \cdot \arctan\left(\frac{y4p(x) + \mathrm{Im}(\omega)}{x - \mathrm{Re}(\omega)}\right)$:

$\Theta4p := (x, xp) \to \theta41p(xp) + \theta42p(xp) - \theta43p(x) - \theta44p(x) - \theta45p(x)$:

$xx40 := Vector[row](1..14, 1)$:

```
for k from 1 to 6 do
 xx40[k] :=-1.11 + 0.005·k :
end do:

for k from 7 to 14 do
 xx40[k] :=-1.14 + 0.01·k :
end do:
```

$\Theta4p(xx40[1], x40p) + \pi + 2 \cdot \pi$;

```
for k from 1 to 13 do
```

```
 print( Θ4p(xx40[k+1], xx40[k]) + π + 2·π );
end do;

print(LowerBound);

Θ4p(x40p, xx40[1]) + π + 2·π;

for k from 1 to 12 do
 print( Θ4p(xx40[k], xx40[k+1]) + π + 2·π );
end do;

−3·π + θ42p(xx40[14]) − θ43p(xx40[13]) − θ44p(xx40[13]) − θ45p(xx40[13]) + π + 2
   ·π;
```

2.95602846971210530183
3.12250275932122988890
3.06430416127340148277
3.05706200297877228386
3.05666806606288602322
3.05814196421500954909
3.14043964842157122532
3.13493893700025508400
3.13166439699199788474
3.12929480824939037865
3.12730207169826467740
3.12544939921513714521
3.12362373165286824043
3.12176981187101298946
LowerBound
2.68999698327280825373
2.62503504403602137402
2.77026140692453803509
2.82279292542446767765
2.85491144304646321843
2.87767715231732167749
2.82199927201674041963
2.85478890237106299088
2.87783640382417034174
2.89501464262163155053
2.90828771252624134977
2.91878694957363216359
2.92722234697431094792

2.93406794464784226659

(A.13*)

```
> xx41 := Vector[row](1..50, 1):

for k from 1 to 20 do
 xx41[k] :=-1 + 0.01·k:
end do:

for k from 21 to 40 do
 xx41[k] :=-1.2 + 0.02·k:
end do:

for k from 41 to 50 do
 xx41[k] :=-2 + 0.04·k:
end do:

3·π + θ42p(-1) − θ43p(xx41[1]) − θ44p(xx41[1]) − θ45p(xx41[1]) − 5·π + 2·π;

for k from 1 to 48 do
 print(Θ4p(xx41[k+1], xx41[k]) − 5·π + 2·π);
end do;

θ41p(-0.04) + θ42p(-0.04) + π/2 - θ44p(0) − θ45p(0) − 5·π + 2·π;

print(LowerBound);

Θ4p(-1.0, -0.99) − 5·π + 2·π;

for k from 1 to 49 do
 print(Θ4p(xx41[k], xx41[k+1]) − 5·π + 2·π);
end do;
```

3.11986117912592778891
3.11788633556232571683
3.11584174221404041107
3.11372811131944475798
3.11154837892330191836
3.10930657183740951650
3.10700716720651390002
3.10465473072827681604
3.10225371567141632926
3.09980835593518856222
3.09732261446691818075

3.0948001642119821074 0
3.09224438794188631833
3.08965838871684955064
3.08704500598474861532
3.08440683429108630159
3.08174624278611651178
3.07906539446432120163
3.07636626453603068144
3.07365065761875464270
3.12847607665879759416
3.12166149405139198945
3.11493136153078032345
3.10827808085844058662
3.10169563750739686874
3.09517920304530120682
3.08872485437694129501
3.08232937241502559646
3.07599009573444450317
3.06970481293709437211
3.06347168270572515467
3.05728917396495461302
3.05115602086040349281
3.04507118882018678718
3.03903384903132055049
3.03304335940899645057
3.02709925066401101180
3.02120121645186925804
3.01534910686222856595
3.00954292471055246890
3.09328112670879206471
3.08117635902018991418
3.06944244840977169416
3.05807738419399362082
3.04708542930814774482
3.03647728050112476287
3.02627054171924103010
3.01649053899729312507
3.00717153884646822717
2.99835847400568785056

LowerBound

2.93965545333877462822
2.94422569795281744226
2.94795878924200011399
2.95099276525822747065
2.95343560981702341768
2.95537328095548763116
2.95687523768408245324
2.95799834399348863848
2.95878968900756047102
2.95928866453648570439
2.95952852232071837582
2.95953755941317927721
2.95934003304971073361
2.95895687558392722740
2.95840625951458969607
2.95770404864309584259
2.95686416170297126958
2.95589886797486809599
2.95481902952024062223
2.95363430113135966058
2.89271640451391757156
2.89112470764974594608
2.88911657168126475723
2.88674943226510803797
2.88407065636241344571
2.88111967926939655183
2.87792961314316002941
2.87452847524829132316
2.87094013805775670102
2.86718507288580847113
2.86328093819816638394
2.85924304964834901384
2.85508475905000342398
2.85081776252322982800
2.84645235304396349336
2.84199762898082823892
2.83746166752075072079
2.83285166988796793741
2.82817408376042589873
2.82343470714939783712

2.72385901379047685149
2.71484425565777062080
2.70548321868466820714
2.69581310372951810620
2.68586457897371459791
2.67566318425344431387
2.66523044991774278161
2.65458481702439968738
2.64374242541517385669
2.63271782418825775505

(A.14*)

```
> xx42 := Vector[row](1..26, 1) :

for k from 1 to 11 do
 xx42[k] := 0.05·k :
end do:

xx42[12] := 0.575 : xx42[13] := x42tilde :

Θ4p(xx42[1], 0) − 4·π + 2·π;

for k from 1 to 12 do
 print(Θ4p(xx42[k+1], xx42[k]) − 4·π + 2·π);
end do;

print(LowerBound);
θ41p(xx42[1]) + θ42p(xx42[1]) - π/2 - θ44p(0) − θ45p(0) − 4·π + 2·π;

for k from 1 to 12 do
 print(Θ4p(xx42[k], xx42[k+1]) − 4·π + 2·π);
end do;
```

3.03527624873707126422
3.02660631808300606271
3.01933184285403232454
3.01375857373412931291
3.01031557649249983487
3.00961949980791808310
3.01258636427326142771
3.02063910852586984936
3.03612749928547383434

3.06328439906700220237
3.11082012409569644233
2.97273934363200946259
3.06668814627445936861
LowerBound
2.57399401823834303741
2.55914957137258897272
2.54391625362160233946
2.52829707756773762063
2.51229138260830286457
2.49589438180441554451
2.47909479093034997444
2.46186708696892617200
2.44414735164792855966
2.42575276241710072715
2.40606945006609641723
2.58257998158703937093
2.54325190923556221369

(A.15*) - (A.17*)

```
> for k from 14 to 20 do
   xx42[k] := 0.625 + 0.005·(k − 14) :
  end do:

  for k from 21 to 25 do
   xx42[k] := 0.659 + 0.001·(k − 21) :
  end do:

  xx42[26] := 0.6635 :

  Θ4phat := (x, xp) → θ41p(xp) + θ42p(x) − θ43p(x) − θ44p(x) − θ45p(x) − 4·π + 2·π :

  for k from 13 to 25 do
   print(Θ4phat(xx42[k + 1], xx42[k]));
  end do;

  print(LowerBound);

  for k from 13 to 25 do
   print(Θ4phat(xx42[k], xx42[k + 1]));
  end do;

  print(xx2627);
```

$\theta 41p(0.6635) + \theta 42p(\mathrm{Re}(\omega)) + 4\cdot\pi - \theta 43p(\mathrm{Re}(\omega)) - 4\cdot\left(\arctan\left(\frac{\mathrm{Im}(\omega) - 0.69}{\mathrm{Re}(\omega) + 0.22}\right) + \frac{\pi}{2}\right)$

$- 4\cdot\frac{\pi}{2.0} + 2\cdot\pi;$

$\Theta 4phat(0.6635, \mathrm{Re}(\omega));$

3.04864590390078985191
2.92979846524619177491
2.94922494426224122128
2.97236324979572445625
3.00068143898326827974
3.03677412369591339084
3.08597976952317262354
3.12419642405816280290
3.07398024341245821796
3.09047156551115431425
3.10994955149503824856
3.13399811255233658212
3.12985612238483782104
LowerBound
2.63734662731020164075
2.80952182742728415085
2.81927780562865660233
2.83017213431550824987
2.84232188635220147931
2.85571565761197632141
2.86976781100039692116
2.90699136228267209832
3.00929254529897338117
3.01941334051279627238
3.03033915558565197351
3.04203775808455068149
3.07710701815640652102
xx2627
3.13310435455568056457
3.09398471246742172765

Proof of Lemma 14.1(c) and Lemma 7.2(b): (A.18*)

$> \omega := \frac{8\sqrt{6.0} - 3}{25} + \frac{6\sqrt{6.0} + 4}{25}\mathrm{i}:$

$$a5 := 0.78 + 0.21\mathrm{i} :$$
$$\varepsilon 5 := |a5 - \omega| :$$
$$x51m := \mathrm{Re}(a5) + \sqrt{1 + \mathrm{Re}(a5)^2 - 2\cdot(\mathrm{Re}(a5)\cdot\mathrm{Re}(\omega) + \mathrm{Im}(a5)\cdot\mathrm{Im}(\omega))} ;$$
$$x52m := 0.78 + \varepsilon 5;$$

$$x51m := 1.28863164896621714360$$
$$x52m := 1.33027825173278758344$$

(A.19*)

$$y5p := x \to \mathrm{Im}(a5) + \sqrt{\varepsilon 5^2 - (x - \mathrm{Re}(a5))^2} :$$
$$\theta 51p := x \to 6\cdot\arctan\left(\frac{y5p(x)}{x+1}\right) : \theta 52p := x \to 4\cdot\arctan\left(\frac{y5p(x)}{x-1}\right) : \theta 53p := x \to \arctan\left(\frac{y5p(x)}{x}\right) :$$
$$\theta 54p := x \to 4\cdot\arctan\left(\frac{y5p(x) - \mathrm{Im}(\omega)}{x - \mathrm{Re}(\omega)}\right) : \theta 55p := x \to 4\cdot\arctan\left(\frac{y5p(x) + \mathrm{Im}(\omega)}{x - \mathrm{Re}(\omega)}\right) :$$
$$\Theta 5p := (x, xp) \to \theta 51p(xp) + \theta 52p(xp) - \theta 53p(x) - \theta 54p(x) - \theta 55p(x) :$$

```
xx50 := Vector[row](1..12, 1) :

3·π/4.0 ;

for k from 1 to 11 do
 xx50[k] := 0.66 + 0.03·k :
end do:

xx50[12] := 1 :

for k from 1 to 10 do
 print(Θ5p(xx50[k], xx50[k+1]) + 4·π);
end do;
```

$$\theta 51p(1) + 4\cdot\left(-\frac{\pi}{2}\right) - \theta 53p(xx50[11]) - \theta 54p(xx50[11]) - \theta 55p(xx50[11]) + 4\cdot\pi;$$

```
print(UpperBound);

for k from 1 to 11 do
 print(Θ5p(xx50[k+1], xx50[k]) + 4·π);
end do;
```

2.35619449019234492885

2.39152641523649320383

2.42360215407717066256
2.45094740795044099067
2.47375903624628573550
2.49216015801002882225
2.50620174545475441739
2.51586026123200589324
2.52103080942900708859
2.52151463955720814658
2.51699890505399865865
2.66361386315768322668
UpperBound
2.77182869339959515275
2.80697205644853398375
2.83822736489737437299
2.86584599820327756448
2.89002811640705719109
2.91092823563501543751
2.92865896605917425413
2.94329317499188139963
2.95486473453927980164
2.96336795294600280759
2.81568713772162422213

(A.20*) - (A.21*)

> $\Theta5p(xx50[1], \mathrm{Re}(\omega)) + 4\cdot\pi;$
$\theta51p(xx50[1]) + \theta52p(xx50[1]) - \theta53p(\mathrm{Re}(\omega)) - 4\cdot\arctan\left(\frac{0.78 - \mathrm{Re}(\omega)}{\mathrm{Im}(\omega) - 0.21}\right) - 4\cdot\frac{\pi}{2} + 4\cdot\pi;$
$\frac{3\cdot\pi}{4.0};$

2.71109759363462642882
2.38130456898431719130
2.35619449019234492885

(A.22*)

> $xx51p := Vector[row](1..23, 1):$
$xx51p[1] := 1.04 : xx51p[2] := 1.08 :$

$\Theta5p(1, xx51p[1]); \Theta5p(xx51p[1], xx51p[2]);$

$\theta 51p(1) + 4\cdot\frac{\pi}{2} - \theta 53p(xx51p[1]) - \theta 54p(xx51p[1]) - \theta 55p(xx51p[1]);$

$\Theta 5p(xx51p[2],\ xx51p[1]);$

$$2.42084290987285622343$$
$$2.38854943594672417393$$
$$3.04826057324547574734$$
$$3.05274579760865584231$$

The real parts of Tilde{x51,x52,x53}

$$s51 := \frac{\mathrm{Im}(a5)}{\mathrm{Re}(a5)+1} : xtilde51 := \mathrm{Re}(a5) + \frac{\varepsilon 5}{\sqrt{1+s51^2}};$$
$$s52 := \frac{\mathrm{Im}(a5)}{\mathrm{Re}(a5)} : xtilde52 := \mathrm{Re}(a5) + \frac{\varepsilon 5}{\sqrt{1+s52^2}};$$
$$s53 := \frac{\mathrm{Im}(a5+\omega)}{\mathrm{Re}(a5-\omega)} : xtilde53 := \mathrm{Re}(a5) + \frac{\varepsilon 5}{\sqrt{1+s53^2}};$$

$$xtilde51 := 1.32648819251098163356$$
$$xtilde52 := 1.31135735144438370168$$
$$xtilde53 := 0.846247722430934546 37$$

(A.23*) - (A.24*) for 2<= k<= 17

$$\xi 51p := x \to \left((x+1)^2 + (y5p(x))^2\right)^3 : \xi 52p := x \to \left((x-1)^2 + (y5p(x))^2\right)^2 : \xi 53p := x \to \left(x^2 + (y5p(x))^2\right)^{\frac{1}{2}} :$$

$$\xi 54p := x \to \left((x - \mathrm{Re}(\omega))^2 + (y5p(x) - \mathrm{Im}(\omega))^2\right)^2 : \xi 55p := x \to \left((x - \mathrm{Re}(\omega))^2 + (y5p(x) + \mathrm{Im}(\omega))^2\right)^2 :$$

$$\Xi Tilde51p := (x,\ xp) \to \frac{\xi 51p(xp)\cdot \xi 52p(x)}{\xi 53p(x)\cdot \xi 54p(x)\cdot \xi 55p(xp)} :$$

$$h5p1 := (x,\ xp) \to \Xi Tilde51p(x,\ xp)\cdot\cos(\Theta 5p(x,\ xp)) :$$

$$v5p1 := (x,\ xp) \to \Xi Tilde51p(x,\ xp)\cdot\sin(\Theta 5p(x,\ xp)) + \left(h5p1(x,\ xp) - 4.2\cdot\sqrt{2.0}\right) :$$

```
for k from 3 to 10 do
 xx51p[k] := 1.1 + 0.02·(k − 3) :
end do:

for k from 11 to 16 do
 xx51p[k] := 1.25 + 0.01·(k − 11) :
```

```
end do:

xx51p[17] := 1.305 : xx51p[18] := xtilde52 :

for k from 2 to 17 do
 print(Θ5p(xx51p[k], xx51p[k+1]));
end do;

print(UpperBound);

for k from 2 to 17 do
 print(Θ5p(xx51p[k+1], xx51p[k]));
end do;

 print(MaxRealPart);

for k from 2 to 17 do
 print(h5p1(xx51p[k], xx51p[k+1]));
end do;

print(Arrayv5p1);

for k from 2 to 17 do
 print(v5p1(xx51p[k], xx51p[k+1]));
end do;
```

2.53005557432599052718
2.50819542152695016748
2.48177731670674647111
2.44995467971836798560
2.41156514979765543618
2.36495736954980414665
2.30767803903729274473
2.23585666369066030763
2.31387731378892642729
2.27012939129948512086
2.21927339353992426237
2.15900320128976299740
2.08558917756014913556
1.99235067619373615660
2.03265590384007504154
1.92199995992324827775
UpperBound
2.87922191503449090204
2.87106298779527608065

2.86062173390056445775
2.84767620243922969032
2.83195952762021669978
2.81315785207097125413
2.79092214258195409031
2.76492635739728973199
2.60061275051600448876
2.57562965057896773939
2.54790237356738034220
2.51705886983239786295
2.48275779291063929153
2.44495921496877556938
2.28855735681263173080
2.28922396728775907007

MaxRealPart

−101.56561817148329782297
−80.63574223741394518756
−63.95656532864819221675
−50.54186468188970887903
−39.65819046884089328948
−30.75167852209770250552
−23.39662076867928615801
−17.25742314212595235710
−14.35465186640587060237
−12.09352581872567473791
−10.00020329053685574288
−8.04869939832353594271
−6.20944095926663754440
−4.44245204836852184649
−3.93560192450630388404
−2.80289182457246038376

Arrayv5p1

−36.28656439117751354769
−27.36212215922521207847
−20.27817931141196040858
−14.62816879402083657475
−10.10511324558615764985
−6.47397531851087906247
−3.55172509041207887208
−1.19198657063555319633
−4.67032631365639531065
−3.65583321196967954625
−2.74360983780773983008
−1.92068900135105123520
−1.17195773506726019171

−0.47565056529004817398
−1.96878381461208140237
−1.09263094285252575675

(A.23*) - (A.24*) for 18<= k<= 20

```
> xx51p[19] := 1.317 : xx51p[20] := 1.323 : xx51p[21] := xtilde51 :

ΞTilde52p := (x, xp) → (ξ51p(xp)·ξ52p(x))/(ξ53p(xp)·ξ54p(x)·ξ55p(xp)) :
h5p2 := (x, xp) → ΞTilde52p(x, xp)·cos(Θ5p(x, xp)) :
v5p2 := (x, xp) → ΞTilde52p(x, xp)·sin(Θ5p(x, xp)) + (h5p2(x, xp) − 4.2·√2.0) :

for k from 18 to 20 do
 print(Θ5p(xx51p[k], xx51p[k+1]));
end do;

print(UpperBound);

for k from 18 to 20 do
 print(Θ5p(xx51p[k+1], xx51p[k]));
end do;

print(MaxRealPart);

for k from 18 to 20 do
 print(h5p2(xx51p[k], xx51p[k+1]));
end do;

print(Arrayv5p2);

for k from 18 to 20 do
 print(v5p2(xx51p[k], xx51p[k+1]));
end do;
```

1.84280874445742173599
1.68332336741951497923
1.63318517295129039208
UpperBound
2.22785140000711627446
2.20446118157263578301
2.05168727695372621941
MaxRealPart
−1.91225591303869002474
−0.70830240359249259916
−0.32097714442158573927

$$Arrayv5p2$$
$$-0.9961683162324374494 9$$

-0.99616831623243744949

-0.38008113412988319313

-1.12256685919536851573

(A.23*) - (A.24*) for 21<= k<= 22

$xx51p[22] := 1.329 : xx51p[23] := 0.78 + \varepsilon5 :$

$$\Xi Tilde53p := (x, xp) \rightarrow \frac{\xi 51p(x) \cdot \xi 52p(x)}{\xi 53p(xp) \cdot \xi 54p(x) \cdot \xi 55p(xp)} :$$

$h5p3 := (x, xp) \rightarrow \Xi Tilde53p(x, xp) \cdot \cos(\Theta 5p(x, xp)) :$

$v5p3 := (x, xp) \rightarrow \Xi Tilde53p(x, xp) \cdot \sin(\Theta 5p(x, xp)) + \left(h5p3(x, xp) - 4.2 \cdot \sqrt{2.0}\right) :$

$\Theta 5p(xx51p[21], xx51p[22]);$

$\Theta 5p(xx51p[22], xx51p[21]);$

$h5p3(xx51p[21], xx51p[22]);$

$v5p3(xx51p[21], xx51p[22]);$

$$\Theta 5p2223 := 6 \cdot \arctan\left(\frac{0.21}{xx51p[23] + 1}\right) + 4 \cdot \arctan\left(\frac{0.21}{xx51p[23] - 1}\right) - \theta 53p(xx51p[22]) - \theta 54p(xx51p[22]) - \theta 55p(xx51p[22]);$$

$$\Theta 5p2322 := \theta 51p(xx51p[22]) + \theta 52p(xx51p[22]) - \arctan\left(\frac{0.21}{xx51p[23]}\right) - 4 \cdot \arctan\left(\frac{0.21 - \mathrm{Im}(\omega)}{xx51p[23] - \mathrm{Re}(\omega)}\right) - 4 \cdot \arctan\left(\frac{0.21 + \mathrm{Im}(\omega)}{xx51p[23] - \mathrm{Re}(\omega)}\right);$$

$$\Xi Tilde53p2223 := (\xi 51p(xx51p[22]) \cdot \xi 52p(xx51p[22])) \Big/ \left(\left(xx51p[23]^2 + 0.21^2\right)^{\frac{1}{2}} \cdot \xi 54p(xx51p[22]) \cdot \left(\left(xx51p[23] - \mathrm{Re}(\omega)\right)^2 + \left(0.21 + \mathrm{Im}(\omega)\right)^2\right)^2\right) :$$

$\Xi Tilde53p2223 \cdot \cos(\Theta 5p2223);$

$\Xi Tilde53p2223 \cdot \sin(\Theta 5p2223) + \left(\Xi Tilde53p2223 \cdot \cos(\Theta 5p2223) - 4.2 \cdot \sqrt{2.0}\right);$

1.50934837796723374940

1.96982677327378363285

0.27504884983351399722

-1.19415582074523830612

$\Theta 5p2223 := 1.27361771560225278523$

$\Theta 5p2322 := 1.92281761797400576403$

1.15837200377495597082

-0.99885578205211488319

(A.25*) - (A.28*)

```
> Q(x51m);
```

$$\frac{1}{4}\cdot\frac{0.7}{\mathrm{D}(Q)(x51m)};$$

$$y5m := x \to \mathrm{Im}(a5) - \sqrt{\varepsilon 5^2 - (x - \mathrm{Re}(a5))^2} :$$

$$x511m := 1.293 :$$

$$x511m - x51m;$$

$$y5m(x511m);$$

$$\frac{0.0175}{\sqrt{2.0}};$$

$$s54 := \frac{\mathrm{Im}(\omega)}{\mathrm{Re}(x51m - \omega)} :$$

$$(1.2 - \mathrm{Re}(a5))^2 + (s54\cdot(1.2 - x51m) - \mathrm{Im}(a5))^2 - \varepsilon 5^2$$

0.9514277570192032314

0.01761655418104595223

0.004368351033782856

0.01090918069862402346

0.01237436867076458168

−0.02649212774475030710

(A.29*)

$$> \theta 51m := x \to 6\cdot\arctan\left(\frac{y5m(x)}{x+1}\right) : \theta 52m := x \to 4\cdot\arctan\left(\frac{y5m(x)}{x-1}\right) : \theta 53m := x \to \arctan\left(\frac{y5m(x)}{x}\right) :$$

$$\theta 54m := x \to 4\cdot\arctan\left(\frac{y5m(x) - \mathrm{Im}(\omega)}{x - \mathrm{Re}(\omega)}\right) : \theta 55m := x \to 4\cdot\arctan\left(\frac{y5m(x) + \mathrm{Im}(\omega)}{x - \mathrm{Re}(\omega)}\right) :$$

$$\Theta 5m := (x, xp) \to \theta 51m(xp) + \theta 52m(xp) - \theta 53m(x) - \theta 54m(x) - \theta 55m(x) :$$

$$xx51m := Vector[row](1..11, 1) :$$

$$xx51m[1] := 1.293 : xx51m[2] := 1.297 : xx51m[3] := 1.3 : xx51m[4] := 1.305 : xx51m[5] := 1.31 : xx51m[6] := 1.315 :$$

$$xx51m[7] := 1.32 : xx51m[8] := 1.325 : xx51m[9] := 1.327 : xx51m[10] := 1.329 : xx51m[11] := 0.78 + \varepsilon 5 :$$

```
for k from 1 to 9 do
 print(Θ5m(xx51m[k+1], xx51m[k]));
end do;
```

$$\Theta 5m1110 := \theta 51m(xx51m[10]) + \theta 52m(xx51m[10]) - \arctan\left(\frac{0.21}{xx51m[11]}\right) - 4$$

$\cdot\arctan\left(\frac{0.21-\mathrm{Im}(\omega)}{xx51m[11]-\mathrm{Re}(\omega)}\right)-4\cdot\arctan\left(\frac{0.21+\mathrm{Im}(\omega)}{xx51m[11]-\mathrm{Re}(\omega)}\right)$;

```
for k from 1 to 9 do
 print(Θ5m(xx51m[k], xx51m[k+1]));
end do;
```

$$\Theta5m1011 := 6\cdot\arctan\left(\frac{0.21}{xx51m[11]+1}\right)+4\cdot\arctan\left(\frac{0.21}{xx51m[11]-1}\right)-\theta53m(xx51m[10])-\theta54m(xx51m[10])-\theta55m(xx51m[10]);$$

0.04714960557648371900
0.1644464740913582650 8
0.20331290844267341912
0.32921388879036368334
0.45721826990362281089
0.58645002293148379539
0.71051527683945466471
0.99270020869487450095
1.05029675112219134934
$\Theta5m1110 := 1.08331220089597670348$
0.27993111227875697857
0.34640296162005112682
0.52369903564919684798
0.67712548516609522809
0.84407889014948345184
1.03441418037988856759
1.27574007924921690635
1.28968885654341956782
1.45884131065334485742
$\Theta5m1011 := 1.74809534058696657880$

(A.30*)

$$\xi51m := x\to\left((x+1)^2+(y5m(x))^2\right)^3 : \xi52m := x\to\left((x-1)^2+(y5m(x))^2\right)^2 : \xi53m := x\to\left(x^2+(y5m(x))^2\right)^{\frac{1}{2}} :$$

$$\xi54m := x\to\left((x-\mathrm{Re}(\omega))^2+(y5m(x)-\mathrm{Im}(\omega))^2\right)^2 : \xi55m := x\to\left((x-\mathrm{Re}(\omega))^2+(y5m(x)+\mathrm{Im}(\omega))^2\right)^2 :$$

$$\Xi5m := (x,\ xp) \rightarrow \frac{\xi51m(xp)\cdot\xi52m(xp)}{\xi53m(x)\cdot\xi54m(xp)\cdot\xi55m(x)}\ :$$

```
for k from 1 to 9 do
 print(Ξ5m(xx51m[k], xx51m[k+1]));
end do;
```

$$\Xi5m1011 := \frac{\left((xx51m[11]+1)^2+0.21^2\right)^3\cdot\left((xx51m[11]-1)^2+0.21^2\right)^2}{\xi53m(xx51m[10])\cdot\left((xx51m[11]-\mathrm{Re}(\omega))^2+(0.21-\mathrm{Im}(\omega))^2\right)^2\cdot\xi55m(xx51m[10])};$$

1.09784895543866663432
1.13044235825676227285
1.24966767965217635988
1.36591561066656938262
1.51813938607480203441
1.73144957602971311354
2.07596185037674711648
2.15635131640509407195
2.49081646174515493937
$\Xi5m1011 := 3.21086664670108366507$

(A.31*)

```
> for k from 1 to 9 do
 print(Ξ5m(xx51m[k], xx51m[k+1])·cos(Θ5m(xx51m[k+1], xx51m[k])));
end do;

Ξ5m1011·cos(Θ5m1110);
```

1.09662887555173383388
1.11519169848239494470
1.22392830876264195092
1.29256155101778649401
1.36220213976584124526
1.44214254583145420273
1.57363285100138623346
1.17829595209612144530
1.23871694430089073817
1.50398458748504774441