Estimates of the Parabolic Renormalization for Local Degree Three

Fei Yang

About this file :

This file is the Mathematica 12.0 worksheet for checking the numerical estimations in the proof of the Main Theorem in the paper :
Yang Fei, Parabolic and near–parabolic renormalizations for local degree three.
Last updated : June 29, 2024

§ 2.2 Preliminaries on *P*

• Proposition 2.2 : the values of μ, cp_P, cv_P, v_1^P and v_2^P

In[ ]:=

```
11 - 4 Sqrt[6] // N
1 - 2/3 Sqrt[6] // N
-16/(3 (8 Sqrt[6] + 3)) // N
v1p = 9 Sqrt[6] - 47/2 - 1/2 Sqrt[996 Sqrt[6] - 2439];
v2p = 9 Sqrt[6] - 47/2 + 1/2 Sqrt[996 Sqrt[6] - 2439];
v1p // N
v2p // N
```

Out[ ]= 1.202041028867289

Out[ ]= −0.6329931618554521

Out[ ]= −0.2360308329566638

Out[ ]= −1.870460025902034

Out[ ]= −1.038724604000762

§ 6 Passing from *P* to *Q*

• Lemma 6.1 : the values of cp, cp', ω, cv, v_1 and v_2

In[ ]:=

$$cp = \frac{1}{5}\left(1 + 4\sqrt{6} + 2\sqrt{2\left(9 + \sqrt{6}\right)}\right);$$

$$cpp = \frac{1}{5}\left(1 + 4\sqrt{6} - 2\sqrt{2\left(9 + \sqrt{6}\right)}\right);$$

$$\omega = \frac{8\sqrt{6} - 3}{25} + \frac{6\sqrt{6} + 4}{25}\,i;$$

$$cv = \frac{3\left(8\sqrt{6} + 3\right)}{4};$$

$$\nu 1 = \frac{2 + \nu 1p}{-\nu 1p} + \frac{2\sqrt{-1 - \nu 1p}}{-\nu 1p}\,i;$$

$$\nu 2 = \frac{2 + \nu 2p}{-\nu 2p} + \frac{2\sqrt{-1 - \nu 2p}}{-\nu 2p}\,i;$$

```
cp // N
cpp // N
Re[ω] // N
Im[ω] // N
cv // N
Re[ν1] // N
Im[ν1] // N
Re[ν2] // N
Im[ν2] // N
```

Out[ ]= 4.073706920777351

Out[ ]= 0.2454766676757338

Out[ ]= 0.6638367176906169

Out[ ]= 0.7478775382679627

Out[ ]= 16.94693845669907

Out[ ]= 0.06925567630641816

Out[ ]= 0.9975989431125819

Out[ ]= 0.9254381693634482

Out[ ]= 0.3788986601787213

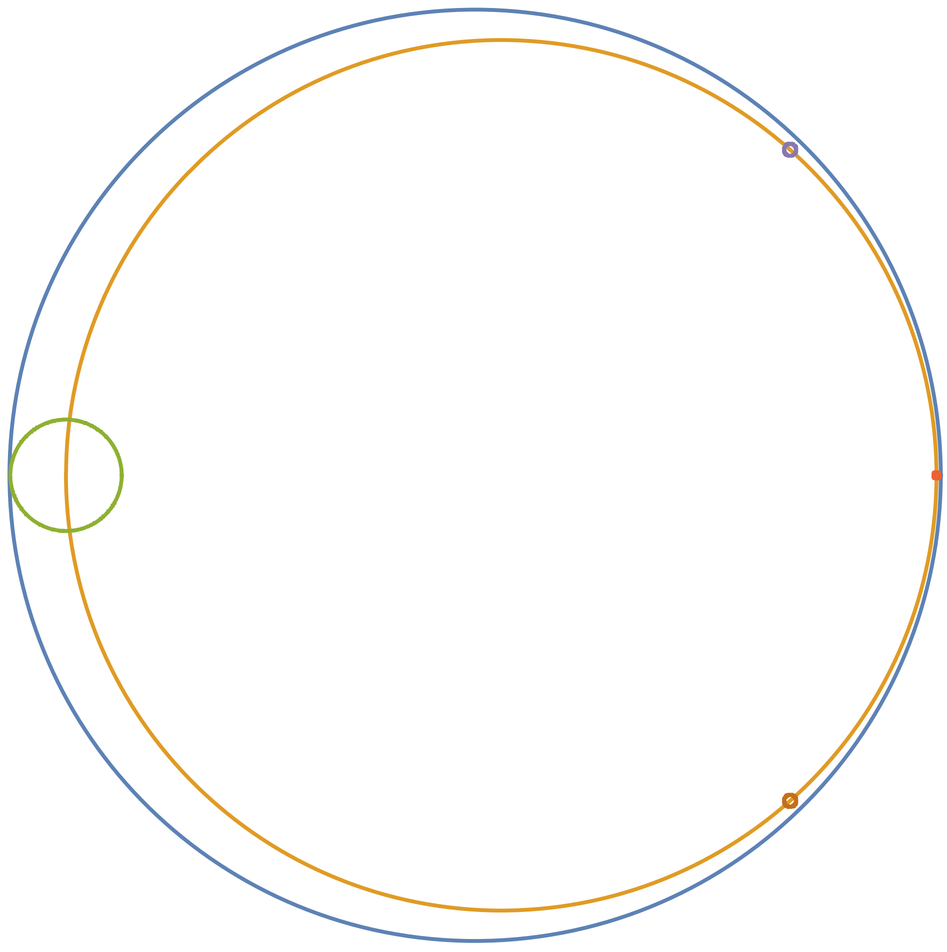
§ 7 Estimates on Q: Part I

• Lemma 7.1 : Figure 9 (left)

In[ ]:=
```
a0 = -0.06; r0 = 1.07; ϵ1 = 0.128; ϵ2 = 0.007; ϵ3 = 0.014;

ParametricPlot[{{-0.06 + r0 Cos[u], r0 Sin[u]},
   {Cos[u], Sin[u]}, {-1 + ϵ1 Cos[u], ϵ1 Sin[u]}, {1 + ϵ2 Cos[u], ϵ2 Sin[u]},
   {Re[ω] + ϵ3 Cos[u], Im[ω] + ϵ3 Sin[u]}, {Re[ω] + ϵ3 Cos[u], -Im[ω] + ϵ3 Sin[u]}},
  {u, 0, 2 Pi}, PlotRange → All, PlotStyle → {Thick},
  Axes → False, AspectRatio → Automatic]
```

Out[ ]=



• Lemma 7.1 : (7.2*)-(7.5*)

In[ ]:=
$$\eta = 3;$$
$$\frac{\epsilon 1^6\,(2-\epsilon 1)^4}{(1+\epsilon 1)\,(2+\epsilon 1)^8}$$
$$cv\, e^{-2\pi\eta}\ \ //\ N$$
$$\frac{(2-\epsilon 2)^6\,\epsilon 2^4}{(1+\epsilon 2)\,(1+\epsilon 2)^8}$$
$$\frac{(2+\epsilon 3)^6\,(1+\epsilon 3)^4}{(1-\epsilon 3)\,\epsilon 3^4\,(1-\epsilon 3)^4}$$
$$cv\, e^{2\pi\eta}\ \ //\ N$$
$$(\mathrm{Re}[\omega]+0.06)^2+\mathrm{Im}[\omega]^2-(r0-\epsilon 3)^2$$

Out[ ]=
$1.138678464038111 \times 10^{-7}$

Out[ ]=
$1.103654476748063 \times 10^{-7}$

Out[ ]=
$1.413092909648051 \times 10^{-7}$

Out[ ]=
$1.970660900297532 \times 10^{9}$

Out[ ]=
$2.602252145992121 \times 10^{9}$

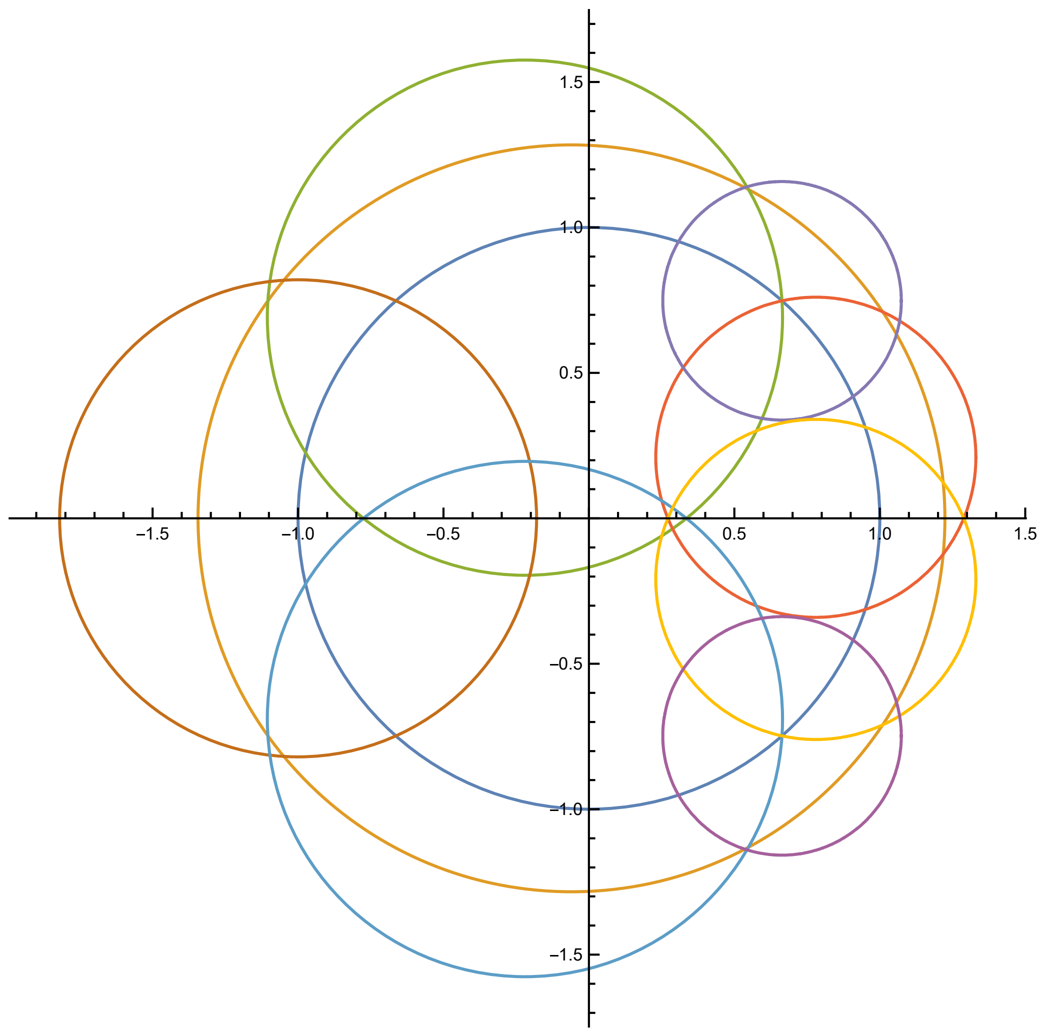
Out[ ]=
-0.03187559387712646

• Lemma 7.3: Figure 9 (right)

In[ ]:=
```
a4 = -0.22 + 0.69 ⅈ;
ϵ4 = Abs[a4 - ω];
a5 = 0.78 + 0.21 ⅈ;
ϵ5 = Abs[a5 - ω];
a6 = ω;
ϵ6 = 0.41;
ϵ7 = 0.82;
r1 = 1.2;

ParametricPlot[{{Cos[u], Sin[u]},
   {a0 + r0 r1 Cos[u], r0 r1 Sin[u]}, {Re[a4] + ϵ4 Cos[u], Im[a4] + ϵ4 Sin[u]},
   {Re[a5] + ϵ5 Cos[u], Im[a5] + ϵ5 Sin[u]}, {Re[ω] + ϵ6 Cos[u], Im[ω] + ϵ6 Sin[u]},
   {-1 + ϵ7 Cos[u], ϵ7 Sin[u]}, {Re[a4] + ϵ4 Cos[u], -Im[a4] + ϵ4 Sin[u]},
   {Re[a5] + ϵ5 Cos[u], -Im[a5] + ϵ5 Sin[u]}, {Re[ω] + ϵ6 Cos[u], -Im[ω] + ϵ6 Sin[u]}},
  {u, 0, 2 Pi}, PlotRange → All, AspectRatio → Automatic]
```

Out[ ]=



- **Lemma 7.3 : (7.8*)-(7.11*)**

In[ ]:=
```
y4p[x_] := Im[a4] + Sqrt[ϵ4^2 - (x - Re[a4])^2];
y4m[x_] := Im[a4] - Sqrt[ϵ4^2 - (x - Re[a4])^2];
y5p[x_] := Im[a5] + Sqrt[ϵ5^2 - (x - Re[a5])^2];
y5m[x_] := Im[a5] - Sqrt[ϵ5^2 - (x - Re[a5])^2];
y6p[x_] := Im[ω] + Sqrt[ϵ6^2 - (x - Re[ω])^2];
y6m[x_] := Im[ω] - Sqrt[ϵ6^2 - (x - Re[ω])^2];
y7p[x_] := Sqrt[ϵ7^2 - (x + 1)^2];
y7m[x_] := -Sqrt[ϵ7^2 - (x + 1)^2];

x60 = 0.54; x61 = Re[ω]; x62p = 1.07; x62m = 1.067; x63 = Re[ω] + ϵ6;
(x60 - Re[ω])^2 + (y4p[x60] - Im[ω])^2 - ϵ6^2
(x62p - Re[ω])^2 + (y5p[x62p] - Im[ω])^2 - ϵ6^2
(x60 - Re[a4])^2 + (y6p[x60] - Im[a4])^2 - ϵ4^2
(x62m - Re[a5])^2 + (y6m[x62m] - Im[a5])^2 - ϵ5^2
```

Out[ ]= 0.004847655527974171

Out[ ]= 0.001799072726483164

Out[ ]= −0.005559894224053563

Out[ ]= −0.005775095520950124

• Lemma 7.3 : (7.12*)-(7.13*)

In[ ]:=
```
y4p[x60] - 23/26 (x60 + 1)

ξ41p[x_] := ((x + 1)^2 + (y4p[x])^2)^3;
ξ42p[x_] := ((x - 1)^2 + (y4p[x])^2)^2;
ξ43p[x_] := (x^2 + (y4p[x])^2)^(1/2);
ξ44p[x_] := ((x - Re[ω])^2 + (y4p[x] - Im[ω])^2)^2;
ξ45p[x_] := ((x - Re[ω])^2 + (y4p[x] + Im[ω])^2)^2;
Ξ4p[x_, xp_] := (ξ41p[xp] * ξ42p[xp]) / (ξ43p[x] * ξ44p[x] * ξ45p[x]);
x601 = 0.6;
Ξ4p[x60, x601]
Ξ4p[x601, x61]
```

Out[ ]= −0.2174262209614934

Out[ ]= 140.6405351777206

Out[ ]= 217.0737524735838

• Lemma 7.3 : (7.14*)-(7.17*)

In[ ]:=

$$s61 = \frac{\text{Im}[a5 + \omega]}{\text{Re}[a5 - \omega]};$$

$$\text{Re}[a5] + \epsilon5 \Big/ \sqrt{1 + s61^2}$$

$$\text{Im}[a5] - \frac{\text{Im}[\omega]}{\text{Re}[\omega] - 1} (\text{Re}[a5] - 1)$$

$$\xi51p[x\_] := \left((x+1)^2 + (y5p[x])^2\right)^3;$$
$$\xi52p[x\_] := \left((x-1)^2 + (y5p[x])^2\right)^2;$$
$$\xi53p[x\_] := \left(x^2 + (y5p[x])^2\right)^{1/2};$$
$$\xi54p[x\_] := \left((x - \text{Re}[\omega])^2 + (y5p[x] - \text{Im}[\omega])^2\right)^2;$$
$$\xi55p[x\_] := \left((x - \text{Re}[\omega])^2 + (y5p[x] + \text{Im}[\omega])^2\right)^2;$$
$$\xi55pmax = \left(\sqrt{\left((\text{Re}[a5] - \text{Re}[\omega])^2 + (\text{Im}[a5] + \text{Im}[\omega])^2\right)} + \epsilon5\right)^4;$$

$$x611 = 0.99;$$
$$\frac{\xi51p[x61] * \xi52p[x611]}{\xi53p[x611] * \xi54p[x611] * \xi55pmax}$$

$$\text{Hat}\Xi5p[x\_, xp\_] := \frac{\xi51p[x] * \xi52p[xp]}{\xi53p[xp] * \xi54p[xp] * \xi55p[x]};$$

```
x612 = 1.05;
x613 = 1.06;
HatΞ5p[x611, x612]
HatΞ5p[x612, x613]
HatΞ5p[x613, x62p]
```

Out[ ]= 0.8462477224309346

Out[ ]= -0.2794438717061494

Out[ ]= 132.5949908473026

Out[ ]= 137.6089514104586

Out[ ]= 141.4762172240462

Out[ ]= 126.5476506283045

• Lemma 7.3 : (7.18*)-(7.23*)

In[ ]:=

$$y6p[x\_] := \text{Im}[\omega] + \sqrt{\epsilon6^2 - (x - \text{Re}[\omega])^2};$$
$$y6m[x\_] := \text{Im}[\omega] - \sqrt{\epsilon6^2 - (x - \text{Re}[\omega])^2};$$

$$s62 = \frac{\text{Im}[\omega]}{\text{Re}[\omega] + 1};$$

$$\text{Re}[\omega] + \epsilon6 \Big/ \sqrt{1 + s62^2}$$

$$s63 = \frac{\text{Im}[\omega]}{\text{Re}[\omega] - 1};$$

$$\text{Re}[\omega] - \epsilon6 \Big/ \sqrt{1 + s63^2}$$

$$s64 = \frac{\text{Im}[\omega]}{\text{Re}[\omega]};$$

$$\text{Re}[\omega] + \epsilon6 \Big/ \sqrt{1 + s64^2}$$

$$\xi61p[x\_] := \left((x+1)^2 + (y6p[x])^2\right)^3;$$
$$\xi62p[x\_] := \left((x-1)^2 + (y6p[x])^2\right)^2;$$
$$\xi63p[x\_] := \left(x^2 + (y6p[x])^2\right)^{1/2};$$

```
ξ64p[x_] := ϵ6^4;
ξ65p[x_] := ((x - Re[ω])^2 + (y6p[x] + Im[ω])^2)^2;

x621 = Re[ω];
x622 = 0.99;
x623 = 1.07;
x63 = Re[ω] + ϵ6;

(ξ61p[x60] × ξ62p[x621]) / (ξ63p[x621] ϵ6^4 ξ65p[x621])

(ξ61p[x621] × ξ62p[x622]) / ((1 + ϵ6) ϵ6^4 ξ65p[x621])

(ξ61p[x622] × ξ62p[x623]) / (ξ63p[x622] ϵ6^4 ξ65p[x622])

(ξ61p[x623] × ξ62p[x623]) / (ξ63p[x622] ϵ6^4 ξ65p[x622])

(((x63 + 1)^2 + (Im[ω])^2)^3 ((x63 - 1)^2 + (Im[ω])^2)^2) / (ξ63p[x623] ϵ6^4 ξ65p[x623])

ξ61m[x_] := ((x + 1)^2 + (y6m[x])^2)^3;
ξ62m[x_] := ((x - 1)^2 + (y6m[x])^2)^2;
ξ63m[x_] := (x^2 + (y6m[x])^2)^(1/2);
ξ64m[x_] := ϵ6^4;
ξ65m[x_] := ((x - Re[ω])^2 + (y6m[x] + Im[ω])^2)^2;
Ξ6m[x_, xp_] := (ξ61m[xp] * ξ62m[xp]) / (ξ63m[x] * ξ64m[x] * ξ65m[x]);

x631 = 1.069; x632 = 1.072;

Ξ6m[x631, x62m]
Ξ6m[x632, x631]
(ξ61m[x632] × ξ62m[x632]) / (((x63)^2 + (Im[ω])^2)^(1/2) ϵ6^4 ((x63 - Re[ω])^2 + (2 Im[ω])^2)^2)
```

Out[ ]= 1.037795908140452

Out[ ]= 0.4957458973639148

Out[ ]= 0.9360097719437699

Out[ ]= 209.6800881065744

Out[ ]= 130.0940237205892

Out[ ]= 130.8795798291456

Out[ ]= 129.1686731928725

Out[ ]= 146.28228102793

Out[ ]= 125.3998012063305

Out[ ]= 126.5506009830773

Out[ ]= 133.0763480077909

• Lemma 7.3 : (7.24*)-(7.31*)

In[◦]:=
```
x70 = -1 - ϵ7; x71 = -1.095; x72 = Re[a4] - ϵ4
x73 = -0.77;
(x71 - Re[a4])^2 + (y7p[x71] - Im[a4])^2 - ϵ4^2
(x71 + 1)^2 + (y4p[x71])^2 - ϵ7^2
y4m[x73]

ξ71p[x_] := ϵ7^6;
ξ72p[x_] := ((x - 1)^2 + (y7p[x])^2)^2;
ξ73p[x_] := (x^2 + (y7p[x])^2)^(1/2);
ξ74p[x_] := ((x - Re[ω])^2 + (y7p[x] - Im[ω])^2)^2;
ξ75p[x_] := ((x - Re[ω])^2 + (y7p[x] + Im[ω])^2)^2;
Ξ7p[x_, xp_] := (ξ71p[xp] * ξ72p[xp]) / (ξ73p[x] * ξ74p[x] * ξ75p[x]);

s71 = -Im[ω] / (Re[ω] + 1);
-1 - ϵ7 / Sqrt[1 + s71^2]

x701 = -1.3;
(ξ71p[x70] * ξ72p[x70]) / (ξ73p[x701] * ξ74p[x701] * ξ75p[x70])
Ξ7p[x701, x70]
Ξ7p[x71, x701]

s72 = Im[a4] / (Re[a4] - 1);
Re[a4] - ϵ4 / Sqrt[1 + s72^2]

(ξ41p[x71] * ξ42p[x71]) / (ξ43p[x72] * ξ44p[x71] * ξ45p[x72])

s73 = Im[a4] / (Re[a4] + 1);
Re[a4] - ϵ4 / Sqrt[1 + s73^2]

s74 = Im[ω - a4] / Re[ω - a4];
Re[a4] - ϵ4 / Sqrt[1 + s74^2]

ξ41m[x_] := ((x + 1)^2 + (y4m[x])^2)^3;
ξ42m[x_] := ((x - 1)^2 + (y4m[x])^2)^2;
ξ43m[x_] := (x^2 + (y4m[x])^2)^(1/2);
ξ44m[x_] := ((x - Re[ω])^2 + (y4m[x] - Im[ω])^2)^2;
ξ45m[x_] := ((x - Re[ω])^2 + (y4m[x] + Im[ω])^2)^2;
Ξ4m[x_, xp_] := (ξ41m[xp] * ξ42m[xp]) / (ξ43m[x] * ξ44m[x] * ξ45m[x]);

x721 = -1;
(ξ41m[x72] * ξ42m[x72]) / (ξ43m[x721] * ξ44m[x72] * ξ45m[x721])
Ξ4m[x721, x72]
Ξ4m[x73, x721]
```

```
      ξ41m[x73] * ξ42m[x721]
---------------------------------
ξ43m[x73] * ξ44m[x73] * ξ45m[x73]
```

Out[ ]= -1.105729729078844

Out[ ]= -0.003397290698944899

Out[ ]= 0.02129639834331976

Out[ ]= -0.004274551581780806

Out[ ]= -1.747918380899671

Out[ ]= 0.01893617709934677

Out[ ]= 0.02274289545324065

Out[ ]= 0.02613246307297593

Out[ ]= -0.990965662781027

Out[ ]= 0.02538634716629378

Out[ ]= -0.8834085223755904

Out[ ]= -1.103836717690617

Out[ ]= 0.01892703739895615

Out[ ]= 0.02071476008895807

Out[ ]= 0.0001797828289986576

Out[ ]= 0.00006825486293344909

• Lemma 7.3 : (7.32*)-(7.41*)

In[ ]:=
```
y[x_] := Sqrt[(r0 * r1)^2 - (x - a0)^2];

x4 = a0 + r0 * r1;
x1 = -1.1; x2 = 0.54; x3 = 1.03;

(x1 + 1)^2 + (y[x1])^2 - ϵ7^2

(x1 - Re[a4])^2 + (y[x1] - Im[a4])^2 - ϵ4^2
(x2 - Re[a4])^2 + (y[x2] - Im[a4])^2 - ϵ4^2

(x2 - Re[ω])^2 + (y[x2] - Im[ω])^2 - ϵ6^2
(x3 - Re[ω])^2 + (y[x3] - Im[ω])^2 - ϵ6^2

(x3 - Re[a5])^2 + (y[x3] - Im[a5])^2 - ϵ5^2
(x4 - Re[a5])^2 + (Im[a5])^2 - ϵ5^2

x1p = -0.5; x2p = 0.3;

(x1p - Re[a4])^2 + (Im[a4])^2 - ϵ4^2
(x2p - Re[a4])^2 + (Im[a4])^2 - ϵ4^2
(x2p - Re[a5])^2 + (Im[a5])^2 - ϵ5^2
```

Out[ ]= -0.09534399999999987

Out[ ]= -0.006144219810720331

Out[ ]= -0.008723148683157711

Out[ ]= -0.002753639011735615

Out[ ]= -0.02923099660524614

Out[ ]= -0.02068011486056104

Out[ ]= -0.06157015433009327

Out[ ]= -0.2300171529740829

Out[ ]= -0.03801715297408281

Out[ ]= -0.0283061543300932

§ 8 Estimates on Q: Part II

• Lemma 8.1

In[∘]:=

$$b0 = \frac{2\left(13+32\sqrt{6}\right)}{25};$$

$$b1 = \frac{2029+256\sqrt{6}}{125};$$

$$a11 = 2\left(617+688\sqrt{6}\right);\ a01 = 25\left(119+16\sqrt{6}\right);$$

$$a12 = 3889250+837000\sqrt{6};\ a02 = 2755539+487396\sqrt{6};$$

$$a13 = 31356325+8965425\sqrt{6};\ a03 = 66811702+23697378\sqrt{6};$$

$$a14 = 102142212+38104768\sqrt{6};\ a04 = 240990025+94826600\sqrt{6};$$

$$Q2max[r\_] := \frac{2^4}{5^5} * \frac{a11*r+a01}{r\,(r-1)^2} + \frac{2^6}{5^{10}} * \frac{a12*r+a02}{(r-1)^4} + \frac{2^{11}}{5^{14}} * \frac{a13*r+a03}{(r-1)^6} + \frac{2^{12}}{5^{16}} * \frac{a14*r+a04}{(r-1)^8};$$

```
b0 // N
b1 // N
```

Out[∘]= 7.310693741524935

Out[∘]= 21.24855499321995

• Lemma 8.2 (8.1*) and Lemma 8.3 (8.2*)

In[∘]:=

```
Q2max[11] // N
```

$$\frac{64\left(617+688\sqrt{6}\right)}{3125} \ // \ N$$

$$LogDQmax[r\_] := \frac{b1}{r^2} + \frac{50}{r^3} + \frac{cp^4}{2\,r^3\,(r-cp)};$$

```
LogDQmax[5.6]
```

Out[∘]= 0.2998031696052446

Out[∘]= 47.15005835335325

Out[∘]= 1.476001571574302

§ 9 Estimates on φ

• Lemma 9.1: some functions

In[∘]:=

```
r0 = 1.07; a0 = -0.06; c00 = 0.06; c01max = 2 * r0;
```

$$\varphi1max[r\_] := r0\sqrt{-Log\left[1-\left(\frac{r0}{r-Abs[a0]}\right)^2\right]};$$

$$LogD\varphi max[r\_] := -Log\left[1-\left(\frac{r0}{r-Abs[a0]}\right)^2\right];$$

• Lemma 9.2 and the value of h0

In[∘]:=

$$Log[11] + \frac{0.06\,\pi}{3}$$

$$\frac{14\sqrt{6}}{25} + 0.45$$

Out[∘]= 2.460727125870167

Out[∘]= 1.82171425595858

• Lemma 9.3: preparation

In[ ]:=
```
α1 = ArcTan[Im[a5] / (Re[a5] - a0)]
α2m = 0.54; α2p = 0.55;
α3 = ArcTan[Im[ω] / (Re[ω] - a0)]
α4 = ArcTan[(Im[ω] + ϵ6) / (Re[ω] - a0)]
h0 = (14 √6)/25 + 0.45;
α5 = π - ArcTan[(h0 + 1)/(a0 + 1)]
r2[θ_] := (ⅇ^(π-θ) + 1)/(ⅇ^(π-θ) - 1);
```

Out[ ]= 0.2449786631268641

Out[ ]= 0.8017319608504963

Out[ ]= 1.012095627387603

Out[ ]= 1.892364613257717

• Lemma 9.3: (9.2*)-(9.7*)

In[ ]:=
```
B5[θ_] := Im[a5] Sin[θ] + (Re[a5] - a0) Cos[θ];
C5[θ_] := (Re[a5] - a0)^2 + Im[a5]^2 - ϵ5^2;
r5[θ_] := 1/r0 (B5[θ] + √(B5[θ]^2 - C5[θ]));
r5[0] - r2[α1]

(a0 + r0 r5[α2p] Cos[α2p] - Re[ω]) ^ 2 + (r0 r5[α2p] Sin[α2p] - Im[ω]) ^ 2 - ϵ6^2
r5[α2p] - r2[α2p]

B6[θ_] := Im[ω] Sin[θ] + (Re[ω] - a0) Cos[θ];
C6[θ_] := (Re[ω] - a0)^2 + Im[ω]^2 - ϵ6^2;
r6[θ_] := 1/r0 (B6[θ] + √(B6[θ]^2 - C6[θ]));
r6[α4] - r2[α4]
(a0 + r0 r6[α2m] Cos[α2m] - Re[a5]) ^ 2 + (r0 r6[α2m] Sin[α2m] - Im[a5]) ^ 2 - ϵ5^2
r6[α2m] - r2[α3]
```

Out[ ]= 0.1435312250557577

Out[ ]= −0.009170714509615624

Out[ ]= 0.06316437307283751

Out[ ]= 0.006296010511505301

Out[ ]= −0.003176019108757477

Out[ ]= 0.01527910994386517

• Lemma 9.3: (9.8*)

In[ ]:=
```
r7[θ_] := (h0 - a0)/(Sqrt[2] r0 Sin[θ + π/4]);
tt4 = UnitVector[24, 1];
tt4[[1]] = α4;
For[k = 2, k ≤ 10, k++, tt4[[k]] = 1 + 0.01 * k]
For[k = 11, k ≤ 15, k++, tt4[[k]] = 1.1 + 0.02 * (k - 10)]
For[k = 16, k ≤ 23, k++, tt4[[k]] = 1.2 + 0.05 * (k - 15)]
tt4[[24]] = α5;

For[k = 1, k ≤ 23, k++, Print[r7[tt4[[k]]] - r2[tt4[[k + 1]]]]]
```

0.003862696425226853
0.003119999727270173
0.003093588303119566
0.003170739944192036
0.003352592617286598
0.003640362273447551
0.004035345097137544
0.004538919897776328
0.005152550650499244
0.002381237793976476
0.004080589683418179
0.006244339828987755
0.008888259408274202
0.01202984025661191
0.003389050665051113
0.01407774098439618
0.02845769936545639
0.0470228700251516
0.07039891324057956
0.09937977760593886
0.1349777461509074
0.1784928624245521
0.008330832923757914

- **Lemma 9.3: (9.9*)**

In[ ]:=
```
θ51 = 2.38; θ52 = 2.6;
r8[θ_] := (h0 + 1)/(r0 Sin[θ]);
r8[α5] - r2[θ51]
r8[θ51] - r2[θ52]
```

Out[ ]= 0.02779813700724887

Out[ ]= 0.03885264547486278

§ 10 Lifting Q and φ to X

- **Proof of Proposition 3.3: (10.1*)**

In[ ]:=
```
c00 + c01max + φ1max[11]
(cv - 11) Sin[π/6] // N
```

Out[ ]= 2.304904234353286

Out[ ]= 2.973469228349533

§ 11 Estimates on F

• Lemma 11.1: some functions

In[ ]:=

```
βmax[r_] := c01max + b1/(2 r) + Q2max[r] + φ1max[r];
σ1[r_, θ_] := (b1 Sin[θ]/(2 r)) / (b0 - c00 + b1 Cos[θ]/(2 r));
σ2[r_, θ_] := Sqrt[(b0 - c00)^2 + (b1/(2 r))^2 + 2 (b0 - c00) (b1/(2 r)) Cos[θ]];
ArgΔFmax[r_, θ_] := -ArcTan[σ1[r, θ]] + ArcSin[βmax[r]/σ2[r, θ]];
ArgΔFmin[r_, θ_] := -ArcTan[σ1[r, θ]] - ArcSin[βmax[r]/σ2[r, θ]];
AbsΔFmax[r_, θ_] := σ2[r, θ] + βmax[r];
AbsΔFmin[r_, θ_] := σ2[r, θ] - βmax[r];

LogDQmax[r_] := b1/r^2 + 50/r^3 + cp^4/(2 r^3 (r - cp)); (* LogDQmax is defined in Lemma 8.3(b) *)
LogDFmax[r_] := LogDQmax[r] + LogDφmax[r];

b0 - c00 - b1/(2*6.1)
βmax[6.1]
```

Out[ ]= 5.509008906015103

Out[ ]= 5.504670577923822

• Lemma 11.2: (11.4*)-(11.9*)

In[ ]:=

```
ϕ1max[r_] := 4 r0 Sqrt[-Log[1 - (4 r0/r)^2]];
c00 + c01max + ϕ1max[cv - 2.35]
2.35/Exp[-LogDφmax[cv - 2.35 - 3.5]]
2.4*Exp[LogDQmax[cv - 2.4]]
Q[ξ_] := ((-1 + ξ)^4 (1 + ξ)^6) / (ξ (1 + (6-16 Sqrt[6])/25 ξ + ξ^2)^4);
2.75 + Q[cv] - 25.5
(25.5 - 22) Sin[7 π/20]
b1/20 + Q2max[20] // N
```

Out[ ]= 3.483254917834224

Out[ ]= 2.372296634067487

Out[ ]= 2.708498706062469

Out[ ]= 2.835999840451148

Out[ ]= 3.118522834659288

Out[ ]= 1.136717860281167

§ 12 Repelling Fatou coordinate Φ_{rep} on X

• (12.1*), (12.2*)

In[ ]:=
```
(ϕ1max[125] + b1/122 + Q2max[122]) / 5
LogDFmax[5 * 25]
```

Out[ ]=
0.06448049609418008

Out[ ]=
0.001459437215851014

§ 13 Attracting Fatou coordinate Φ_{attr}

• Lemma 13.1: (13.1*) - (13.5*)

In[ ]:=
```
θ1 = 3π/20; θ2 = π/4;
u1θ1 = 8.5; u2θ1 = 6.1; u3 = 22 Cos[θ1]; u4 = 17.3;
u1θ2 = 9; u2θ2 = 6.6;

u0θ1 = u1θ1 / Cos[θ1]
u0θ2 = u1θ2 / Cos[θ2] // N

u2θ1 + c00 Cos[θ1] + c01max + φ1max[u2θ1]
u2θ2 + c00 Cos[θ2] + c01max + φ1max[u2θ2]

u4 + c00 Cos[θ1] + c01max + φ1max[u4]
u3 // N

ArgΔFmax[u2θ1, θ1]
ArgΔFmax[u2θ1, -θ1]
-ArgΔFmin[u2θ1, θ1]
-ArgΔFmin[u2θ1, -θ1]

ArgΔFmax[u2θ2, θ2]
ArgΔFmax[u2θ2, -θ2]
-ArgΔFmin[u2θ2, θ2]
-ArgΔFmin[u2θ2, -θ2]
π/2 - θ2 // N
```

Out[ ]= 9.539773019892067

Out[ ]= 12.72792206135786

Out[ ]= 8.484526381809028

Out[ ]= 8.958676365900757

Out[ ]= 19.5599339913554

Out[ ]= 19.60214353214409

Out[ ]= 0.5827833574379442

Out[ ]= 0.7619569410691078

Out[ ]= 0.7619569410691078

Out[ ]= 0.5827833574379442

Out[ ]= 0.5069961130447143

Out[ ]= 0.7767195201032997

Out[ ]= 0.7767195201032997

Out[ ]= 0.5069961130447143

Out[ ]= 0.7853981633974483

- **Lemma 13.2: (13.8*) - (13.12*)**

In[ ]:=
```
r4 = 0.34;
u5 = u3 - u1θ1;
Abs[b0 - c00 + (b1 e^(-θ1 i))/(2 u4) - (2 u5 r4^2 e^(θ1 i))/(1 - r4^2)] + βmax[u4] - (2 u5 r4)/(1 - r4^2)
-ArgΔFmin[u4, θ1] + 1/2 LogDFmax[u4] - 1/2 Log[1 - r4^2]
π/5 // N
-ArgΔFmax[u4, θ1] - 1/2 LogDFmax[u4] + 1/2 Log[1 - r4^2]
-(3 π)/20 // N
Exp[1/2 LogDFmax[u4]] / (AbsΔFmin[u4, θ1] Sqrt[1 - r4^2])
Sqrt[1 - r4^2] / (AbsΔFmax[u4, θ1] Exp[1/2 LogDFmax[u4]])
```

Out[ ]= -0.1616861638858023

Out[ ]= 0.5244859101242096

Out[ ]= 0.6283185307179586

Out[ ]= -0.453008529542726

Out[ ]= -0.471238898038469

Out[ ]= 0.227569895310661

Out[ ]= 0.08396094459145347

- **Lemmas 13.3 and 13.4: (13.14*) - (13.18*)**

In[ ]:=
```
Tan[1.245]
Sqrt[1 + 8^2] / 0.083
u6 = 10.7;
b1/u6 + Q2max[u6]
(22 - b0) Cos[θ1] - u6
LogDQmax[u6]
LogDφmax[u6]
```

Out[ ]= 2.960027220229311

Out[ ]= 97.13563552166926

Out[ ]= 2.306767929529839

Out[ ]= 2.388267712102307

Out[ ]= 0.2433711778035649

Out[ ]= 0.01016458475250777

§ 14 Locating domains

- **Proof of Lemma 14.1: (14.2*)**

In[∘]:=
```
t0 = 6.5 Sqrt[2] - cp;
Varθ1[t_] := 3 ArcTan[t/(cp - cpp)];
Varθ2[t_] := ArcTan[(t - Im[ν1])/(cp - Re[ν1])]; Varθ3[t_] := ArcTan[(t + Im[ν1])/(cp - Re[ν1])];
Varθ4[t_] := ArcTan[(t - Im[ν2])/(cp - Re[ν2])]; Varθ5[t_] := ArcTan[(t + Im[ν2])/(cp - Re[ν2])];
Varθ6[t_] := ArcTan[t/cp];
Varθ7[t_] := 4 ArcTan[(t - Im[ω])/(cp - Re[ω])]; Varθ8[t_] := 4 ArcTan[(t + Im[ω])/(cp - Re[ω])];
Varθ[tp_, tpp_] := Varθ1[tp] + Varθ2[tp] + Varθ3[tp] +
   Varθ4[tp] + Varθ5[tp] - Varθ6[tpp] - Varθ7[tpp] - Varθ8[tpp];

tt0 = UnitVector[12, 1];
For[k = 1, k ≤ 8, k++, tt0[[k]] = 0.5 * k]
For[k = 9, k ≤ 10, k++, tt0[[k]] = 0.8 + 0.4 * k]
For[k = 11, k ≤ 12, k++, tt0[[k]] = 2.8 + 0.2 * k]

-3π/4 // N
Varθ[0, tt0[[1]]]
For[k = 1, k ≤ 11, k++, Print[Varθ[tt0[[k]], tt0[[k + 1]]]]]

Print[UpperBound]

π/2 // N
Varθ[tt0[[1]], 0]
For[k = 1, k ≤ 11, k++, Print[Varθ[tt0[[k + 1]], tt0[[k]]]]]
```

Out[∘]=
-2.356194490192345

Out[∘]=
-1.235149997511832

-1.496065418216154
-1.7150825709315
-1.894315445112851
-2.038574214411201
-2.153887641712506
-2.246315472393094
-2.321189071112118
-2.282467522621119
-2.334917212267798
-2.213024338401398
-2.242727660531918
UpperBound

Out[∘]=
1.570796326794897

Out[∘]=
0.9345568853698125

0.6022970209681111
0.2523397885429439
−0.09776367079251891
−0.4312570352070035
−0.7360421809091702
−1.006062958465805
−1.240382864364038
−1.51909522077491
−1.65060363785032
−1.897548461186938
−1.943567605990372

• Proof of Lemma 14.1: (14.4*) - (14.6*)

In[∘]:=
```
ArcTan[t0/cp]
0.25 π
ArcTan[(t0 - Im[ω])/(cp - Re[ω])]
ArcTan[(t0 + Im[ω])/(cp - Re[ω])]
```

Out[∘]= 0.8985904665608595

Out[∘]= 0.7853981633974483

Out[∘]= 0.9082785288462393

Out[∘]= 1.044286239245108

• Proof of Lemma 14.1: (14.7*)

In[∘]:=
$$\text{Q3max[r\_]} := \frac{2^4}{5^5} * \frac{a01}{r\,(r-1)^2} + \frac{2^6}{5^{10}} * \frac{a12*r + a02}{(r-1)^4} + \frac{2^{11}}{5^{14}} * \frac{a13*r + a03}{(r-1)^6} + \frac{2^{12}}{5^{16}} * \frac{a14*r + a04}{(r-1)^8};$$
$$(6.5 + b0)\,\text{Cos}\left[\frac{\pi}{4}\right] + \frac{2^4}{5^5}\,\frac{a11}{(6.5-1)^2}\,\text{Cos}\left[\frac{\pi}{4}\right] + \text{Q3max}[6.5]$$
$$\text{cv Cos}\left[\frac{\pi}{4}\right]\ //\ \text{N}$$

Out[∘]= 10.7302450222952

Out[∘]= 11.98329510308299

• Proof of Lemma 14.1: (14.9*), (14.10*)

In[∘]:=
$$\text{cp} + \text{c00} - \text{c01max} - \varphi\text{1max[cp]}$$
$$6.5 + \text{c00 Cos}\left[\frac{\pi}{4}\right] - \text{c01max} - \varphi\text{1max}[6.5]$$

Out[∘]= 1.703186416902996

Out[∘]= 4.223401285936191

• Proof of Lemma 14.1: (14.12*)

In[ ]:=
```
y2[x_] := h0 - x;
θ21[x_] := 6 ArcTan[y2[x]/(x + 1)];
θ22[x_] := 4 ArcTan[y2[x]/(x - 1)];
θ23[x_] := ArcTan[y2[x]/x];
θ24[x_] := 4 ArcTan[(y2[x] - Im[ω])/(x - Re[ω])];
θ25[x_] := 4 ArcTan[(y2[x] + Im[ω])/(x - Re[ω])];
Θ2[x_, xp_] := θ21[xp] + θ22[xp] - θ23[x] - θ24[x] - θ25[x];

xx1 = UnitVector[16, 1];
For[k = 1, k ≤ 16, k++, xx1[[k]] = 0.04 * k];(* Evalution *)

-5π/4.0

θ21[xx1[[1]]] + (θ22[xx1[[1]]] + 4 π) - π/2 - (θ24[0] + 4 π) - (θ25[0] + 4 π)
(* Θ2[0,xx1[[1]]]-4π+π *)
For[k = 1, k ≤ 15, k++, Print[Θ2[xx1[[k]], xx1[[k + 1]]] - 4 π]]
θ21[Re[ω]] + θ22[Re[ω]] - θ23[xx1[[16]]] - θ24[xx1[[16]]] - θ25[xx1[[16]]] - 4 π
(* Θ2[xx1[[16]],Re[ω]]-4π+π *)

Text[UpperBound]

θ21[0] + θ22[0] - θ23[xx1[[1]]] - θ24[xx1[[1]]] - θ25[xx1[[1]]] - 4 π
(* Θ2[xx1[[1]],0]-4π+π *)
For[k = 1, k ≤ 15, k++, Print[Θ2[xx1[[k + 1]], xx1[[k]]] - 4 π]]
θ21[0.64] + (θ22[0.64] + 4 π) - θ23[Re[ω]] - 2 π - 2 π
-3π/4.0
```

Out[ ]= -3.926990816987241

Out[ ]= -2.848504466967304

-2.932097957396893
-3.01280662533488
-3.08985845339687
-3.162360302433372
-3.229272069578384
-3.289372474145273
-3.341213315054461
-3.383057837365483
-3.412797222601565
-3.427837128934693
-3.424943776092123
-3.400036944426608
-3.34791750071523
-3.261925281801133
-3.13355327301948

Out[ ]= -2.848503310031376

Out[ ]= UpperBound

Out[ ]=
-2.549687100121751
-2.622567152068562
-2.691881446915357
-2.756736427450852
-2.81608473685359
-2.868689737139231
-2.913078426791332
-2.947478398963657
-2.969732855912746
-2.977185603301647
-2.966525513653476
-2.933577821256732
-2.873029856141434
-2.778087052560814
-2.640085194608419
-2.448168401935149

Out[ ]=
-2.404227851757005

Out[ ]=
-2.356194490192345

- **Proof of Lemma 14.1: (14.13*)**

In[ ]:=
```
Θ2[-1, -0.975] - 5 π
Θ2[-0.975, -0.95] - 5 π
Θ2[-0.95, -0.925] - 5 π
Θ2[-0.925, -0.9] - 5 π

6 * π/2 + θ22[-1] - θ23[-0.975] - θ24[-0.975] - θ25[-0.975] - 5 π
Θ2[-0.95, -0.975] - 5 π
Θ2[-0.925, -0.95] - 5 π
Θ2[-0.9, -0.925] - 5 π
```

Out[ ]=
-0.8132163103184613

Out[ ]=
-0.8514843701511623

Out[ ]=
-0.8903912268694434

Out[ ]=
-0.9299428527202238

Out[ ]=
-0.7292386521231791

Out[ ]=
-0.7659217831774967

Out[ ]=
-0.8032074698973766

Out[ ]=
-0.8411009216607681

- **Proof of Lemma 14.1: (14.14*)**

In[ ]:=
```
ξ21[x_] := ((x + 1)^2 + y2[x]^2)^3;
ξ22[x_] := ((x - 1)^2 + y2[x]^2)^2;
ξ23[x_] := (x^2 + y2[x]^2)^(1/2);
ξ24[x_] := ((x - Re[ω])^2 + (y2[x] - Im[ω])^2)^2;
ξ25[x_] := ((x - Re[ω])^2 + (y2[x] + Im[ω])^2)^2;
Ξ2[x_, xp_] := (ξ21[xp] × ξ22[xp]) / (ξ23[x] × ξ24[x] × ξ25[x]);
h2[x_, xp_] := Ξ2[xp, x] Cos[Θ2[xp, x] - 5 π];
v2[x_, xp_] := Ξ2[xp, x] Sin[Θ2[xp, x] - 5 π] - (h2[x, xp] - 4.2 √2);

Ξ2[-0.975, -1] Cos[6 * π/2 + θ22[-1] - θ23[-0.975] - θ24[-0.975] - θ25[-0.975] - 5 π]
(*h2[-1,-0.975]*)
h2[-0.975, -0.95]
h2[-0.95, -0.925]
h2[-0.925, -0.9]

-Ξ2[-0.975, -1] - (1.7 - 4.2 √2)
-Ξ2[-0.95, -0.975] - (1.7 - 4.2 √2)
-Ξ2[-0.925, -0.95] - (1.7 - 4.2 √2)
-Ξ2[-0.9, -0.925] - (1.7 - 4.2 √2)
```

Out[ ]= 1.650188739528938

Out[ ]= 1.603276906542011

Out[ ]= 1.553759858791987

Out[ ]= 1.501491772427759

Out[ ]= 2.026704094611583

Out[ ]= 2.015221123518758

Out[ ]= 2.002146751878581

Out[ ]= 1.987378088450074

• Proof of Lemma 14.1: (14.15*)

In[ ]:=
```
xx2 = UnitVector[22, 1];
For[k = 4, k ≤ 22, k++, xx2[[k]] = -1.1 + 0.05 * k]; (* Evalution *)

For[k = 4, k ≤ 21, k++, Print[Θ2[xx2[[k]], xx2[[k + 1]]] - 5 π]]

Text[UpperBound]
For[k = 4, k ≤ 20, k++, Print[Θ2[xx2[[k + 1]], xx2[[k]]] - 5 π]]
θ21kmax = θ21[-0.05] + (θ22[-0.05] + 4 π) - π/2 - (θ24[0] + 4 π) - (θ25[0] + 4 π)
```

-1.036435003297353
-1.121243552973741
-1.208785098684384
-1.299073635460587
-1.392105265443584
-1.487854624226731
-1.586270823525819
-1.687272866178365
-1.790744486353997
-1.896528362421087
-2.004419640235625
-2.11415868773825
-2.225422973236007
-2.337817913092753
-2.450866460280505
-2.563997089631203
-2.676529658778044
-2.787658356653157

Out[ ]=
UpperBound

-0.8536247491292386
-0.9313371962998698
-1.011458479521693
-1.093989735031322
-1.178913911401375
-1.266192085491277
-1.355759223029084
-1.447519312578606
-1.541339790528038
-1.637045159192015
-1.734409677572973
-1.833148970860954
-1.932910354472453
-2.033261592558281
-2.133677696417806
-2.233525195825104
-2.332043057185377

Out[ ]=
-2.428319033778724

• Proof of Lemma 14.1: (14.16*)

In[ ]:=
```
For[k = 4, k ≤ 20, k++, Print[h2[xx2[[k]], xx2[[k + 1]]]]]
Ξ2[0, -0.05] Cos[θ21kmax]

Print[v2⁺]
For[k = 4, k ≤ 20, k++, Print[v2[xx2[[k]], xx2[[k + 1]]]]]
Ξ2[0, -0.05] Sin[θ21kmax] - (Ξ2[0, -0.05] Cos[θ21kmax] - 4.2 √2)
```

1.653459957502067
1.530736716264315
1.392787445597598
1.237551315143829
1.062590117416331
0.8650002166817413
0.6412992692742827
0.3872790742434156
0.09781244970309524
−0.2334030296670541
−0.6141890884719623
−1.054354494780103
−1.566339499950579
−2.166116422734387
−2.874467311783215
−3.718825732059974
−4.735976734215344

Out[•]=

−5.976084947577951

$v2^+$

2.39023277001321
2.350698057194612
2.322107162199116
2.306384082797175
2.305850081476969
2.323321375291061
2.362235106922499
2.426813079920167
2.52227636260345
2.655129035406191
2.833536803291579
3.06783701317049
3.371232458597732
3.760744715032073
4.258537372616981
4.893770948851858
5.705227345254315

Out[•]=

6.745052796594834

• Proof of Lemma 14.1: (14.17*)

In[ ]:=
```
y3[x_] := h0 + 1;
ξ31[x_] := ((x + 1)^2 + y3[x]^2)^3;
ξ32[x_] := ((x - 1)^2 + y3[x]^2)^2;
ξ33[x_] := (x^2 + y3[x]^2)^(1/2);
ξ34[x_] := ((x - Re[ω])^2 + (y3[x] - Im[ω])^2)^2;
ξ35[x_] := ((x - Re[ω])^2 + (y3[x] + Im[ω])^2)^2;
θ31[x_] := 6 ArcTan[y3[x] / (x + 1)];
θ32[x_] := 4 ArcTan[y3[x] / (x - 1)];
θ33[x_] := ArcTan[y3[x] / x];
θ34[x_] := 4 ArcTan[(y3[x] - Im[ω]) / (x - Re[ω])];
θ35[x_] := 4 ArcTan[(y3[x] + Im[ω]) / (x - Re[ω])];
Ξ3[x_, xp_] := (ξ31[xp] × ξ32[xp]) / (ξ33[x] × ξ34[x] × ξ35[x]);
Θ3[x_, xp_] := θ31[xp] + θ32[xp] - θ33[x] - θ34[x] - θ35[x] + π;

xx3 = UnitVector[40, 1];
For[k = 1, k ≤ 20, k++, xx3[[k]] = -1 - 0.04 * k];
For[k = 21, k ≤ 29, k++, xx3[[k]] = 2.2 - 0.2 * k];
For[k = 30, k ≤ 40, k++, xx3[[k]] = 8 - 0.4 * k];

θ31kmin =
  6 * π/2 + (θ32[-1] + 4 π) - (θ33[-1.04] + π) - (θ34[-1.04] + 4 π) - (θ35[-1.04] + 4 π)
For[k = 1, k ≤ 7, k++, Print[Θ3[xx3[[k + 1]], xx3[[k]]]]]

Print[UpperBound]
θ31kmax = Θ3[-1, -1.04]
For[k = 1, k ≤ 7, k++, Print[Θ3[xx3[[k]], xx3[[k + 1]]]]]
```

Out[ ]=

-0.8483444234535398

-0.8202010997341453
-0.7912975896961454
-0.761673711110844
-0.7313696257968143
-0.7004257103929987
-0.6688824314422472
-0.6367802251715933
UpperBound

Out[ ]=

-0.6301048216498284

-0.603801254530083
-0.5767590810159042
-0.5490169163553373
-0.5206137275085814
-0.4915887011449254
-0.4619811164439458
-0.4318302230910636

• Proof of Lemma 14.1: (14.18*)

```
In[◦]:= 4.2 √2 - 1.7
        Ξ3[-1, -1.04]
        For[k = 1, k ≤ 7, k++, Print[Ξ3[xx3⟦k⟧, xx3⟦k + 1⟧]]]
```

Out[◦]= 4.239696961967

Out[◦]= 2.062656932623018

2.000158528645964
1.941091485675486
1.885300377774866
1.832634700624883
1.782949116643569
1.736103628648885
1.691963690617395

• Proof of Lemma 14.1: (14.19*), (14.20*)

```
In[◦]:= Ξ3[-1, -1.04] Cos[Θ3[-1, -1.04]]
        For[k = 1, k ≤ 7, k++, Print[Ξ3[xx3⟦k⟧, xx3⟦k + 1⟧] Cos[Θ3[xx3⟦k⟧, xx3⟦k + 1⟧]]]]

        For[k = 8, k ≤ 28, k++, Print[Ξ3[xx3⟦k⟧, xx3⟦k + 1⟧]]]
```

Out[◦]= 1.66655615297622

1.64649711307032
1.627089569635225
1.608232779346009
1.589836382923896
1.571819523780651
1.554110015080064
1.536643557628328
1.650400263554264
1.611289823914753
1.574514331423322
1.539961162538852
1.507523015225385
1.477097790116736
1.448588452617401
1.421902879966312
1.3969536968078
1.373658102367293
1.351937691918844
1.331718274857302
1.625459929611302
1.557560885921774
1.517519563537789
1.500386724152471
1.502358184833961
1.520499292926841
1.552527761793999
1.596648042012086
1.651427530228283

• Proof of Lemma 14.1: (14.21*)

```
In[◦]:= For[k = 29, k ≤ 39, k++, Print[Θ3[xx3⟦k + 1⟧, xx3⟦k⟧]]]
        Print[UpperBound]
        For[k = 29, k ≤ 39, k++, Print[Θ3[xx3⟦k⟧, xx3⟦k + 1⟧]]]
```

0.884141088859848
1.138858710767368
1.356501746695108
1.542329997662133
1.701277963651245
1.837692212015978
1.955272119034947
2.057106970329619
2.145750833662824
2.223305769372729
2.291500034680302
UpperBound
1.821305193342377
1.960286022835424
2.079226124377412
2.181037128677439
2.268405886758178
2.343675986854685
2.408830357766981
2.465519256520085
2.515104555076192
2.558706267347742
2.597245222644116

• Proof of Lemma 14.1: (14.22*)

In[ ]:=
```
h3[x_, xp_] := Ξ3[x, xp] Cos[Θ3[xp, x]];
v3[x_, xp_] := Ξ3[x, xp] Sin[Θ3[xp, x]] - (4.2 √2 - h3[x, xp]);

For[k = 29, k ≤ 39, k++, Print[h3[xx3[[k]], xx3[[k + 1]]]]]

Print[v3⁺]
For[k = 29, k ≤ 39, k++, Print[v3[xx3[[k]], xx3[[k + 1]]]]]
```

1.520696588654552
1.085807215004933
0.5992554559850657
0.0872688138889453
−0.4338080054391968
−0.9544019433845865
−1.469297041111466
−1.975957585700163
−2.473434163250646
−2.961677470918296
−3.441119772327906
$v3^{+}$
−2.563877927769326
−2.498397454389978
−2.586969099508198
−2.787570716896793
−3.067727309061849
−3.403480009187443
−3.77761994787183
−4.177971720038952
−4.595990571968006
−5.025704008933354
−5.462945748971062

• Proof of Lemma 14.1: (14.23*)

In[ ]:=
```
2 * Q2max[8] - 8 - 1.6 + b0
```

Out[ ]=

-0.9396354866549823

§ Appendix A. The position of D(a4,ϵ4) and D(a5,ϵ5)

• Proof of Lemma 7.2(a): (A.1*) - (A.5*)

In[ ]:=

$$x42m = \frac{2\,Re[a4]\,Im[a4]\,Im[\omega] + \left((Re[a4])^2 - (Im[a4])^2\right) Re[\omega]}{(Re[a4])^2 + (Im[a4])^2}$$

y4m[x42m]

x40m = -0.22 - ϵ4

x41m = -1;

$$s41 = -\frac{(1 + Re[a4])\,Im[a4] + \epsilon 4 \sqrt{(1 + Re[a4])^2 + (Im[a4])^2 - \epsilon 4^2}}{\epsilon 4^2 - (1 + Re[a4])^2}$$

$$x41tilde = \frac{Re[a4] + Im[a4]\,s41 - s41^2}{1 + s41^2}$$

$$s42 = -\frac{(1 - Re[a4])\,Im[a4] + \epsilon 4 \sqrt{(1 - Re[a4])^2 + (Im[a4])^2 - \epsilon 4^2}}{(1 - Re[a4])^2 - \epsilon 4^2};$$

$$x42tilde = \frac{Re[a4] + Im[a4]\,s42 + s42^2}{1 + s42^2}$$

Out[ ]=

-0.9742203713525839

Out[ ]=

0.2255984663991158

Out[ ]=

-1.105729729078844

Out[ ]=

-5.810452110055706

Out[ ]=

-1.092896611036714

Out[ ]=

0.605140619626996

• Proof of Lemma 7.2(a): (A.6*) - (A.11*)

In[ ]:=
```
θ41m[x_] := 6 ArcTan[y4m[x]/(x + 1)];
θ42m[x_] := 4 ArcTan[y4m[x]/(x - 1)];
θ43m[x_] := ArcTan[y4m[x]/x];
θ44m[x_] := 4 ArcTan[(y4m[x] - Im[ω])/(x - Re[ω])];
θ45m[x_] := 4 ArcTan[(y4m[x] + Im[ω])/(x - Re[ω])];
Θ4m[x_, xp_] := θ41m[xp] + θ42m[xp] - θ43m[x] - θ44m[x] - θ45m[x] + π;

x401m = -1.096;
Θ4m[x40m, x401m] + 2 π
Θ4m[x401m, x40m] + 2 π

6 ArcTan[s41] + θ42m[-1] - θ43m[x401m] - θ44m[x401m] - θ45m[x401m] + π + 2 π
6 (-π/2) + θ42m[x401m] - θ43m[-1] - θ44m[-1] - θ45m[-1] + π + 2 π

y41m[x_] := s41 (x + 1);
x411m = -1.04;
Θ41m[x_, xp_] := 6 ArcTan[s41] + 4 ArcTan[y41m[xp]/(xp - 1)] - ArcTan[y41m[x]/x] -
    4 ArcTan[(y41m[x] - Im[ω])/(x - Re[ω])] - 4 ArcTan[(y41m[x] + Im[ω])/(x - Re[ω])] + 3 π;
Θ41m[x41tilde, x411m]
Θ41m[x411m, x41tilde]
Θ41m[x411m, -1]
Θ41m[-1, x411m]
```

Out[ ]= 3.134180762530336

Out[ ]= 2.245023829985335

Out[ ]= 3.084524715307098

Out[ ]= 0.300058755957096

Out[ ]= 3.086010654787155

Out[ ]= 1.146123142926024

Out[ ]= 2.155747032634826

Out[ ]= 0.5688388483560214

- **Proof of Lemma 7.2(a): (A.12*)**

In[ ]:=

```
x40p = -0.22 - ϵ4;
x41p = -1;
x42p = 0;
x43p = Re[ω];

θ41p[x_] := 6 ArcTan[y4p[x] / (x + 1)];
θ42p[x_] := 4 ArcTan[y4p[x] / (x - 1)];
θ43p[x_] := ArcTan[y4p[x] / x];
θ44p[x_] := 4 ArcTan[(y4p[x] - Im[ω]) / (x - Re[ω])];
θ45p[x_] := 4 ArcTan[(y4p[x] + Im[ω]) / (x - Re[ω])];
Θ4p[x_, xp_] := θ41p[xp] + θ42p[xp] - θ43p[x] - θ44p[x] - θ45p[x];

xx40 = UnitVector[14, 1];
For[k = 1, k ≤ 6, k++, xx40[[k]] = -1.11 + 0.005 * k];
For[k = 7, k ≤ 14, k++, xx40[[k]] = -1.14 + 0.01 * k];

Θ4p[xx40[[1]], x40p] + π + 2 π
For[k = 1, k ≤ 13, k++, Print[Θ4p[xx40[[k + 1]], xx40[[k]]] + π + 2 π]]

Print[LowerBound]
Θ4p[x40p, xx40[[1]]] + π + 2 π
For[k = 1, k ≤ 12, k++, Print[Θ4p[xx40[[k]], xx40[[k + 1]]] + π + 2 π]]
-3 π + θ42p[xx40[[14]]] - θ43p[xx40[[13]]] - θ44p[xx40[[13]]] -
  θ45p[xx40[[13]]] + π + 2 π (*Θ4p[xx40[[13]],xx40[[14]]]+2π*)
```

Out[ ]=

2.95602846971208

3.122502759321247
3.064304161273398
3.057062002978774
3.056668066062883
3.058141964215009
3.140439648421572
3.134938937000255
3.131664396991997
3.129294808249391
3.127302071698265
3.125449399215134
3.123623731652869
3.121769811871014
LowerBound

Out[ ]=

2.68999698327282

2.625035044035996
2.770261406924538
2.82279292542446
2.854911443046462
2.877677152317318
2.821999272016734
2.854788902371062
2.877836403824171
2.89501464262163
2.90828771252624
2.918786949573634
2.927222346974309

Out[ ]=
2.934067944647843

• Proof of Lemma 7.2(a): (A.13*)

In[ ]:=
```
xx41 = UnitVector[50, 1];
For[k = 1, k ≤ 20, k++, xx41⟦k⟧ = -1 + 0.01 * k];
For[k = 21, k ≤ 40, k++, xx41⟦k⟧ = -1.2 + 0.02 * k];
For[k = 41, k ≤ 50, k++, xx41⟦k⟧ = -2 + 0.04 * k];

3 π + θ42p[-1] - θ43p[xx41⟦1⟧] - θ44p[xx41⟦1⟧] - θ45p[xx41⟦1⟧] - 5 π + 2 π
For[k = 1, k ≤ 48, k++, Print[Θ4p[xx41⟦k + 1⟧, xx41⟦k⟧] - 5 π + 2 π]]
θ41p[-0.04] + θ42p[-0.04] + π/2 - θ44p[0] - θ45p[0] - 5 π + 2 π

Print[LowerBound]

Θ4p[-1.0, -0.99] - 5 π + 2 π
For[k = 1, k ≤ 49, k++, Print[Θ4p[xx41⟦k⟧, xx41⟦k + 1⟧] - 5 π + 2 π]]
```

Out[ ]=
3.119861179125929

3.117886335562325
3.115841742214041
3.113728111319444
3.111548378923301
3.109306571837408
3.107007167206513
3.104654730728274
3.102253715671417
3.099808355935188
3.097322614466918
3.094800164211982
3.092244387941886
3.089658388716849
3.08704500598475
3.084406834291086
3.081746242786117
3.079065394464319
3.076366264536032
3.073650657618755
3.128476076658798
3.121661494051391
3.114931361530781
3.10827808085844
3.101695637507396
3.095179203045301
3.088724854376942
3.082329372415025
3.075990095734444
3.069704812937097
3.063471682705725
3.057289173964954
3.051156020860406
3.045071188820185
3.039033849031322
3.033043359408996
3.027099250664008
3.021201216451871
3.015349106862228
3.009542924710551
3.093281126708792
3.081176359020187
3.06944244840977
3.058077384193995
3.047085429308151
3.036477280501128
3.026270541719242
3.016490538997296
3.007171538846468

Out[ ]=
2.998358474005688

LowerBound

Out[ ]=
2.939655453338775

2.944225697952819
2.947958789242
2.950992765258228
2.953435609817021
2.955373280955486
2.956875237684081
2.95799834399349
2.95878968900756
2.959288664536485
2.959528522320719
2.959537559413178
2.95934003304971
2.958956875583926
2.95840625951459
2.957704048643098
2.956864161702971
2.95589886797487
2.954819029520238
2.953634301131359
2.892716404513918
2.891124707649746
2.889116571681264
2.886749432265109
2.884070656362415
2.881119679269396
2.877929613143159
2.874528475248292
2.870940138057758
2.867185072885807
2.863280938198166
2.859243049648349
2.855084759050001
2.850817762523231
2.846452353043963
2.841997628980828
2.837461667520751
2.832851669887969
2.828174083760427
2.823434707149398
2.723859013790476
2.71484425565777
2.705483218684668
2.695813103729519
2.685864578973717
2.675663184253446
2.665230449917743
2.6545848170244
2.643742425415177
2.632717824188259

- **Proof of Lemma 7.2(a): (A.14*)**

In[ ]:=
```
xx42 = UnitVector[26, 1];
For[k = 1, k ≤ 11, k++, xx42[[k]] = 0.05 * k];
xx42[[12]] = 0.575;
xx42[[13]] = x42tilde;

Θ4p[xx42[[1]], 0] - 4 π + 2 π
For[k = 1, k ≤ 12, k++, Print[Θ4p[xx42[[k + 1]], xx42[[k]]] - 4 π + 2 π]]

Print[LowerBound]

θ41p[xx42[[1]]] + θ42p[xx42[[1]]] - π/2 - θ44p[0] - θ45p[0] - 4 π + 2 π
For[k = 1, k ≤ 12, k++, Print[Θ4p[xx42[[k]], xx42[[k + 1]]] - 4 π + 2 π]]
```

Out[ ]=
```
3.03527624873707
3.026606318083006
3.019331842854033
3.013758573734128
3.0103155764925
3.009619499807918
3.012586364273261
3.02063910852587
3.036127499285476
3.063284399067003
3.110820124095698
2.972739343632009
3.06668814627446
LowerBound
```

Out[ ]=
```
2.573994018238343
2.559149571372588
2.543916253621603
2.528297077567739
2.512291382608302
2.495894381804416
2.479094790930351
2.461867086968926
2.444147351647928
2.425752762417101
2.406069450066097
2.582579981587042
2.543251909235561
```

- **Proof of Lemma 7.2(a): (A.15*) - (A.17*)**

In[ ]:=
```
For[k = 14, k ≤ 20, k++, xx42[[k]] = 0.625 + 0.005 * (k - 14)];
For[k = 21, k ≤ 25, k++, xx42[[k]] = 0.659 + 0.001 * (k - 21)];
xx42[[26]] = 0.6635;

Θ4phat[x_, xp_] := θ41p[xp] + θ42p[x] - θ43p[x] - θ44p[x] - θ45p[x] - 4 π + 2 π;

For[k = 13, k ≤ 25, k++, Print[Θ4phat[xx42[[k + 1]], xx42[[k]]]]]
Print[LowerBound]
For[k = 13, k ≤ 25, k++, Print[Θ4phat[xx42[[k]], xx42[[k + 1]]]]]

Print[x2627]
θ41p[0.6635] + (θ42p[Re[ω]] + 4 π) -
  θ43p[Re[ω]] - 4 (ArcTan[(Im[ω] - 0.69)/(Re[ω] + 0.22)] + π/2) - 4 * π/2 + 2 π
Θ4phat[0.6635, Re[ω]]
```

3.048645903900791
2.929798465246194
2.949224944262245
2.972363249795727
3.000681438983269
3.036774123695915
3.085979769523176
3.124196424058166
3.073980243412462
3.090471565511161
3.109949551495042
3.133998112552355
3.129856122384835
LowerBound
2.637346627310203
2.809521827427286
2.819277805628657
2.830172134315511
2.842321886352202
2.855715657611977
2.869767811000399
2.906991362282675
3.009292545298976
3.0194133405128
3.030339155585656
3.042037758084554
3.077107018156426
x2627

Out[ ]=
3.133104354555679

Out[ ]=
3.093984712467421

• Proof of Lemma 14.1(c) and Lemma 7.2(b): (A.18*)

In[◦]:=
```
ω = (8 √6 - 3)/25 + (6 √6 + 4)/25 ⅈ;
a5 = 0.78 + 0.21 ⅈ;
ϵ5 = Abs[a5 - ω];
x51m = Re[a5] + √(1 + (Re[a5])^2 - 2 (Re[a5] Re[ω] + Im[a5] Im[ω]))
x52m = 0.78 + ϵ5
```

Out[◦]=
1.288631648966217

Out[◦]=
1.330278251732788

• Proof of Lemma 14.1(c) and Lemma 7.2(b): (A.19*)

In[◦]:=
```
y5p[x_] := Im[a5] + √(ϵ5^2 - (x - Re[a5])^2);
θ51p[x_] := 6 ArcTan[y5p[x]/(x + 1)]; θ52p[x_] := 4 ArcTan[y5p[x]/(x - 1)];
θ53p[x_] := ArcTan[y5p[x]/x]; θ54p[x_] := 4 ArcTan[(y5p[x] - Im[ω])/(x - Re[ω])];
θ55p[x_] := 4 ArcTan[(y5p[x] + Im[ω])/(x - Re[ω])];
Θ5p[x_, xp_] := θ51p[xp] + θ52p[xp] - θ53p[x] - θ54p[x] - θ55p[x];

3 π/4 // N

xx50 = UnitVector[12, 1];
For[k = 1, k ≤ 11, k++, xx50[[k]] = 0.66 + 0.03 * k];  (* Evalution *)
xx50[[12]] = 1;

For[k = 1, k ≤ 10, k++, Print[Θ5p[xx50[[k]], xx50[[k + 1]]] + 4 π]]
θ51p[1] + 4 * (-π/2) - θ53p[xx50[[11]]] - θ54p[xx50[[11]]] - θ55p[xx50[[11]]] + 4 π

Print[UpperBound]

For[k = 1, k ≤ 11, k++, Print[Θ5p[xx50[[k + 1]], xx50[[k]]] + 4 π]]
```

Out[◦]=
2.356194490192345

2.391526415236472
2.423602154077161
2.450947407950434
2.473759036246282
2.492160158010023
2.506201745454751
2.515860261232001
2.521030809429007
2.521514639557207
2.516998905053997

Out[◦]=
2.66361386315768

UpperBound
2.771828693399584
2.806972056448528
2.838227364897371
2.865845998203273
2.890028116407054
2.910928235635012
2.928658966059174
2.943293174991879
2.954864734539278
2.963367952945999
2.815687137721621

• Proof of Lemma 14.1(c) and Lemma 7.2(b): (A.20*) - (A.21*)

In[∘]:=
```
Θ5p[xx50⟦1⟧, Re[ω]] + 4 π
θ51p[xx50⟦1⟧] + θ52p[xx50⟦1⟧] - θ53p[Re[ω]] - 4 ArcTan[(0.78 - Re[ω])/(Im[ω] - 0.21)] - 4 * π/2 + 4 π
3 π/4 // N
```

Out[∘]= 2.711097593634605

Out[∘]= 2.381304568984316

Out[∘]= 2.356194490192345

• Proof of Lemma 14.1(c) and Lemma 7.2(b): (A.22*)

In[∘]:=
```
xx51p = UnitVector[23, 1];
xx51p⟦1⟧ = 1.04;
xx51p⟦2⟧ = 1.08;

Θ5p[1, xx51p⟦1⟧]
Θ5p[xx51p⟦1⟧, xx51p⟦2⟧]

θ51p[1] + 4 * π/2 - θ53p[xx51p⟦1⟧] - θ54p[xx51p⟦1⟧] - θ55p[xx51p⟦1⟧]
Θ5p[xx51p⟦2⟧, xx51p⟦1⟧]
```

Out[∘]= 2.420842909872854

Out[∘]= 2.388549435946722

Out[∘]= 3.048260573245474

Out[∘]= 3.052745797608655

• Proof of Lemma 14.1(c) and Lemma 7.2(b): The real parts of Tilde{x51,x52,x53}

In[ ]:=
```
s51 = Im[a5] / (Re[a5] + 1);
xTilde51 = Re[a5] + ϵ5 / Sqrt[1 + s51^2]
s52 = Im[a5] / Re[a5];
xTilde52 = Re[a5] + ϵ5 / Sqrt[1 + s52^2]
s53 = Im[a5 + ω] / Re[a5 - ω];
xTilde53 = Re[a5] + ϵ5 / Sqrt[1 + s53^2]
```

Out[ ]=
1.326488192510982

Out[ ]=
1.311357351444384

Out[ ]=
0.8462477224309346

• Proof of Lemma 14.1(c) and Lemma 7.2(b): (A.23*) - (A.24*) for 2≤ k≤ 17

In[ ]:=
```
ξ51p[x_] := ((x + 1)^2 + (y5p[x])^2)^3; ξ52p[x_] := ((x - 1)^2 + (y5p[x])^2)^2;
ξ53p[x_] := (x^2 + (y5p[x])^2)^(1/2);
ξ54p[x_] := ((x - Re[ω])^2 + (y5p[x] - Im[ω])^2)^2;
ξ55p[x_] := ((x - Re[ω])^2 + (y5p[x] + Im[ω])^2)^2;
TildeΞ51p[x_, xp_] := (ξ51p[xp] × ξ52p[x]) / (ξ53p[x] × ξ54p[x] × ξ55p[xp]);
h5p1[x_, xp_] := TildeΞ51p[x, xp] Cos[Θ5p[x, xp]];
v5p1[x_, xp_] := TildeΞ51p[x, xp] Sin[Θ5p[x, xp]] - (4.2 Sqrt[2] - h5p1[x, xp]);

For[k = 3, k ≤ 10, k++, xx51p[[k]] = 1.1 + 0.02 * (k - 3)];  (* Evalution *)
For[k = 11, k ≤ 16, k++, xx51p[[k]] = 1.25 + 0.01 * (k - 11)];  (* Evalution *)
xx51p[[17]] = 1.305;
xx51p[[18]] = xTilde52;

For[k = 2, k ≤ 17, k++, Print[Θ5p[xx51p[[k]], xx51p[[k + 1]]]]]

Print[UpperBound]
For[k = 2, k ≤ 17, k++, Print[Θ5p[xx51p[[k + 1]], xx51p[[k]]]]]

Print[MaxRealPart]
For[k = 2, k ≤ 17, k++, Print[h5p1[xx51p[[k]], xx51p[[k + 1]]]]]

Print[v5^+]
For[k = 2, k ≤ 17, k++, Print[v5p1[xx51p[[k]], xx51p[[k + 1]]]]]
```

2.530055574325989
2.50819542152695
2.481777316706745
2.449954679718366
2.411565149797653
2.364957369549803
2.307678039037291
2.235856663690659
2.313877313788928
2.270129391299486
2.219273393539924
2.159003201289761
2.085589177560149

1.992350676193736
2.032655903840079
1.921999959923241
UpperBound
2.879221915034489
2.871062987795274
2.860621733900564
2.847676202439228
2.831959527620216
2.81315785207097
2.790922142581954
2.764926357397289
2.600612750516001
2.575629650578966
2.547902373567379
2.517058869832398
2.482757792910638
2.444959214968776
2.288557356812629
2.289223967287766
MaxRealPart
−101.5656181714831
−80.63574223741388
−63.95656532864803
−50.54186468188957
−39.65819046884079
−30.7516785220976
−23.39662076867922
−17.25742314212589
−14.35465186640587
−12.09352581872568
−10.00020329053686
−8.048699398323537
−6.209440959266645
−4.442452048368521
−3.935601924506333
−2.802891824572424
$(v_5)^+$
−36.28656439117717
−27.36212215922517
−20.27817931141173
−14.62816879402065
−10.105113245586
−6.473975318510789
−3.551725090411985
−1.19198657063551
−4.670326313656458
−3.655833211969704
−2.743609837807719
−1.920689001351001
−1.171957735067261
−0.4756505652900387
−1.96878381461213
−1.092630942852415

• Proof of Lemma 14.1(c) and Lemma 7.2(b): (A.23*) - (A.24*) for 18≤ k≤ 20

In[ ]:=

```
xx51p[[19]] = 1.317;
xx51p[[20]] = 1.323;
xx51p[[21]] = xTilde51;
TildeΞ52p[x_, xp_] := (ξ51p[xp] × ξ52p[x]) / (ξ53p[xp] × ξ54p[x] × ξ55p[xp]);
h5p2[x_, xp_] := TildeΞ52p[x, xp] Cos[Θ5p[x, xp]];
v5p2[x_, xp_] := TildeΞ52p[x, xp] Sin[Θ5p[x, xp]] - (4.2 Sqrt[2] - h5p2[x, xp]);

For[k = 18, k ≤ 20, k++, Print[Θ5p[xx51p[[k]], xx51p[[k + 1]]]]]

Print[UpperBound]
For[k = 18, k ≤ 20, k++, Print[Θ5p[xx51p[[k + 1]], xx51p[[k]]]]]

Print[MaxRealPart]
For[k = 18, k ≤ 20, k++, Print[h5p2[xx51p[[k]], xx51p[[k + 1]]]]]

Print[v5^+]
For[k = 18, k ≤ 20, k++, Print[v5p2[xx51p[[k]], xx51p[[k + 1]]]]]
```

```
1.84280874445743
1.683323367419519
1.633185172951285
UpperBound
2.227851400007109
2.204461181572635
2.051687276953734
MaxRealPart
-1.912255913038738
-0.7083024035925227
-0.3209771444215581
(v5)^+
-0.9961683162325237
-0.3800811341298944
-1.1225668591953
```

• Proof of Lemma 14.1(c) and Lemma 7.2(b): (A.23*) - (A.24*) for 21≤ k≤ 22

In[ ]:=
```
xx51p[[22]] = 1.329;
xx51p[[23]] = 0.78 + ϵ5;
TildeΞ53p[x_, xp_] := (ξ51p[x] × ξ52p[x]) / (ξ53p[xp] × ξ54p[x] × ξ55p[xp]);
h5p3[x_, xp_] := TildeΞ53p[x, xp] Cos[Θ5p[x, xp]];
v5p3[x_, xp_] := TildeΞ53p[x, xp] Sin[Θ5p[x, xp]] - (4.2 Sqrt[2] - h5p3[x, xp]);

Θ5p[xx51p[[21]], xx51p[[22]]]
Θ5p[xx51p[[22]], xx51p[[21]]]
h5p3[xx51p[[21]], xx51p[[22]]]
v5p3[xx51p[[21]], xx51p[[22]]]

Θ5p1[x_, xp_] := 6 ArcTan[Im[a5] / (xp + 1)] + 4 ArcTan[Im[a5] / (xp - 1)] - θ53p[x] - θ54p[x] - θ55p[x];
Θ5p2[x_, xp_] := θ51p[xp] + θ52p[xp] - ArcTan[Im[a5] / x] -
    4 ArcTan[(Im[a5] - Im[ω]) / (x - Re[ω])] - 4 ArcTan[(Im[a5] + Im[ω]) / (x - Re[ω])];
Θ5p1[xx51p[[22]], xx51p[[23]]]
Θ5p2[xx51p[[23]], xx51p[[22]]]

TildeΞ53p3[x_, xp_] :=
   (ξ51p[x] × ξ52p[x]) / ((xp^2 + (Im[a5])^2)^(1/2) ξ54p[x] ((xp - Re[ω])^2 + (Im[a5] + Im[ω])^2)^2);

h5p33[x_, xp_] := TildeΞ53p3[x, xp] Cos[Θ5p1[x, xp]];
v5p33[x_, xp_] := TildeΞ53p3[x, xp] Sin[Θ5p1[x, xp]] - (4.2 Sqrt[2] - h5p33[x, xp]);
h5p33[xx51p[[22]], xx51p[[23]]]
v5p33[xx51p[[22]], xx51p[[23]]]
```

Out[ ]= 1.509348377967255

Out[ ]= 1.96982677327377

Out[ ]= 0.275048849833418

Out[ ]= −1.194155820745349

Out[ ]= 1.273617715602237

Out[ ]= 1.922817617974028

Out[ ]= 1.158372003775037

Out[ ]= −0.9988557820519852

• Proof of Lemma 14.1(c) and Lemma 7.2(b): (A.25*) - (A.28*)

In[ ]:=
```
x51m = Re[a5] + Sqrt[1 + (Re[a5])^2 - 2 (Re[a5] Re[ω] + Im[a5] Im[ω])];
Q[z_] := (-1 + z)^4 (1 + z)^6 / (z (1 + (6-16 Sqrt[6])/25 z + z^2)^4);
Q[x51m]
1/4 * 0.7/Q'[x51m]
y5m[x_] := Im[a5] - Sqrt[ϵ5^2 - (x - Re[a5])^2];
x511m = 1.293;
x511m - x51m
y5m[x511m]
0.0175/Sqrt[2]
s54 = Im[ω]/Re[x51m - ω];
(1.2 - Re[a5])^2 + (s54 (1.2 - x51m) - Im[a5])^2 - ϵ5^2
```

$$x51m = Re[a5] + \sqrt{1 + (Re[a5])^2 - 2\,(Re[a5]\,Re[\omega] + Im[a5]\,Im[\omega])};$$

$$Q[z\_] := \frac{(-1+z)^4 (1+z)^6}{z\left(1 + \frac{6-16\sqrt{6}}{25} z + z^2\right)^4};$$

Out[ ]= 0.9514277757019218

Out[ ]= 0.01761655418104594

Out[ ]= 0.004368351033782636

Out[ ]= 0.01090918069862348

Out[ ]= 0.01237436867076458

Out[ ]= −0.02649212774475035

• Proof of Lemma 14.1(c) and Lemma 7.2(b): (A.29*)

In[ ]:=
```
θ51m[x_] := 6 ArcTan[y5m[x]/(x + 1)];
θ52m[x_] := 4 ArcTan[y5m[x]/(x - 1)];
θ53m[x_] := ArcTan[y5m[x]/x];
θ54m[x_] := 4 ArcTan[(y5m[x] - Im[ω])/(x - Re[ω])];
θ55m[x_] := 4 ArcTan[(y5m[x] + Im[ω])/(x - Re[ω])];
Θ5m[x_, xp_] := θ51m[xp] + θ52m[xp] - θ53m[x] - θ54m[x] - θ55m[x];

xx51m = UnitVector[11, 1];
xx51m[[1]] = 1.293;
xx51m[[2]] = 1.297;
xx51m[[3]] = 1.30;
xx51m[[4]] = 1.305;
xx51m[[5]] = 1.31;
xx51m[[6]] = 1.315;
xx51m[[7]] = 1.32;
xx51m[[8]] = 1.325;
xx51m[[9]] = 1.327;
xx51m[[10]] = 1.329;
xx51m[[11]] = 0.78 + ϵ5;

For[k = 1, k ≤ 9, k++, Print[Θ5m[xx51m[[k + 1]], xx51m[[k]]]]]
Θ5m1110 = θ51m[xx51m[[10]]] + θ52m[xx51m[[10]]] - ArcTan[Im[a5]/xx51m[[11]]] -
  4 ArcTan[(Im[a5] - Im[ω])/(xx51m[[11]] - Re[ω])] - 4 ArcTan[(Im[a5] + Im[ω])/(xx51m[[11]] - Re[ω])]
Print[LowerBound]
For[k = 1, k ≤ 9, k++, Print[Θ5m[xx51m[[k]], xx51m[[k + 1]]]]]
6 ArcTan[Im[a5]/(xx51m[[11]] + 1)] + 4 ArcTan[Im[a5]/(xx51m[[11]] - 1)] -
 θ53m[xx51m[[10]]] - θ54m[xx51m[[10]]] - θ55m[xx51m[[10]]]
```

0.0471496055764784
0.1644464740913508
0.2033129084426742
0.329213888790356
0.4572182699036245
0.5864500229314751
0.7105152768394565
0.9927002086948682
1.050296751122191

Out[ ]=
1.08331220089595

LowerBound
0.2799311122787523
0.3464029616200521
0.523699035649189
0.6771254851660957
0.8440788901494742
1.034414180379889
1.275740079249205
1.289688856543412
1.458841310653325

Out[∘]=
1.748095340586978

• Proof of Lemma 14.1(c) and Lemma 7.2(b): (A.30*)

```
In[∘]:= ξ51m[x_] := ((x + 1)^2 + (y5m[x])^2)^3; ξ52m[x_] := ((x - 1)^2 + (y5m[x])^2)^2;
ξ53m[x_] := (x^2 + (y5m[x])^2)^(1/2);
ξ54m[x_] := ((x - Re[ω])^2 + (y5m[x] - Im[ω])^2)^2;
ξ55m[x_] := ((x - Re[ω])^2 + (y5m[x] + Im[ω])^2)^2;
TildeΞ5m[x_, xp_] := (ξ51m[xp] × ξ52m[xp]) / (ξ53m[x] × ξ54m[xp] × ξ55m[x]);

For[k = 1, k ≤ 9, k++, Print[TildeΞ5m[xx51m[[k]], xx51m[[k + 1]]]]]
TildeΞ5m1011 =
  ((xx51m[[11]] + 1)^2 + (Im[a5])^2)^3 ((xx51m[[11]] - 1)^2 + (Im[a5])^2)^2 /
  (ξ53m[xx51m[[10]]] ((xx51m[[11]] - Re[ω])^2 + (Im[a5] - Im[ω])^2)^2 ξ55m[xx51m[[10]]])
```

1.097848955438665
1.130442358256763
1.249667679652172
1.365915610666572
1.518139386074795
1.731449576029716
2.075961850376732
2.15635131640508
2.490816461745119

Out[∘]=
3.210866646701111

• Proof of Lemma 14.1(c) and Lemma 7.2(b): (A.31*)

```
In[∘]:= For[k = 1, k ≤ 9, k++,
  Print[TildeΞ5m[xx51m[[k]], xx51m[[k + 1]]] Cos[Θ5m[xx51m[[k + 1]], xx51m[[k]]]]]]
TildeΞ5m1011 * Cos[Θ5m1110]
```

1.096628875551733
1.115191698482398
1.223928308762638
1.292561551017792
1.362202139765834
1.442142545831465
1.573632851001372
1.178295952096125
1.238716944300875

Out[∘]=
1.503984587485136