Introduction to complex geometry

Yalong Shi*

Abstract

Notes for 2020 BICMR Summer School for Differential Geometry.

Contents

1 Preliminaries in complex analysis

Let Ω be a domain of $\mathbb{C}$ with piecewise C^1 boundary. $f = u + \sqrt{-1}v : \Omega \to \mathbb{C}$ be a C^1 map. We generally regard this as a complex-valued function. It is usually convenient to introduce the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\Big).$$

Then it is easy to see that

$$df := \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}.$$

Then it is direct to check that the Cauchy-Riemann equation can be expressed as

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Now we assume f is only C^1, not necessarily holomorphic. For any $z \in \Omega$, let $\Delta(z, \epsilon)$ be a small disc with center z and radius ϵ. By Green formula (written in the form of differential forms), we have

$$\begin{aligned}\int_{\partial\left(\Omega\setminus\Delta(z,\epsilon)\right)} \frac{f(w)}{w-z}dw &= \int_{\Omega\setminus\Delta(z,\epsilon)} d\Big(\frac{f(w)}{w-z}dw\Big) \\ &= \int_{\Omega\setminus\Delta(z,\epsilon)} d\Big(\frac{f(w)}{w-z}\Big) \wedge dw \\ &= \int_{\Omega\setminus\Delta(z,\epsilon)} \frac{\frac{\partial f}{\partial \bar{w}}(w)}{w-z} d\bar{w} \wedge dw.\end{aligned}$$

In polar coordinates around z, the final integrand is in fact bounded, so we can let $\epsilon \to 0$ to get

$$\int_{\Omega} \frac{\frac{\partial f}{\partial \bar{w}}(w)}{w-z} d\bar{w} \wedge dw.$$

On the other hand, we have

$$\begin{aligned}\int_{\partial\left(\Omega\setminus\Delta(z,\epsilon)\right)} \frac{f(w)}{w-z}dw &= \int_{\partial\Omega} \frac{f(w)}{w-z}dw - \int_{|w-z|=\epsilon} \frac{f(w)}{w-z}dw \\ &= \int_{\partial\Omega} \frac{f(w)}{w-z}dw - \sqrt{-1}\int_0^{2\pi} f(z + \epsilon e^{\sqrt{-1}\theta})d\theta \\ &\to \int_{\partial\Omega} \frac{f(w)}{w-z}dw - 2\pi\sqrt{-1}f(z).\end{aligned}$$

So we finally get

$$f(z) = \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Omega}\frac{f(w)}{w-z}dw + \frac{1}{2\pi\sqrt{-1}}\int_{\Omega}\frac{\frac{\partial f}{\partial \bar{z}}(w)}{w-z}dw\wedge d\bar{w}. \tag{1.1}$$

One direct corollary is the following solution formula for 1-dimensional $\bar{\partial}$-equation:

Lemma 1.1. *Let $f \in C_0^\infty(\mathbb{C})$ be a complex-valued function, then the function defined by*

$$u(z) := \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{C}}\frac{f(w)}{w-z}dw\wedge d\bar{w}$$

is a smooth function on $\mathbb{C}$ and satisfies the equation

$$\frac{\partial u}{\partial \bar{z}} = f.$$

Proof. Assume supp$f \subset \Delta(0,R)$, then for any $z \in \Delta(0,R')$ we have

$$\begin{aligned} u(z) &= \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{C}}\frac{f(z+w)}{w}dw\wedge d\bar{w} \\ &= \frac{1}{2\pi\sqrt{-1}}\int_{\Delta(0,R+R')}\frac{f(z+w)}{w}dw\wedge d\bar{w}. \end{aligned}$$

We can taking derivative with respect to $\bar{z}$ under the integration sign to get

$$\begin{aligned} \frac{\partial u}{\partial \bar{z}}(z) &= \frac{1}{2\pi\sqrt{-1}}\int_{\Delta(0,R+R')}\frac{\frac{\partial f}{\partial \bar{z}}(z+w)}{w}dw\wedge d\bar{w} \\ &= \frac{1}{2\pi\sqrt{-1}}\int_{\Delta(0,R)}\frac{\frac{\partial f}{\partial \bar{z}}(w)}{w-z}dw\wedge d\bar{w}. \end{aligned}$$

By (1.1), this equals

$$f(z) - \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta(0,R)}\frac{f(w)}{w-z}dw = f(z),$$

since supp$f \subset \Delta(0,R)$. □

Now we use the following conventions: $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, with $z_i = x_i + \sqrt{-1}y_i$, and

$$|z|^2 := |z_1|^2 + \cdots + |z_n|^2 = x_1^2 + y_1^2 + \cdots + x_n^2 + y_n^2.$$

For multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, we write

$$z^\alpha := z_1^{\alpha_1} \ldots z_n^{\alpha_n},$$

with $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\alpha! := \alpha_1! \cdots \alpha_n!$.

In one variable complex analysis, the unit disc plays a special role. The higher dimensional generalizations are balls and polydiscs:

- A *complex ball* with center $\mathbf{a} = (a_1, \ldots, a_n)$ and radius $r > 0$ is defined by

  $$B(\mathbf{a}, r) := \{z \in \mathbb{C}^n \big| \, |z - \mathbf{a}| < r\}.$$

  This is nothing but the Euclidean ball in $\mathbb{R}^{2n}$.

- A *polydic* with center $\mathbf{a} = (a_1, \ldots, a_n)$ and multi-radius $\mathbf{r} = (r_1, \ldots, r_n)$ with $r_i > 0, \forall i = 1, \ldots, n$ is defined by

  $$\Delta(\mathbf{a}, \mathbf{r}) := \{z \in \mathbb{C}^n \big| \, |z_i - a_i| < r_i, \forall i = 1, \ldots, n\}.$$

  This is the product of n 1-dimensional discs. When all the r_i equal $r > 0$, we usually abuse the notation to write it as $\Delta(\mathbf{a}, r)$.

Definition 1.2. *Let $\Omega \subset \mathbb{C}^n$ be a non-empty open set (we call it a "region"), $f = u + \sqrt{-1}v : \Omega \to \mathbb{C}$ a C^1 map. We call f a* holomorphic function*, denoted by $f \in \mathcal{O}(\Omega)$, if f satisfies the Cauchy-Riemann equation:*

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_i}, \quad \frac{\partial u}{\partial y_i} = -\frac{\partial v}{\partial x_i}, \quad i = 1, \ldots, n. \tag{1.2}$$

This is equivalent to say f is holomorphic in each of its complex variables.

Remark 1.3. *By a deep theorem of Hartogs, we can remove the C^1 assumption in the above definition. For a proof, see Hörmander's book.*

As in the one-variable case, we introduce

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\Big(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\Big), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\Big(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\Big).$$

We also define

$$\partial := \sum_{i=1}^n \frac{\partial}{\partial z_i} dz_i, \quad \bar{\partial} := \sum_{i=1}^n \frac{\partial}{\partial \bar{z}_i} d\bar{z}_i,$$

then it is direct to check that $df = \partial f + \bar{\partial} f$, and f is holomorphic if and only if $\bar{\partial} f = 0$.

A 1-form of the form

$$\varphi = \varphi_1 dz_1 + \cdots + \varphi_n dz_n$$

with φ_i functions on Ω is called a $(1,0)$-form on Ω, and a 1-form of the form

$$\eta = \eta_1 d\bar{z}_1 + \cdots + \eta_n d\bar{z}_n$$

with η_i functions on Ω is called a $(0,1)$-form on Ω. A central technique in the modern theory of complex analysis is to use the $\bar{\partial}$-equation $\bar{\partial} u = \eta$ with good estimates to construct holomorphic objects.

As in one variable case, Cauchy formula is very important in several complex variables:

Theorem 1.4 (Cauchy formula)**.** *If $f \in \mathscr{O}(\Delta(\mathbf{a}, \mathbf{r})) \cap C^0(\overline{\Delta(\mathbf{a}, \mathbf{r})})$, then we have*

$$f(z) = \Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n \int_{|\zeta_1|=r_1} \cdots \int_{|\zeta_n|=r_n} \frac{f(\zeta_1, \cdots, \zeta_n)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n, \quad \forall z \in \Delta(\mathbf{a}, \mathbf{r}). \tag{1.3}$$

In particular, $f \in C^\infty(\Delta(\mathbf{a}, \mathbf{r}))$.

Proof. If f is holomorphic in a neighborhood of $\overline{\Delta(\mathbf{a}, \mathbf{r})}$, then (1.3) follows from repeating use of 1-dimensional Cauchy formula. In general, we work on $\Delta(\mathbf{a}, \theta\mathbf{r})$ for $0 < \theta < 1$ and let $\theta \to 1$.

The last claim follows from Cauchy formula by taking derivatives with respect to z under the integration sign. □

Remark 1.5. *An interesting feature of this formula is that the interior value of f depends only on its value on a part of the boundary. We write*

$$\partial_0 \Delta(\mathbf{a}, \mathbf{r}) := \{z | \, |z_i - a_i| = r_i, i = 1, \ldots, n\}.$$

It is called "characteristic boundary" or "distinguished boundary" or "Shilov boundary" of $\Delta(\mathbf{a}, \mathbf{r})$. If f is a given continuous function in a neighborhood of $\partial_0\Delta(\mathbf{a}, \mathbf{r})$, then the integral (1.3) *defines a holomorphic function in $\Delta(\mathbf{a}, \mathbf{r})$, since it is easy to see that the function is C^1 in z, and we can take derivatives under the integration sign.*

A direct corollary is the following useful derivative estimate:

Corollary 1.6 (Cauchy estimate)**.** *If $f \in \mathscr{O}(\Omega)$ and $\Delta(\mathbf{a}, \mathbf{r}) \subset \Omega$, then we have*

$$|\partial^\alpha f(a)| \leq \frac{\alpha!}{\mathbf{r}^\alpha} \sup_{\Delta(\mathbf{a}, \mathbf{r})} |f|. \tag{1.4}$$

Moreover, if $K \subset \Omega$ is compact, then for any relatively compact open neighborhood U, we have

$$\sup_K |\partial^\alpha f| \leq C_\alpha \sup_U |f|, \quad \forall f \in \mathscr{O}(\Omega),$$

where C_α is a constant depending only on α, K and U.

Proof. Again if $\overline{\Delta(\mathbf{a},\mathbf{r})} \subset \Omega$, by (1.3), we have for any $z \in \Delta(\mathbf{a},\mathbf{r})$:

$$\partial^\alpha f(z) = \frac{\alpha!}{(2\pi\sqrt{-1})^n} \int_{\partial_0\Delta(\mathbf{a},\mathbf{r})} \frac{f(\zeta_1,\cdots,\zeta_n)}{(\zeta_1 - z_1)^{\alpha_1+1}\cdots(\zeta_n - z_n)^{\alpha_n+1}} d\zeta_1\cdots d\zeta_n. \tag{1.5}$$

This implies that

$$|\partial^\alpha f(a)| \le \frac{\alpha!}{(2\pi)^n} \sup_{\Delta(\mathbf{a},\mathbf{r})} |f| \Big(\Pi_i(2\pi r_i)\Big)\Big(\Pi_i \frac{1}{r_i^{\alpha_i+1}}\Big) = \frac{\alpha!}{\mathbf{r}^\alpha} \sup_{\Delta(\mathbf{a},\mathbf{r})} |f|.$$

Again if $\overline{\Delta(\mathbf{a},\mathbf{r})} \not\subset \Omega$, we work on $\Delta(\mathbf{a},\theta\mathbf{r})$ for $0 < \theta < 1$ and let $\theta \to 1$.

The second statement follows directly from (1.4) by a compactness argument. □

Remark 1.7. *By* (1.4)*, we can bound* $\frac{\partial f}{\partial z_i}$ *by* $\sup|f|$*. On the other hand, we always have* $\frac{\partial f}{\partial \bar{z}_i} = 0$*, so we can bound the real partial derivatives of* f *by* $\sup|f|$*.*

There are several interesting corollaries of Cauchy formula and Cauchy estimates:

We say a series of functions $\sum_i f_i$ converges normally to f in a domain Ω, if it converges uniformly and absolutely on any compact subset $K \subset \Omega$ to f. Then we have

Corollary 1.8. *If* $f \in \mathscr{O}(\Delta(\mathbf{a},\mathbf{r}))$*, then we can expand* f *into a power series, converging normally in* $\Delta(\mathbf{a},\mathbf{r})$*:*

$$f(z) = \sum_\alpha \frac{\partial^\alpha f(\mathbf{a})}{\alpha!}(z-\mathbf{a})^\alpha, \quad \forall z \in \Delta(\mathbf{a},\mathbf{r}).$$

Proof. For any compact subset $K \subset \Delta(\mathbf{a},\mathbf{r})$, we can find a $\theta \in (0,1)$ such that $K \subset \Delta(\mathbf{a},\theta\mathbf{r})$. So we can assume without loss of generality that f is holomorphic in a neighborhood of $\overline{\Delta(\mathbf{a},\mathbf{r})}$. Also we assume $\mathbf{a} = 0$. Note that we have

$$\frac{1}{\Pi_i(\zeta_i - z_i)} = \frac{1}{\Pi_i\zeta_i} \cdot \frac{1}{1-\frac{z_1}{\zeta_1}} \cdots \frac{1}{1-\frac{z_n}{\zeta_n}} = \frac{1}{\Pi_i\zeta_i} \sum_{\alpha\in\mathbb{Z}^n_{\ge 0}} \frac{z^\alpha}{\zeta^\alpha},$$

with the right hand side converging normally in $\Delta(0,\mathbf{r})$ when $\zeta \in \partial_0\Delta(0,\mathbf{r})$. So we have by Cauchy formula:

$$\begin{aligned} f(z) &= \Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n \int_{\partial_0\Delta(0,\mathbf{r})} \frac{f(\zeta_1,\cdots,\zeta_n)}{(\zeta_1-z_1)\cdots(\zeta_n-z_n)} d\zeta_1\cdots d\zeta_n \\ &= \Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n \int_{\partial_0\Delta(0,\mathbf{r})} \frac{f(\zeta_1,\cdots,\zeta_n)}{\zeta_1\cdots\zeta_n} \sum_\alpha \frac{z^\alpha}{\zeta^\alpha} d\zeta_1\cdots d\zeta_n \\ &= \sum_\alpha c_\alpha z^\alpha, \end{aligned}$$

where

$$c_\alpha = \Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n \int_{\partial_0\Delta(0,\mathbf{r})} \frac{f(\zeta_1,\cdots,\zeta_n)}{\zeta_1^{\alpha_1+1}\cdots\zeta_n^{\alpha_n+1}} d\zeta_1\cdots d\zeta_n.$$

Comparing with (1.5), we have $\partial^\alpha f(0) = \alpha! c_\alpha$. □

Remark 1.9. *It is a good exercise for the readers to develop a theory of power series in more than one variables. The corresponding Abel's lemma also holds. One can find it in Grauert and Fritzsche's book "Several Complex Variables" (GTM38).*

Corollary 1.10 (Weierstrass theorem)**.** *If* $\{f_i\} \subset \mathscr{O}(\Omega)$*, and* f_i *converges to a function* f*, uniformly on any compact subset of* Ω *, then* $f \in \mathscr{O}(\Omega)$*.*

Proof. For any $\overline{\Delta(\mathbf{a}, \mathbf{r})} \subset \Omega$, we have Cauchy formula for each f_i. By uniform convergence of f_i, we can take limit inside the integration to get

$$f(z) = \Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n \int_{\partial_0 \Delta(\mathbf{a},\mathbf{r})} \frac{f(\zeta_1, \cdots, \zeta_n)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n, \quad \forall z \in \Delta(\mathbf{a}, \mathbf{r}).$$

Since f is continuous and the right hand side of the above formula is holomorphic in z, we conclude that $f \in \mathscr{O}(\Omega)$. □

Corollary 1.11 (Montel theorem)**.** *Let* $\{f_\alpha\} \subset \mathscr{O}(\Omega)$*. If they are uniformly bounded on any compact subset* $K \subset \Omega$*, then* $\{f_\alpha\}$ *is a normal family, i.e., any sequence of* $\{f_\alpha\}$ *contains a subsequence that converges to a holomorphic function* $f \in \mathscr{O}(\Omega)$*, uniformly on any compact subset of* Ω*.*

Proof. By (1.4), we can bound the first order derivatives of f_i uniformly on any compact set $K \subset \Omega$. So $\{f_i\}_{i=1}^{\infty}$ is equi-continuous. Then the corollary follows from Arzela-Ascoli theorem and Corollary 1.10. □

Corollary 1.12 (Uniqueness theorem)**.** *Let* $\Omega \subset \mathbb{C}^n$ *be a domain and* $f \in \mathscr{O}(\Omega)$*. If there is a non-empty open set* $U \subset \Omega$ *such that* $f|_U \equiv 0$*, then* $f \equiv 0$ *in* Ω*.*

Proof. Define the set

$$N := \{z \in \Omega \big| \, \partial^\alpha f(z) = 0, \forall \alpha \in \mathbb{Z}^n_{\geq 0}\}.$$

By definition it is a closed subset of Ω. By Cauchy formula, N is also open. Since by assumption $N \neq \emptyset$, the connectivity assumption of Ω implies that $N = \Omega$, so $f \equiv 0$. □

Corollary 1.13 (Maximum Principle)**.** *Let* $\Omega \subset \mathbb{C}^n$ *be a domain. If* $f \in \mathscr{O}(\Omega) \cap C^0(\bar{\Omega})$*, then*

$$\max_{\bar{\Omega}} |f| = \max_{\partial\Omega} |f|,$$

and $\max |f|$ *can not be achieved at an interior point unless* f *is a constant.*

Proof. Suppose $\max_{\bar{\Omega}} |f|$ is achieved at $\mathbf{a} \in \Omega$, choose $r > 0$ such that $\overline{\Delta(\mathbf{a}, r)} \subset \Omega$. Repeating the 1-dimensional maximum principle, we conclude that $f|_{\Delta(\mathbf{a},r)} \equiv f(\mathbf{a})$. By Corollary 1.12, $f \equiv f(\mathbf{a})$. □

One of the first examples showing that complex analysis in higher dimensions is drastically different form the 1-dimensional case is the following phenomenon discovered by Hartogs.

Example 1.14 (Hartogs phenomenon)**.** *Define a domain $H \subset \Delta(0,1) \subset \mathbb{C}^2$ by*

$$H := \{(z,w) \in \mathbb{C}^2 \;\big|\; |z| < \frac{1}{2}, |w| < 1\} \cup \{(z,w) \in \mathbb{C}^2 \;\big|\; |z| < 1, \frac{1}{2} < |w| < 1\}.$$

Then the restriction map $\mathscr{O}(\Delta(0,1)) \to \mathscr{O}(H)$ is always surjective, i.e. any holomorphic functions on H can be continued holomorphically to the larger domain $\Delta(0,1)$. In fact, for $f \in \mathscr{O}(H)$, we choose a $\frac{1}{2} < \beta < 1$, and define

$$\tilde{f}(z,w) := \frac{1}{2\pi\sqrt{-1}} \int_{|\xi|=\beta} \frac{f(z,\xi)}{\xi - w} d\xi, \quad |z| < 1, |w| < \beta.$$

Then by uniqueness, $\tilde{f}$ is independent of β, hence defines a function $\tilde{f} \in \mathscr{O}(\Delta(0,1))$. Again by uniqueness, we have $\tilde{f}|_H = f$.

Note that for a pair of domains $\Omega \subsetneq \Omega' \subset \mathbb{C}$, we can always find a $f \in \mathscr{O}(\Omega)$ such that f can not be continued holomorphically to Ω'. For example, choose any $a \in \partial\Omega \cap \Omega'$, then $\frac{1}{z-a}$ is what we want.

We have another extension theorem, also due to Hartogs:

Theorem 1.15 (Hartogs's extension theorem)**.** *Let K be a compact subset of the open set $\Omega \subset \mathbb{C}^n$. Assume $\Omega \setminus K$ is connected, then any $f \in \mathscr{O}(\Omega \setminus K)$ extends holomorphically to Ω.*

Proof. We need a lemma:

Lemma 1.16. *Let $\eta := \eta_1 d\bar{z}_1 + \ldots \eta_n d\bar{z}_n$ be a smooth $(0,1)$-form with compact support on $\mathbb{C}^n$. If*

$$\frac{\partial \eta_i}{\partial \bar{z}_j} = \frac{\partial \eta_j}{\partial \bar{z}_i} \tag{1.6}$$

for any pair $i, j = 1, \ldots, n$, then we can always find a smooth function $u \in C_0^\infty(\mathbb{C}^n)$ such that $\bar{\partial} u = \eta$.

Assuming the lemma at present. Choose a real-valued smooth function with compact support $\varphi \in C_0^\infty(\Omega)$ such that φ is identically 1 in a small neighborhood of K. Then $v := (1-\varphi)f$ can be viewed as a smooth function on Ω, vanishing near K and holomorphic outside $\mathrm{supp}\varphi$. We define a smooth $(0,1)$-form with compact support on $\mathbb{C}^n$ by

$$\eta := \begin{cases} \bar{\partial} v = -f\bar{\partial}\varphi, & \text{on } \Omega, \\ 0, & \text{on } \mathbb{C}^n \setminus \Omega. \end{cases}$$

Then it is easy to see that η satisfies (1.6), so by Lemma1.16, we can find $u \in C_0^\infty(\mathbb{C}^n)$ such that $\bar{\partial} u = \eta$. We define a function F on Ω by

$$F(z) := v(z) - u(z), \quad \forall z \in \Omega.$$

Then we have $\bar{\partial} F = 0$, so $F \in \mathscr{O}(\Omega)$.

Finally we need to check that $F|_{\Omega \setminus K} = f$. Since $\Omega \setminus K$ is connected, by uniqueness theorem, we only need to show that they coincide on an open subset of $\Omega \setminus K$.

Note that u is in fact holomorphic on $\mathbb{C}^n \setminus \operatorname{supp}\varphi$ (which may not be connected). Since it also has compact support, it necessarily vanishes on the unbounded component of $\mathbb{C}^n \setminus \operatorname{supp}\varphi$ by the uniqueness theorem. But the boundary of this unbounded component must belong to $\Omega \setminus K$, so we can find open subset of $\Omega \setminus K$ on which $u = 0$ and $v = f$, thus $F = f$ there. □

Proof of Lemma 1.16: We define

$$u(z) := \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{\eta_1(w_1, z_2, \dots, z_n)}{w_1 - z_1} dw_1 \wedge d\bar{w}_1.$$

Then it is easy to see that $u \in C^\infty(\mathbb{C}^n)$ and. since η_1 has compact support, vanishes when $|z_2| + \cdots + |z_n|$ is large enough. By Lemma1.1,

$$\frac{\partial u}{\partial \bar{z}_1} = \eta_1.$$

Also, for $k = 2, \dots, n$, by (1.6)

$$\begin{aligned}
\frac{\partial u}{\partial \bar{z}_k}(z) &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{\partial_{\bar{k}} \eta_1(w_1, z_2, \dots, z_n)}{w_1 - z_1} dw_1 \wedge d\bar{w}_1 \\
&= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{\partial_{\bar{1}} \eta_k(w_1, z_2, \dots, z_n)}{w_1 - z_1} dw_1 \wedge d\bar{w}_1 \\
&= \eta_k(z).
\end{aligned}$$

The last equality also uses (1.1) as in the proof of Lemma1.1. So we proved that $\bar{\partial} u = \eta$.

Finally we prove that u has compact support. We already knew that u vanishes when one of $|z_2|, \dots, |z_n|$ is large enough. Now choose $R > 0$ large enough and apply (1.1) to u as a function of z_2:

$$\begin{aligned}
u(z) &= \frac{1}{2\pi\sqrt{-1}} \int_{|w|<R} \frac{\frac{\partial u}{\partial \bar{z}_2}(z_1, w, z_3, \dots, z_n)}{w - z_2} dw \wedge d\bar{w} \\
&= \frac{1}{2\pi\sqrt{-1}} \int_{|w|<R} \frac{\eta_2(z_1, w, z_3, \dots, z_n)}{w - z_2} dw \wedge d\bar{w}.
\end{aligned}$$

From this expression, we conclude that u also vanishes when $|z_1|$ is large enough, hence u has compact support. □

Remark 1.17. *It is interesting to compare Lemma 1.16 with Lemma1.1. One could say that many of the "strange" properties in higher dimensional complex analysis are caused by the fact that we can solve the $\bar{\partial}$-equation with a solution also with* **compact** *support.*

As a direct corollary of Theorem 1.15, we see that all isolated singularities of holomorphic functions with more than one variable are always removable.

Definition 1.18. *Let $U \subset \mathbb{C}^n$ be a domain, then a map $f = (f_1, \ldots, f_m) : U \to \mathbb{C}^m$ is called holomorphic, if all its components are holomorphic, i.e. $f_i \in \mathcal{O}(U), \forall i = 1, \ldots, m$. If f is bijective onto its image and its inverse is also holomorphic, then we say it is biholomorphic*[1]*, and U is biholomorphic to $f(U)$.*

Example 1.19. *If $\Omega \subsetneq \mathbb{C}$ is a simply connected domain, then Ω is biholomorphic to $\Delta(1) \subset \mathbb{C}$. This is the famous "Riemann mapping theorem".*

Example 1.20.

1. *Any polydisc $\Delta(\mathbf{a}, \mathbf{r})$ is biholomorphic to $\Delta(0, 1)$: we can choose the biholomorphic map to be*

$$f(z_1, \ldots, z_n) = \Big(\frac{z_1 - a_1}{r_1}, \ldots, \frac{z_n - a_n}{r_n}\Big).$$

2. *The ball $B(0, 1) \subset \mathbb{C}^n$ is biholomorphic to the unbounded domain*

$$H := \{w \in \mathbb{C}^n \mid Im\ w_n > \sum_{i=1}^{n-1} |w_i|^2\}$$

by the map

$$w = f(z) = (\frac{z_1}{1 + z_n}, \ldots, \frac{z_{n-1}}{1 + z_n}, \sqrt{-1}\frac{1 - z_n}{1 + z_n}).$$

The boundary of H is called the "Heisenberg group", which plays important roles in CR-geometry and harmonic analysis.

Another example showing that complex analysis in higher dimensions is drastically different form the 1-dimensional case is the following theorem discovered by H. Poincaré.

Theorem 1.21 (H. Poincaré)**.** *Let $n \geq 2$, then $B(0, 1) \subset \mathbb{C}^n$ is not biholomorphic to $\Delta(0, 1) \subset \mathbb{C}^n$.*

Proof. I learnt the following proof from the book of R. Narasimhan , where the author attributes the idea to Remmert and Stein. Poincaré's original proof is to show that the groups of automorphisms (means biholomorphic maps onto itself) of these two domains are not isomorphic. For simplicity, we only prove the $n = 2$ case and left the general case to readers.

[1] The we necessarily have $m = n$

Step 1: Suppose we have a biholomorphic map $f(z,w) : \Delta(0,1) \to B(0,1)$. Then for any sequence $\{z_i\} \subset \Delta \subset \mathbb{C}$ with $|z_i| \to 1$, the sequence of one-variable holomorphic functions $g_i(w) = f(z_i,w) : \Delta \to B(0,1)$ is uniformly bounded. By Montel's theorem, we can assume that g_i converges uniformly on compact subsets of Δ to a holomorphic map $g(w) = (g_1(w), g_2(w)) : \Delta \to \overline{B(0,1)}$.

Step 2: We have $|g(w)| \equiv 1$ on Δ.

In fact, if there is a point $w_0 \in \Delta$ such that $g(w_0) \in B(0,1)$. Choose a small $\epsilon > 0$ such that $\overline{B(g(w_0),\epsilon)} \subset B(0,1)$. Since a biholomorphic map is necessarily proper (i.e. the preimage of any compact set is also compact), $f^{-1}(\overline{B(g(w_0),\epsilon)})$ is a compact subset of $\Delta(0,1)$. Since $(z_i,w_0) \to \partial\Delta(0,1)$ as $i \to \infty$, we have $(z_i,w_0) \notin f^{-1}(\overline{B(g(w_0),\epsilon)})$ when i is large enough. This means

$$f(z_i,w_0) \notin \overline{B(g(w_0),\epsilon)}$$

when i is large enough, contradicting the fact $f(z_i,w_0) \to g(w_0)$.

Step 3: From Step 2, we further conclude that $g(w)$ is a constant map, i.e. $g'(w) \equiv (0,0)$.

One way of seeing this is to use the fact that a non-constant holomorphic function in one variable is always an open map. Alternatively, we can compute the derivatives:

$$\begin{aligned}
0 &= \frac{\partial^2 |g(w)|^2}{\partial w \partial \bar{w}} \\
&= \frac{\partial}{\partial w}\Big(g_1(w)\frac{\partial \bar{g}_1}{\partial \bar{w}}(w)\Big) + \frac{\partial}{\partial w}\Big(g_2(w)\frac{\partial \bar{g}_2}{\partial \bar{w}}(w)\Big) \\
&= |g_1'(w)|^2 + |g_2'(w)|^2.
\end{aligned}$$

It follows that

$$\lim_{i\to\infty} \frac{\partial f}{\partial w}(z_i,w) = g'(w) \equiv 0.$$

This implies that for each fixed $w \in \Delta$, $\frac{\partial f}{\partial w}(z,w)$, as a function of z, is holomorphic in $\Delta \subset \mathbb{C}$ and continuous on $\bar{\Delta}$ with boundary value 0. By maximum principle we get $\frac{\partial f}{\partial w}(z,w) \equiv 0$ on $\Delta(0,1)$. This implies f is independent of w, contradicts the fact that f is a biholomorphic map. $\square$

Many theorems in multi-variable calculus have "holomorphic" versions, for example, the inverse function theorem and implicit function theorem. Let Ω be a non-empty domain of $\mathbb{C}^n$ and $f : \Omega \to \mathbb{C}^m$ be a holomorphic map. Then we can define the holomorphic Jacobian of f at $z \in \Omega$ to be the $m \times n$ matrix:

$$J_f^{\mathbb{C}}(z) := \frac{\partial(f_1, \dots, f_m)}{\partial(z_1, \dots, z_n)} := \Big(\frac{\partial f_i}{\partial z_j}(z)\Big)_{1\le i\le m, 1\le j\le n}.$$

Theorem 1.22 (The inverse function theorem)**.** *Let $f : \Omega \to \mathbb{C}^n$ be a holomorphic map and $J_f^{\mathbb{C}}(z_0)$ is non-degenerate for some point $z_0 \in \Omega$, then f has a local holomorphic inverse g in a neighborhood of $f(z_0)$, and we have*

$$J_g^{\mathbb{C}}(f(z)) = J_f^{\mathbb{C}}(z)^{-1}.$$

Proof. We shall first apply the traditional inverse function theorem to get an inverse map. For this, we need to study the real Jacobian of f at z_0:

$$J_f^{\mathbb{R}}(z) := \begin{pmatrix} \Big(\frac{\partial u_i}{\partial x_j}\Big)_{1\le i,j\le n} & \Big(\frac{\partial u_i}{\partial y_j}(z)\Big)_{1\le i,j\le n} \\ \Big(\frac{\partial v_i}{\partial x_j}\Big)_{1\le i,j\le n} & \Big(\frac{\partial v_i}{\partial y_j}(z)\Big)_{1\le i,j\le n} \end{pmatrix},$$

where we write $z_i = x_i + \sqrt{-1}y_i$ and $f_i = u_i + \sqrt{-1}v_i$.

Claim: For holomorphic f, we have

$$\det J_f^{\mathbb{R}}(z) = |\det J_f^{\mathbb{C}}(z)|^2.$$

The reason is simple. For short, we write

$$J_f^{\mathbb{R}}(z) =: \begin{pmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} \end{pmatrix}.$$

Then the Cauchy-Riemann equation can be written as

$$\frac{\partial U}{\partial X} = \frac{\partial V}{\partial Y}, \quad \frac{\partial U}{\partial Y} = -\frac{\partial V}{\partial X},$$

and hence

$$J_f^{\mathbb{C}} = \frac{\partial f}{\partial Z} = \frac{\partial U}{\partial X} - \sqrt{-1}\frac{\partial U}{\partial Y}.$$

So we have

$$\begin{aligned} \det J_f^{\mathbb{R}} &= \det\begin{pmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} \end{pmatrix} = \det\begin{pmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} \\ -\frac{\partial U}{\partial Y} & \frac{\partial U}{\partial X} \end{pmatrix} \\ &= \det\begin{pmatrix} J_f^{\mathbb{C}} & \frac{\partial U}{\partial Y} \\ -\sqrt{-1}J_f^{\mathbb{C}} & \frac{\partial U}{\partial X} \end{pmatrix} = \det\begin{pmatrix} J_f^{\mathbb{C}} & \frac{\partial U}{\partial Y} \\ 0 & \overline{J_f^{\mathbb{C}}} \end{pmatrix} \\ &= |\det J_f^{\mathbb{C}}|^2. \end{aligned}$$

Now we have $\det J_f^{\mathbb{R}}(z_0) = |\det J_f^{\mathbb{C}}(z_0)|^2 \neq 0$. By classical inverse function theorem, we have a local C^1 inverse of f near $w_0 := f(z_0)$. We write it as $z = g(w)$. We shall prove that it is holomorphic.

In fact, from the identity $w_i = f_i(g_1(w), \ldots, g_n(w))$, we have, by the chain rule,

$$0 = \frac{\partial w_i}{\partial \bar{w}_j} = \sum_k \frac{\partial f_i}{\partial z_k}(g(w))\frac{\partial g_k}{\partial \bar{w}_j}(w).$$

Since the matrix $J_f^{\mathbb{C}}$ is invertible near z_0, we conclude that $\frac{\partial g_k}{\partial \bar{w}_j} = 0$ for all k, j. So g is holomorphic. Again taking $\frac{\partial}{\partial w_j}$ on both sides of $w_i = f_i(g_1(w), \ldots, g_n(w))$, we get $I_n = J_f^{\mathbb{C}}(g(w))J_g^{\mathbb{C}}(w)$. ◻

Theorem 1.23 (The implicit function theorem)**.** *Let $f : \Omega \to \mathbb{C}^m$ be a holomorphic map with $m < n$. Suppose $f(z_0, w_0) = 0$ with $z_0 \in \mathbb{C}^{n-m}, w_0 \in \mathbb{C}^m$ and $(z_0, w_0) \in \Omega$. If*

$$\det \frac{\partial(f_1, \ldots, f_m)}{\partial(z_{n-m+1}, \ldots, z_n)}(z_0, w_0) \neq 0,$$

then we can find a holomorphic map $g : \Delta(z_0, \epsilon) \to \Delta(w_0, \delta) \subset \mathbb{C}^m$ such that $g(z_0) = w_0$ and

$$f(z, g(z)) \equiv 0, \quad \forall z \in \Delta(z_0, \epsilon).$$

Moreover, we have

$$\{(z, w) \in \Delta(z_0, \epsilon) \times \Delta(w_0, \delta)|\, f(z, w) = 0\} = \{(z, w)|\, z \in \Delta(z_0, \epsilon), w = g(z)\}.$$

Proof. There are at least two ways of proof. For example, we can argue as in the inverse function theorem by reducing it to the classical implicit function theorem, or we can consider the map $\tilde{f}(z, w) = (z, f(z, w)) : \Omega \to \mathbb{C}^n$ and apply Theorem 1.22. We leave the detail as an exercise. ◻

Remark 1.24. *The implicit function theorem says that if a holomorphic map is non-degenerate at a given zero point, then its zero locus is locally a graph near that point. What happens if the Jacobian degenerates at a given point? For example, consider the $m = 1$ case. If a holomorphic function $f(z_1, \ldots, z_{n-1}, w)$ satisfies $\frac{\partial^k f}{\partial w^k}(z_0, w_0) \neq 0$ but $\frac{\partial^i f}{\partial w^i}(z_0, w_0) = 0, \forall i = 0, \ldots, k-1$. What can we say about the zero locus of f near (z_0, w_0)? Weierstrass's "preparation theorem" answers this question. This theorem is fundamental to the local theory of several complex variables.*

2 Complex manifolds and complex vector bundles

2.1 Complex manifolds

Roughly speaking, a complex manifold is a topological space X on which we can talk about "holomorphic" functions. Since we know what does a holomorphic function means in Euclidean spaces, the first condition we impose on X is:

Condition 1:(existence of coordinate charts) X is locally homeomorphic to open sets of $\mathbb{C}^n$. To be precise, we require that there is an open covering $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ of X such that for each U_i we have a homeomorphism $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{C}^n$ onto an open set $\varphi_i(U_i)$ of $\mathbb{C}^n$.

Given these coordinates, we should define a function $f : \Omega \to \mathbb{C}$ to be holomorphic if all its coordinate-representations $f \circ \varphi_i^{-1} \in \mathscr{O}(\varphi_i(U_i \cap \Omega))$. But is this a well-defined notion? For example if $\Omega \subset U_i \cap U_j \neq \emptyset$, then on Ω we have two sets of coordinates. Is it possible that $f \circ \varphi_i^{-1} \in \mathscr{O}(\varphi_i(U_i \cap \Omega))$ but $f \circ \varphi_j^{-1} \notin \mathscr{O}(\varphi_i(U_j \cap \Omega))$? To avoid this, note that $f \circ \varphi_j^{-1} = (f \circ \varphi_i^{-1}) \circ (\varphi_i \circ \varphi_j^{-1})$, so we require:

Condition 2:(compatibility) Coordinate changes of Condition 1 should be holomorphic. To be precise, we require that whenever $U_i \cap U_j \neq \emptyset$, we have $\varphi_i \circ \varphi_j^{-1}$ is a biholomorphic map from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cap U_j)$.

Given these 2 conditions, one can check easily that the notion of "holomorphic function" makes perfect sense. However, to avoid pathology and use more analytic tools such as metrics and integration, we also require a complex manifold to be a nice topological space:

Condition 3: X satisfies T_2 and C_2 axioms, i.e. X is a Hausdorff space, and has a countable topological basis.

Definition 2.1. *A complex (analytic) manifold of dimension n is a topological space X satisfying Conditions 1,2,3 above. A 1-dimensional complex manifold is also known as a "Riemann surface". A map $f : X \to \mathbb{C}$ from a complex manifold X is called a "holomorphic function", if $f \circ \varphi_i^{-1} \in \mathscr{O}(\varphi_i(U_i))$ for all $i \in \Lambda$. In this case, we write $f \in \mathscr{O}(X)$.*

If X, Y are both complex manifolds of dimensions n and m respectively, a map $F : X \to Y$ is called "holomorphic", if for all coordinate charts (U, φ) of X and (V, ψ) of Y, the map $\psi \circ F \circ \varphi^{-1}$ is a holomorphic map on $\varphi(U \cap F^{-1}(V)) \subset \mathbb{C}^n$ whenever $U \cap F^{-1}(V) \neq \emptyset$. A holomorphic map with a holomorphic inverse is called "biholomorphic".

Remark 2.2. *In standard textbooks, the set of coordinate charts $\{(U_i, \varphi_i)\}_{i\in\Lambda}$ is assumed to be maximal, i.e., whenever a homeomorphism from an open set V, $\psi : V \to \psi(V) \subset \mathbb{C}^n$ is compatible with (U_i, φ_i) for all $U_i \cap V \neq \emptyset$, we have $(V, \psi) \in \{(U_i, \varphi_i)\}_{i\in\Lambda}$. It is easy* *to check that from the coordinate charts in our definition, one can always enlarge it to a unique maximal one satisfying the compatibility condition.*

Example 2.3. 1. *Open subsets of $\mathbb{C}^n$ are complex manifolds.*

2. *Let $\{e_1, \ldots, e_{2n}\}$ be any fixed $\mathbb{R}$-basis of $\mathbb{C}^n$, and let $\Lambda := \{m_1 e_1 + \cdots + m_{2n} e_{2n} | \ m_i \in \mathbb{Z}\}$ be a lattice of rank $2n$. Then we can define the quotient space $\mathbb{C}^n/\Lambda$, it is a compact Hausdorff space equipped with quotient topology. There is a natural complex manifold structure on $\mathbb{C}^n/\Lambda$, we call this complex manifold a "complex torus".*

3. *Let $P \in \mathbb{C}[z, w]$ be a polynomial of degree d. Define*

$$C := \{(z, w) | \ P(z, w) = 0\}.$$

We call it an "affine plane algebraic curve". Assume P is irreducible and $\frac{\partial P}{\partial z}, \frac{\partial P}{\partial w}$ have no common zeroes on C. Then C is a natural complex manifold. The coordinates can be chosen in the following way: if $\frac{\partial P}{\partial w}(z_0, w_0) \neq 0$, then we can apply the implicit function theorem 1.23 to find a neighborhood $\Delta(z_0, \epsilon) \times \Delta(w_0, \delta)$ and a holomorphic function $g(z)$ such that $U := C \cap (\Delta(z_0, \epsilon) \times \Delta(w_0, \delta)) = \{(z, w) | \ z \in \Delta(z_0, \epsilon), w = g(z)\}$. We choose $\varphi : U \to \mathbb{C}$ to be $\varphi(z, w) = z$. If $\frac{\partial P}{\partial z}(z_0, w_0) \neq 0$, we use w as local coordinate. Exercise: what's the coordinates transformation function?

The last example is a special case of "complex submanifold" we now define:

Definition 2.4. *A closed subset Y of a n-dimensional complex manifold X is called a "complex (analytic) submanifold" of dimension k, if for any $p \in Y$, we can find a compatible chart (U, φ) of X such that $p \in U$ and*

$$\varphi(U \cap Y) = \{(z_1, \ldots, z_n) \in \varphi(U) | \ z_{k+1} = \cdots = z_n = 0\}.$$

One can check that the restriction of such charts (we call them "adapted charts") to Y makes Y a complex manifold and the inclusion $Y \subset X$ is a holomorphic map.

Example 2.5 (The complex projective space)**.** *We define an equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$: $(z_0, \ldots, z_n) \sim (w_0, \ldots, w_n)$ if and only if we can find a non-zero $\lambda \in \mathbb{C}$ (write $\lambda \in \mathbb{C}^*$ for short) such that $w_i = \lambda z_i$ for all $i = 0, \ldots, n$. The equivalent class of $(z_0, \ldots, z_n)$ is denoted by $[z_0, \ldots, z_n]$. The n-dimensional complex projective space $\mathbb{C}P^n$ is defined to be the space of all equivalent classes, endowed with quotient topology. Then it is a compact Hausdorff space. We choose the holomorphic coordinate charts as follows: Define*

$$U_i := \{[z_0, \ldots, z_n] \in \mathbb{C}P^n | \ z_i \neq 0\}, \quad i = 0, \ldots, n.$$

These are open sets, and we define

$$\varphi_i : U_i \to \mathbb{C}^n, \quad \varphi_i([z_0, \ldots, z_n]) := (\frac{z_0}{z_i}, \ldots, \frac{\hat{z}_i}{z_i}, \ldots, \frac{z_n}{z_i}).$$

The checking of compatibility is left to readers. Also it is easy to check that $\mathbb{C}P^1$ is our familiar S^2.

Let $F_1, \dots, F_k \in \mathbb{C}[z_0, \dots, z_n]$ be a set of irreducible homogeneous polynomials of degrees $d_1, \dots, d_k$ respectively. Then the set

$$V(F_1, \dots, F_k) := \{[z_0, \dots, z_n] | \, F_1(z_0, \dots, z_n) = \cdots = F_k(z_0, \dots, z_n) = 0\}$$

is well-defined and is called a (complex) projective algebraic variety. If we assume that $V(F_1, \dots, F_k)$ is a complex submanifold of $\mathbb{C}P^n$, then it will be called a "projective algebraic manifold".

Example 2.6. *If $F \in \mathbb{C}[z_0, \dots, z_n]$ is irreducible and homogeneous of degree d. If we assume that the only common zero of $\frac{\partial F}{\partial z_0}, \dots, \frac{\partial F}{\partial z_n}$ in $\mathbb{C}^{n+1}$ is $(0, \dots, 0)$. Then $V(F)$ is a complex submanifold of dimension $n-1$. We check this on U_0. $V(F) \cap U_0$ is the zero locus of the holomorphic function $F(1, z_1, \dots, z_n) \in \mathscr{O}(U_0)$. We shall show that $\frac{\partial F}{\partial z_1}(1, z_1, \dots, z_n), \dots \frac{\partial F}{\partial z_n}(1, z_1, \dots, z_n)$ have no common zeroes on $V(F) \cap U_0$.*

Suppose

$$F(1, z_1^0, \dots z_n^0) = \frac{\partial F}{\partial z_1}(1, z_1^0, \dots, z_n^0) = \cdots = \frac{\partial F}{\partial z_n}(1, z_1^0, \dots, z_n^0) = 0.$$

By Euler's theorem on homogeneous functions, we have

$$\frac{\partial F}{\partial z_0}(1, z_1^0, \dots, z_n^0) + z_1^0 \frac{\partial F}{\partial z_1}(1, z_1^0, \dots, z_n^0) + \cdots + z_n^0 \frac{\partial F}{\partial z_n}(1, z_1^0, \dots, z_n^0) = dF(1, z_1^0, \dots, z_n^0) = 0.$$

This implies $\frac{\partial F}{\partial z_0}(1, z_1^0, \dots, z_n^0) = 0$, so $(1, z_1^0, \dots, z_n^0)$ is a common zero of $\frac{\partial F}{\partial z_0}, \dots, \frac{\partial F}{\partial z_n}$ in $\mathbb{C}^{n+1}$ different from $(0, \dots, 0)$.

For example, $V(z_0^d + \cdots + z_n^d)$ is a smooth submanifold of $\mathbb{C}P^n$, called the "Fermat hypersurface" of degree d.

A generalization of submanifold is the following:

Definition 2.7. *A closed subset A of a complex manifold X is called an "analytic subvariety", if it is locally the common zeroes of finitely many holomorphic functions, i.e. $\forall p \in A$, there is an open set $U \subset X$ and $f_1, \dots, f_k \in \mathscr{O}(U)$ such that $A \cap U = \{z \in U | \, f_1(z) = \cdots = f_k(z) = 0\}$.*

An analytic subvariety A is called a "hypersurface" if it is locally the zero locus of a holomorphic function.

Note that a complex submanifold is an analytic subvariety, we just choose U to be the domain of the adapted chart and f_i to be $z_{k+1}, \dots, z_n$.

Let $A \subset X$ be an analytic subvariety. $p \in A$ is called a *"regular point"*, if we can find open $U \subset X$ and $f_1, \dots, f_k \in \mathscr{O}(U)$ such that $A \cap U = \{z \in U | \, f_1(z) = \cdots = f_k(z) = 0\}$ and

$$rank \frac{\partial(f_1, \dots, f_k)}{\partial(z_1, \dots, z_n)}(p) = k.$$

In this case, A is locally near p a complex submanifold of dimension $n-k$: without loss of generality, assume

$$\det \frac{\partial(f_1,\ldots,f_k)}{\partial(z_1,\ldots,z_k)}(p) \neq 0,$$

then we can choose a new compatible coordinate system $(f_1,\ldots,f_k,z_{k+1},\ldots,z_n)$. This is an adapted chart for A near p.

The locus of regular points of A is denoted by A_{reg}. Its complement in A is called the *"singular locus"*, and its elements are called *"singular points of A"*.

Chow's theorem says that any complex analytic subvariety of $\mathbb{C}P^n$ is projective algebraic, i.e., the common zeroes of finitely many homogeneous polynomials.

To end this section, we say something about the existence of complex structures on a given differential manifold. A complex manifold is an even dimensional orientable differential manifold. However, for a given even dimensional oriented manifold, it is not always clear whether or not we can make it a complex manifold. There are topological obstructions to "almost complex structure", this can rule out all even dimensional spheres except S^2 and S^6. We already knew S^2 is a complex manifold. But the S^6 case is still open. In this view, we give an example of complex structures on product of odd dimensional spheres:

Example 2.8 (Calabi-Eckman)**.** *We can make $S^{2p+1}\times S^{2q+1}$ into a complex manifold. The idea is that we can write*

$$S^{2p+1}=\{z\in\mathbb{C}^{p+1}|\ \sum_{i=0}^{p}|z_i|^2=1\},\quad S^{2q+1}=\{z\in\mathbb{C}^{q+1}|\ \sum_{j=0}^{q}|z_j|^2=1\},$$

and we have the Hopf fibration maps:

$$\pi_p: S^{2p+1}\to\mathbb{C}P^p,\quad \pi_q: S^{2q+1}\to\mathbb{C}P^q,$$

each with fiber S^1. So if we consider the map $\pi=(\pi_p,\pi_q): S^{2p+1}\times S^{2q+1}\to\mathbb{C}P^p\times\mathbb{C}P^q$, then we can view $S^{2p+1}\times S^{2q+1}$ as a fiber bundle on $\mathbb{C}P^p\times\mathbb{C}P^q$, which is a complex manifold, with fiber $S^1\times S^1=T^2$, which can also be made a complex manifold.

To be precise, fix a $\tau\in\mathbb{C}$ with $Im\tau>0$. We donote by T_τ the complex torus $\mathbb{C}/<1,\tau>$. Consider the open sets:

$$U_{kj}:=\{(z,z')\in S^{2p+1}\times S^{2q+1}|\ z_k z'_j\neq 0\},$$

and the map $h_{kj}: U_{kj}\to\mathbb{C}^{p+q}\times T_\tau$ given by

$$h_{kj}(z,z')=(\frac{z_0}{z_k},\ldots,\frac{\hat{z_k}}{z_k},\ldots,\frac{z_p}{z_k},\frac{z'_0}{z'_j},\ldots,\frac{\hat{z'_j}}{z'_j},\ldots,\frac{z'_q}{z'_j},t_{kj}),$$

where $t_{kj}:=\frac{1}{2\pi\sqrt{-1}}(\log z_k+\tau\log z'_j)\mod <1,\tau>$. Exercise: check that these charts makes $S^{2p+1}\times S^{2q+1}$ a complex manifold.

A direct application of the maximum principle gives:

Theorem 2.9. *Any holomorphic function on a compact connected complex manifold should be a constant.*

Let M be a complex submanifolds of $\mathbb{C}^n$. Since the restriction of complex coordinate functions of $\mathbb{C}^n$ to M are holomorphic functions on M, we get:

Corollary 2.10. *There are no compact complex submanifolds of $\mathbb{C}^n$ of positive dimension.*

Remark 2.11. *Those non-compact complex manifolds which admit proper holomorphic embeddings into $\mathbb{C}^N$ for some large N are precisely "Stein manifolds" in complex analysis (Remmert's theorem).*

The triumph of this short course is Kodaira's "projective embedding theorem", characterizing those compact complex manfolds which admit holomorphic embeddings into $\mathbb{C}P^N$ for some large N, i.e., projective algebraic manifolds.

2.2 Vector bundles

Roughly speaking, a holomorphic vector bundle over a complex manifold is a family of vector spaces, varying holomorphically.

Definition 2.12. *A holomorphic vector bundle of rank r over a n-dimensional complex manifold X is a complex manifold E of dimension $n + r$, together with a holomorphic surjective map $\pi : E \to X$ satisfying:*

1. *(**Fiberwise linear**) Each fiber $E_p := \pi^{-1}(p)$ has the structure of r-dimensional vector space over $\mathbb{C}$;*
2. *(**Locally trivial**) There is an open cover of X, $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ such that each $\pi^{-1}(U_i)$ is biholomorphic to $U_i \times \mathbb{C}^r$ via $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$, and $E_p \hookrightarrow \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ is a linear isomorphism onto $\{p\} \times \mathbb{C}^r$ for any $p \in U_i$. φ_i is called a "local trivialization".*

In this case, whenever $U_i \cap U_j \neq \emptyset$, we have a holomorphic map, called the "transition map", $\psi_{ij} : U_i \cap U_j \to GL(r, \mathbb{C})$ (viewed as an open subset of $\mathbb{C}^{r^2}$) such that $\varphi_i \circ \varphi_j^{-1}(z, v) = (z, \psi_{ij}(z)v)$. These families of transition maps satisfies the "cocycle condition":

(1) $\psi_{ij}\psi_{ji} = I_r$ on $U_i \cap U_j$;

(2) Whenever $U_i \cap U_j \cap U_k \neq \emptyset$, we have $\psi_{ij}\psi_{jk}\psi_{ki} = I_r$ on $U_i \cap U_j \cap U_k$.

The name "cocycle" is no coincidence. In fact we will see later that $\{\psi_{ij}\}$ above is indeed a cocycle in Čech's approach to sheaf cohomology theory.

Remark 2.13. *On the other hand, if we are given a set of holomorphic transition maps $\psi_{ij} : U_i \cap U_j \to GL(r, \mathbb{C})$ satisfying the cocycle condition, we can construct a holomorphic vector bundle by setting $E = \coprod_{i\in\Lambda}(U_i \times \mathbb{C}^r)/\sim$, where $(z, v) \sim (z', w)$ for $(z, v) \in U_i \times \mathbb{C}^r$ and $(z', w) \in U_j \times \mathbb{C}^r$ if and only if $z = z'$ and $v = \psi_{ij}(z)w$. We leave the detail as an exercise.*

A holomorphic vector bundle of rank 1 is usually called a *"holomorphic line bundle"*.

Definition 2.14 (holomorphic section)**.** *Let $\pi : E \to X$ be a holomorphic vector bundle over X. Let $U \subset X$ be an open set. A holomorphic section of E over U is a holomorphic map $s : U \to E$ such that $\pi \circ s = id_U$, i.e., $s(p) \in E_p$ for any $p \in U$. The set of holomorphic sections over U is usually denoted by $\Gamma(U, \mathcal{O}(E))$ or $\mathcal{O}(E)(U)$.*

One of the fundamental problem for the theory of vector bundles is the construction of global holomorphic sections of a given bundle. An important tool is the L^2-method for the $\bar{\partial}$-equation. One can find the basics from Hörmander's book. It is interesting that whether or not we can solve the equation depends on the geometry, in particular, the curvature of the bundle.

Definition 2.15 (bundle map)**.** *Let $\pi^E : E \to X$ and $\pi^F : F \to X$ are holomorphic vector bundles of ranks r and s respectively. A bundle map from E to F is a holomorphic map $f : E \to F$ such that f maps E_p to F_p for any $p \in X$ and $f|_{E_p} : E_p \to F_p$ is linear. When a bundle map has an inverse bundle map, we will say that these two bundles are isomorphic.*

Another fundamental problem is the classification problem. One important tool is the theory of characteristic classes that we shall discuss later. Also the set of isomorphic classes of holomorphic vector bundles over a given complex manifold has rich structures and is an important invariant for the complex manifold.

Example 2.16 (trivial bundle)**.** *$X \times \mathbb{C}^r$ with $\pi_1 : X \times \mathbb{C}^r \to X$ is a holomorphic vector bundle over X, called the "trivial bundle" over X, denoted by $\underline{\mathbb{C}^r}$.*

Example 2.17 (holomorphic tangent bundle)**.** *Let X be a complex manifold of dimension n. We shall now construct its "holomorphic tangent bundle" TX as follows:*

Let $p \in X$, we first define the ring

$$\mathcal{O}_{X,p} := \varinjlim \mathcal{O}_X(U),$$

where the direct limit is taken with respect to open sets $p \in U$. For persons not familiar with direct limit, this is $\coprod_{U \ni p} \mathcal{O}_X(U)/\sim$, with $f \in \mathcal{O}_X(U)$ equivalent to $g \in \mathcal{O}_X(V)$ iff we can find another open set $p \in W \subset U \cap V$ such that $f|_W = g|_W$. As an exercise, we can see that $\mathcal{O}_{X,p}$ is isomorphic to the ring of convergent power series $\mathbb{C}\{z_1, \ldots, z_n\}$. An element of $\mathcal{O}_{X,p}$ is called a "germ of holomorphic function" at p.

A tangent vector at p is a derivation $v : \mathcal{O}_{X,p} \to \mathbb{C}$, i.e., a $\mathbb{C}$-linear map satisfying the Leibniz rule

$$v(fg) = v(f)g(p) + f(p)v(g).$$

The set of tangent vectors at p is easily seen to be a $\mathbb{C}$-vector space. We call it the (holomorphic) tangent space of X at p, denoted by T_pX.

If $\varphi : U_i \to \mathbb{C}^n$ is a holomorphic coordinate chart with $\varphi_i = (z_1, \ldots, z_n)$. Then we can define $\frac{\partial}{\partial z_i}|_p \in T_pX$ to be

$$\frac{\partial}{\partial z_i}|_p(f) := \frac{\partial(f \circ \varphi_i^{-1})}{\partial z_i}(\varphi_i(p)).$$

Then one can show that $\{\frac{\partial}{\partial z_i}|_p\}_{i=1}^n$ is a basis of T_pX.

Let $TX := \coprod_{p \in X} T_pX$, and define $\pi : TX \to X$ in the obvious way. We can make it a holomorphic vector bundle of rank n over X as follows: Let (U_i, φ_i) be a holomorphic chart. Then we can define the local trivialization $\tilde{\varphi}_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$ to be

$$\tilde{\varphi}_i(q, \sum_i a_i \frac{\partial}{\partial z_i}|_q) := (q, a_1, \ldots, a_n).$$

This gives a complex structure on TX and at the same time gives a local trivialization of TX over U_i.

A holomorphic section of TX over U is called a "holomorphic vector field" *on U.*

Example 2.18 (holomorphic cotangent bundle). *Any $f \in \mathscr{O}_{X,p}$ defines a linear functional on T_pX by $v \mapsto v(f)$. We call this $df|_p \in (T_pX)^* =: T_p^*X$. T_p^*X is called the (holomorphic) cotangent space of X at p. It is easy to see that if (U_i, φ_i) is a holomorphic chart, then $\{dz_i|_p\}_{i=1}^n$ is the basis of T_p^*X dual to $\{\frac{\partial}{\partial z_i}|_p\}_{i=1}^n$.*

We can similarly give $T^*X := \coprod_{p\in X} T_p^*X$ a holomorphic bundle structure, called the "(holomorphic) cotangent bundle" of X. We leave this as an exercise.

A holomorphic section of T^*X over U is called a "holomorphic 1-form" on U.

In this course, holomorphic line bundles play very important roles. Let $\pi : L \to X$ be a holomorphic line bundle and $\{U_i\}_{i\in\Lambda}$ an open cover by trivialization neighborhoods, and $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ the trivialization map. Since $GL(1, \mathbb{C}) = \mathbb{C}^*$, now the transition maps ψ_{ij} become non-vanishing holomorphic functions on $U_i \cap U_j$. Let $s \in \Gamma(X, \mathscr{O}(L))$, then $\varphi_i \circ s|_{U_i} : U_i \to U_i \times \mathbb{C}$ could be represented by a holomorphic function $f_i \in \mathscr{O}(U_i)$, such that $\varphi_i \circ s|_{U_i}(p) = (p, f_i(p))$. When $U_i \cap U_j \neq \emptyset$, since $s|_{U_i} = s|_{U_j}$ on $U_i \cap U_j$, we have for any $p \in U_i \cap U_j$:

$$\begin{aligned}(p, f_i(p)) &= \varphi_i(s(p)) \\ &= (\varphi_i \circ \varphi_j^{-1}) \circ \varphi_j(s(p)) \\ &= (\varphi_i \circ \varphi_j^{-1})(p, f_j(p)) \\ &= (p, \psi_{ij}(p) f_j(p)).\end{aligned}$$

So we have $f_i = \psi_{ij} f_j$ on $U_i \cap U_j$. On the other hand, it is direct to check that given a family of holomorphic functions $f_i \in \mathscr{O}(U_i)$, satisfying $f_i = \psi_{ij} f_j$ on $U_i \cap U_j$, then there corresponds a unique $s \in \Gamma(X, \mathscr{O}(L))$.

Example 2.19 (Universal line bundle over $\mathbb{C}P^n$). [2] *We define a holomorphic line bundle $U \to \mathbb{C}P^n$ as follows: As a set,*

$$U = \{([z], v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} | \ v \in [z]\},$$

where we view $[z]$ as the 1-dimensional subspace of $\mathbb{C}^{n+1}$ determined by z. As one can check easily, we can write

$$U = \{([z], v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} | \ v_i z_j - z_j z_i = 0, \forall i, j = 0, \ldots, n\}.$$

From this, it is easy to see that U is a complex submanifold of $\mathbb{C}P^n \times \mathbb{C}^{n+1}$, and hence a complex manifold. The projection onto its first component $\mathbb{C}P^n$ is clearly a holomorphic map, with fiber the 1-dimensional linear subspace of $\mathbb{C}^{n+1}$ generated by $(z_0, \ldots, z_n)$.

For local triviality, we use the holomorphic charts $\{(U_i, \varphi_i)\}_{i=0}^n$ defined before. On $\pi^{-1}(U_i)$, each $v \in U_{[z]}$ can be uniquely write as $t \cdot (\frac{z_0}{z_i}, \ldots, 1, \ldots, \frac{z_n}{z_i})$, so we define

$$\tilde{\varphi}_i([z_0, \ldots, z_n], t \cdot (\frac{z_0}{z_i}, \ldots, 1, \ldots, \frac{z_n}{z_i})) = ([z_0, \ldots, z_n], t) \in U_i \times \mathbb{C}.$$

This is easily seen to be a biholomorphic map.

It is easy to write down the transition functions: $\psi_{ij}([z]) = \frac{z_i}{z_j}$.

[2] Also called the "tautological bundle"

Construct new bundles from old ones: The usual constructions in linear algebra all have counterparts in the category of vector bundles over X.

Direct sum

Let E, F be vector bundles over X of rank r and s respectively. Then their direct sum is a vector bundle of rank $r+s$ with fiber $E_p \oplus F_p$. To describe it, it suffices to write down the transition maps: if $\{U_i\}_{i\in\Lambda}$ is a common trivializing covering of X for E and F. The transition maps are ψ_{ij} and η_{ij} respectively, then the transition maps for $E\oplus F$ are precisely $diag(\psi_{ij}, \eta_{ij})$ with values in $GL(r+s, \mathbb{C})$.

Tensor product

Let E, F be vector bundles over X of rank r and s respectively. Then their tensor product is a vector bundle of rank rs with fiber $E_p \otimes F_p$. In applications, we only use the tensor product of a line bundle L with a general vector bundle E. In this case, if the transition maps for E and L with respect to a common trivializing covering are ψ_{ij} and η_{ij}, then the transition maps of $E \otimes L$ are $\eta_{ij}\psi_{ij}$.

$\underline{Hom}(E, F)$

Let E, F be vector bundles over X of rank r and s respectively. Then $\underline{Hom}(E, F)$ is a vector bundle of rank rs with fiber $Hom(E_p, F_p)$, the space of linear maps from E_p to F_p. In particular, we define the dual of E to be $E^* := \underline{Hom}(E, \underline{\mathbb{C}})$, whose fiber over p is exactly the dual space of E_p, $(E_p)^*$.

When $L \to X$ is a holomorphic line bundle, we can easily describe L^* in terms of transition functions: if the transition functions of L are ψ_{ij}, then the transition functions of L^* are ψ_{ij}^{-1}. For this reason, we usually also write L^{-1} for L^*.

Exercise: Prove that the bundle $\underline{Hom}(E, F)$ is isomorphic to $E^* \otimes F$.

Example 2.20. *Let $U \to \mathbb{C}P^n$ be the universal bundle, its dual is usually denoted by H, we call it the "hyperplane line bundle".* [3] *Another common notation for H is $\mathscr{O}(1)$. We also write the H^k, or $\mathscr{O}(k)$, short for the k-times tensor product of H, $H^k := H^{\otimes k} = H \otimes \cdots \otimes H$, and $\mathscr{O}(-k) := H^{-k} := U^{\otimes k}$.*

We now study the holomorphic sections of H^k for $k > 0$. Let $s \in \Gamma(\mathbb{C}P^n, \mathscr{O}(H^k))$, we know that s can be represented by a family of holomorphic functions $f_\alpha \in \mathscr{O}(U_\alpha)$, where $U_\alpha = \{[z] \in \mathbb{C}P^n | \, z_\alpha \neq 0\}$. These f_α's satisfy the condition

$$f_\alpha([z]) = \left(\frac{z_\beta}{z_\alpha}\right)^k f_\beta([z])$$

on $U_\alpha \cap U_\beta$.

[3] The reason for this name should be clear after we find out what are the zero locus of its holomorphic sections.

Pulling back to $\mathbb{C}^{n+1} \setminus \{0\}$, we can view $z_\alpha^k f_\alpha([z])$ as a homogeneous function of degree k on $\mathbb{C}^{n+1} \setminus \{z_\alpha = 0\}$, which is also holomorphic. Now the above compatibility condition means that these $z_\alpha f_\alpha([z])$'s could be "glued" together to form a holomorphic function on $\mathbb{C}^{n+1} \setminus \{0\}$, homogeneous of degree k. By Hartogs extension theorem 1.15, this function extends to a holomorphic function $F(z_0, \ldots, z_n) \in \mathscr{O}(\mathbb{C}^{n+1})$. We necessarily have $F(0) = 0$ by homogeneity and continuity. From this we easily conclude that F is a homogeneous polynomial of degree k.

On the other hand, it is easy to see that any homogeneous polynomial of degree k in $\mathbb{C}[z_0, \ldots, z_n]$ determines uniquely a holomorphic section of H^k. So we have

$$\dim_{\mathbb{C}} \Gamma(\mathbb{C}P^n, \mathscr{O}(H^k)) = \binom{n+k}{n}.$$

Exercise: Prove that when $k < 0$, $\Gamma(\mathbb{C}P^n, \mathscr{O}(H^k)) = \{0\}$.

Definition 2.21. *The isomorphic classes of holomorphic line bundles over X is called the "Picard group" of X, denoted by $Pic(X)$.*

$Pic(X)$ is indeed a group: we define $[L_i] \cdot [L_2] := [L_1 \otimes L_2]$, then $\underline{\mathbb{C}}$ is the identity element and $[L]^{-1}$ is just $[L^*]$.

For $\mathbb{C}P^n$, we have $Pic(\mathbb{C}P^n) \cong \mathbb{Z}$, and any holomorphic line bundle is isomorphic to $\mathscr{O}(k)$ for some $k \in \mathbb{Z}$. However, this is rather deep, and we can not prove it here. One can find a proof in Chapter 1 of [2].

Wedge product

Let E be vector bundles over X of rank r, for $k \in \mathbb{N}$ and $k \leq r$, the degree k wedge product of E is a vector bundle $\Lambda^k E$ with fiber $\Lambda^k E_p$ at p. The highest degree wedge product $\Lambda^r E$ is also called the "determinant line bundle" of E, since its transition functions are precisely $\det \psi_{ij}$.

$\Omega^p(X) := \Lambda^p T^* X$ is the bundle of holomorphic p-forms.

Pull back via holomorphic map

Let $E \to X$ be a holomorphic vector bundle of rank r, $f : Y \to X$ be a holomorphic map between complex manifolds, then we can define a "pull back" holomorphic vector f^*E over Y. In fact, we can simply define the total space of f^*E to be

$$f^*E := \{(y, (x, v)) \in Y \times E |\ x = f(y)\},$$

and $p : f^*E \to Y$ is just the projection to its first component.

We can also describe f^*E via transition maps: if $\{U_i\}_{i \in \Lambda}$ is a trivializing covering of X for E with transition maps $\psi_{ij} : U_i \cap U_j \to GL(r, \mathbb{C})$, and we choose an open covering $\{V_\alpha\}_{\alpha \in I}$ such that $f(V_\alpha) \subset U_i$ for some $i \in \Lambda$. We fix a map $\tau : I \to \Lambda$ such that $f(V_\alpha) \subset U_{\tau(\alpha)}$. Then the transition maps for f^*E with respect to $\{V_\alpha\}_{\alpha \in I}$ are just $f^*\psi_{\tau(\alpha)\tau(\beta)} = \psi_{\tau(\alpha)\tau(\beta)} \circ f : V_\alpha \cap V_\beta \to GL(r, \mathbb{C})$.

2.3 Almost complex manifolds

The definition of a n-dimensional differential manifold is similar to that of complex manifolds. Just replace every $\mathbb{C}^n$ by $\mathbb{R}^n$ and every "holomorphic" by "smooth" or C^∞. Similar for differential vector bundles over a differential manifold. A differential manifold is called orientable, if we can find a coordinate covering such that whenever two coordinate charts intersect, the Jacobian determinant of the coordinate transform is positive.

Lemma 2.22. *A n-dimensional complex manifold X is also a $2n$-dimensional orientable differential manifold.*

This follows from the computation we did before in the proof of Theorem 1.22. Here if we have a holomorphic coordinate chart (U, φ) with $\varphi = (z_i, \ldots, z_n)$, then the corresponding chart to define the oriented differential structure is $(x_1, \ldots, x_n, y_1, \ldots, y_n)$.

For $p \in X$, we can define a real tangent vector at p and the corresponding real tangent space at p, $T_p^{\mathbb{R}}X$. In terms of coordinate chart $\varphi = (z_1, \ldots, z_n)$, we have

$$T_p^{\mathbb{R}}X = \mathbb{R} < \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} >_{i=1}^n .$$

We can give $\coprod_{p\in X} T_p^{\mathbb{R}}X$ a structure of $\mathbb{R}$-vector bundle of rank $2n$, called the "real tangent bundle" of X, and denoted by $T^{\mathbb{R}}X$. Similarly, we can define the real cotangent bundle $T^{*\mathbb{R}}X$.

There are two ways to get from this our previous holomorphic tangent and cotangent bundles.

Recall that any real vector space V of dimension $2n$ can be regarded as $\mathbb{C}$-vector space of dimension n once we know what does it mean to multiply $\sqrt{-1}$ to an element of V. This is equivalent to giving a $\mathbb{R}$-linear map $J : V \to V$ such that $J^2 := J \circ J = -id$. We call such a J a "complex structure" on V. In this case, V can be regarded as a $\mathbb{C}$-vector space by defining

$$(\alpha + \sqrt{-1}\beta)v := \alpha v + \beta J v, \quad \forall \alpha, \beta \in \mathbb{R}, \forall v \in V.$$

Definition 2.23. *Let M be a real orientable differential manifold of dimension $2n$. An almost complex structure on M is a bundle map $J : TM \to TM$ satisfying $J^2 = -id$.*

Note that a complex manifold X has a natural almost complex structure: just define

$$J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}, \quad J\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}.$$

If an almost complex structure is induced from a complex structure as above, we will call it "integrable".

Example 2.24. *For S^2, we can define $J : TS^2 \to TS^2$ as follows: we identify T_xS^2 with the subspace of $\mathbb{R}^3$:*

$$T_xS^2 \cong \{y \in R^3 | \, x \cdot y = 0\}.$$

Then we define $J_x : T_xS^2 \to T_xS^2$ by

$$J_x(y) := x \times y.$$

On can check that this is an integrable almost complex structure, induced by the complex structure of $S^2 \cong \mathbb{C}P^1$.

Example 2.25. *For S^6, we have a similar almost complex structure given by "wedge product" in $\mathbb{R}^7$. Note that the wedge product in $\mathbb{R}^3$ can be defined as the product of purely imaginary quaternions. To define this wedge product in $\mathbb{R}^7$, we shall use Cayley's theory of octonions.*

We write $\mathbb{H} \cong \mathbb{R}^4$ the space of quaternions $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ with $a, b, c, d \in \mathbb{R}$, satisfying $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, and $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. Then this multiplication is still associative but not commutative. For $q \in \mathbb{H}$, we define $\bar{q} := a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$, then $|q|^2 = q\bar{q}$.

Now we define the space of octonians, $\mathbb{O} \cong \mathbb{R}^8$, as $\mathbb{O} := \{x = (q_1, q_2) | \, q_1, q_2 \in \mathbb{H}\}$. The multiplication is defined by

$$(q_1, q_2)(q_1', q_2') := (q_1q_1' - \bar{q}_2'q_2, q_2'q_1 + q_2\bar{q}_1').$$

And we also define $\bar{x} := (\bar{q}_1, -q_2)$. Then we still have $x\bar{x} = x \cdot x = |x|^2$, here the $\cdot$ means the usual inner product in $\mathbb{R}^8$. Note that this multiplication is even not associative.

We identify $\mathbb{R}^7$ as the space of purely imaginary octonians. If $x, x' \in \mathbb{R}^7$, we define $x \times x'$ as the imaginary part of xx'. Then one can check that $xx = -|x|^2$, $x \times x' = -x' \times x$, and $(x \times x') \cdot x'' = x \cdot (x' \times x'')$.

From this, one can define an almost complex structure on $S^6 \subset \mathbb{R}^7$ in a similar way as S^2: identify T_xS^6 with $\{y \in \mathbb{R}^7 | \, x \cdot y = 0\}$, then define

$$J_x(y) := x \times y.$$

Remark 2.26. *For spheres of even dimension $2n$, it is known (Borel-Serre) that there are no almost complex structures unless $n = 1, 3$. A modern proof of this fact using characteristic classes can be found in P. May's book on algebraic topology. It is generally believed that there are no integrable almost complex structures on S^6, however S.T. Yau has a different conjecture saying that one can make S^6 into a complex manifold. This is still open.*

Now given $J : T^{\mathbb{R}}X \to T^{\mathbb{R}}X$, we can view $T^{\mathbb{R}}X$ as a $\mathbb{C}$-vector bundle. One can check that, when X is a complex manifold, $(T^{\mathbb{R}}X, J)$ is isomorphic to the holomorphic tangent bundle TX as $\mathbb{C}$-vector bundles. This is the first approach.

The second approach also uses J. Let again V be a real vector space with complex structure J. But now we simply complexify V to get

$$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}.$$

We also extend J $\mathbb{C}$-linearly to $V_{\mathbb{C}}$, again $J^2 = -id$.

There is a direct sum decomposition of $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, which are $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of J respectively. In fact we have a very precise description of $V^{1,0}$ and $V^{0,1}$:

$$V^{1,0} = \{v - \sqrt{-1}Jv |\ v \in V\}, \quad V^{0,1} = \{v + \sqrt{-1}Jv |\ v \in V\}.$$

It is direct to check that they are both $\mathbb{C}$-linear subspaces of $V_{\mathbb{C}}$ and $V^{0,1} = \overline{V^{1,0}}$.

Now apply this to $(T^{\mathbb{R}}X, J)$ for a manifold with an almost complex structure: define the complexified tangent bundle to be

$$T^{\mathbb{C}}X := T^{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C}$$

and we have the decomposition

$$T^{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X,$$

which are the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of J, respectively. When J is integrable, $T^{1,0}X$ is locally generated by $\{\frac{\partial}{\partial z_i}\}_{i=1}^n$, so we can again identify it with our previous holomorphic vector bundle TX.

We define $T^{*1,0}X$ to be the subspace of $T^{*\mathbb{C}}X := T^{*\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C}$ that annihilates $T^{0,1}X$. And similarly define $T^{*0,1}X$. Then

$$T^{*\mathbb{C}}X = T^{*1,0}X \oplus T^{*0,1}X.$$

When J is integrable, $T^{*1,0}X$ is locally generated by $\{dz_i\}_{1\leq i\leq n}$ and $T^{*0,1}X$ is generated by $\{d\bar{z}_i\}_{1\leq i\leq n}$. We define the vector bundle $\Lambda^{p,q}T^*X$, the bundle of (p,q)-forms to be the sub-bundle of $\Lambda^{p+q}T^{*\mathbb{C}}X$, generated by $\Lambda^p T^{*1,0}X$ and $\Lambda^q T^{*0,1}X$. Then we have

$$\Lambda^k T^{*\mathbb{C}}X = \bigoplus_{p=0}^{k} \Lambda^{p,k-p}T^*X,$$

and we denote the projection map of $\Lambda^{p+q}T^{*\mathbb{C}}X$ onto $\Lambda^{p,q}T^*X$ by $\Pi_{p,q}$. The set of smooth sections of $\Lambda^{p,q}T^*X$ over an open set U is denoted by $A^{p,q}(U)$, while the set of smooth sections of $\Lambda^k T^{*\mathbb{C}}X$ is denoted by $A^k(U)$.

When J is integrable, a smooth section of $\Lambda^{p,q}T^*X$ over a coordinate open set U is of the forms

$$\sum_{1\leq i_1<\cdots<i_p\leq n, 1\leq j_1<\cdots<j_q\leq n} a_{i_1\ldots i_p,\bar{j}_1\ldots\bar{j}_q} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q},$$

where $a_{i_1\ldots i_p,\bar{j}_1\ldots\bar{j}_q} \in C^\infty(U;\mathbb{C})$.

The exterior differential operator d extends $\mathbb{C}$-linearly to $d : A^k(U) \to A^{k+1}(U)$. We define the operators

$$\partial := \Pi_{p+1,q} \circ d : A^{p,q}(U) \to A^{p+1,q}(U),$$

and

$$\bar{\partial} := \Pi_{p,q+1} \circ d : A^{p,q}(U) \to A^{p,q+1}(U).$$

When J is integrable, then for $\eta = \sum_{|I|=p,|J|=q} a_{I\bar{J}} dz_I \wedge d\bar{z}_J \in A^{p,q}(U)$, we have

$$\begin{aligned} d\eta &= \sum_{I,J} da_{I\bar{J}} \wedge dz_I \wedge d\bar{z}_J \\ &= \sum_{I,J} \partial a_{I\bar{J}} \wedge dz_I \wedge d\bar{z}_J + \sum_{I,J} \bar{\partial} a_{I\bar{J}} \wedge dz_I \wedge d\bar{z}_J \in A^{p+1,q}(U) \oplus A^{p,q+1}(U). \end{aligned}$$

So we always have $d = \partial + \bar{\partial}$. Conversely, we have:

Theorem 2.27 (Newlander-Nirenberg)**.** *An almost complex structure is integrable if and only if $d = \partial + \bar{\partial}$ (equivalently, $[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$) for any $A^{p,q}(U)$.*

Besides the original proof of Newlander-Nirenberg, there is another proof by J.J. Kohn based on techniques for solving the "$\bar{\partial}$-equation", which can be found in Hörmander's book.

2.4 De Rham cohomology and Dolbeault cohomology

In the following, we always assume the almost complex structure J is integrable, i.e., X is a complex manifold.

Now $d = \partial + \bar{\partial}$. Since we always have $d^2 = 0$, a fact first noticed by Poincaré, we have

$$0 = \partial^2 + \bar{\partial}^2 + (\partial\bar{\partial} + \bar{\partial}\partial),$$

acting on $A^{p,q}(X)$. Comparing types, we get

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

We can define from these identities several differential cochain complexes:
The de Rham complex

$$0 \to A^0(X) \xrightarrow{d} A^1(X) \xrightarrow{d} \dots \xrightarrow{d} A^{2n}(X) \to 0$$

We define the *de Rham cohomology* (with coefficient $\mathbb{C}$)

$$H^k_{dR}(X, \mathbb{C}) := Ker\Big(A^k(X) \xrightarrow{d} A^{k+1}(X)\Big)/dA^{k-1}(X).$$

The Dolbeault complex

$$0 \to A^{p,0}(X) \xrightarrow{\bar{\partial}} A^{p,1}(X) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} A^{p,n}(X) \to 0.$$

We define the *Dolbeault cohomology*

$$H^{p,q}_{\bar{\partial}}(X) := Ker\Big(A^{p,q}(X) \xrightarrow{\bar{\partial}} A^{p,q+1}(X)\Big)/\bar{\partial}A^{p,q-1}(X).$$

The holomorphic de Rham complex

$$0 \to \Omega^0(X) \xrightarrow{d=\partial} \Omega^1(X) \xrightarrow{d=\partial} \dots \xrightarrow{d=\partial} \Omega^n(X) \to 0$$

We define the *holomorphic de Rham cohomology*

$$H^k_{dR}(X, hol) := Ker\Big(\Omega^k(X) \xrightarrow{d} \Omega^{k+1}(X)\Big)/d\Omega^{k-1}(X).$$

The relation between these cohomology theories, as well as computational tools will be discussed when we finish sheaf cohomology theory and Hodge theorem.

3 A brief introduction to sheaf theory

3.1 Basic concepts in sheaf theory

Recall that a presheaf $\mathscr{F}$ of abelian groups over a topological space X is a rule assigning an abelian group $\mathscr{F}(U)$ for each open set $U \subset X$, and for each pair $V \subset U$ a homomorphism $r^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$ (called "restriction homomorphism"), satisfying $r^U_U = id$ and for any $W \subset V \subset U$, we have $r^U_W = r^V_W \circ r^U_V$. An element of $\mathscr{F}(U)$ is usually called a "section" of $\mathscr{F}$ over U. We also defined the stalk of $\mathscr{F}$ at a point $p \in X$ to be

$$\mathscr{F}_p := \varinjlim \mathscr{F}(U),$$

where the direct limit is taken with respect to open sets $p \in U$. This is $\coprod_{U \ni p} \mathscr{F}(U)/\sim$, with $s \in \mathscr{F}(U)$ equivalent to $t \in \mathscr{F}(V)$ iff we can find another open set $p \in W \subset U \cap V$ such that $r^U_W(s) = r^V_W(t)$.

By a morphism f between two presheaves $\mathscr{F}$ and $\mathscr{G}$ over X, we mean for each U open, we are given a homomorphism of abelian groups $f_U : \mathscr{F}(U) \to \mathscr{G}(U)$, such that whenever we have open sets $V \subset U$, we have a commutative diagram:

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{f_U} & \mathscr{G}(U) \\ {\scriptstyle r^U_V}\downarrow & & \downarrow{\scriptstyle \rho^U_V} \\ \mathscr{F}(V) & \xrightarrow[f_V]{} & \mathscr{G}(V). \end{array}$$

Definition 3.1. *A presheaf of abelian groups $\mathscr{F}$ over X is called a sheaf, if it satisfies the following two properties:*

(S1) Assume we have a family of open sets $U_i \subset U, i \in \Lambda$ and $\cup_i U_i = U$. If $s \in \mathscr{F}(U)$ satisfies $r^U_{U_i}(s) = 0, \forall i \in \Lambda$, then $s = 0$.

(S2) Assume we have a family of open sets $U_i \subset U, i \in \Lambda$ and $\cup_i U_i = U$. If we also have a family of sections $s_i \in \mathscr{F}(U_i), \forall i \in \Lambda$, satisfying $r^{U_i}_{U_i \cap U_j}(s_i) = r^{U_j}_{U_i \cap U_j}(s_i)$ whenever $U_i \cap U_j \neq \emptyset$, then there is a section $s \in \mathscr{F}(U)$ such that $r^U_{U_i}(s) = s_i, \forall i \in \Lambda$.

A morphism between two sheaves is just a morphism between presheaves.

Note that by (S1), the section in (S2) is also unique.

Example 3.2. *Let X be a complex manifold, then $\mathscr{O}_X$ is a sheaf of commutative rings over X. We call it the "structure sheaf" of X.*

We can also define other sheaves on X. For example, define $\mathcal{E}(U) := C^\infty(U; \mathbb{C})$, then it is easy to see that $\mathcal{E}$ is a sheaf, called the "sheaf of smooth functions". Similarly, we can define the sheaf of continuous functions on X.

If $E \to X$ is a holomorphic vector bundle, then $\mathscr{O}(E)(U)$ defines a sheaf of abelian groups. It can also be viewed as a sheaf of $\mathscr{O}_X$-modules.

Example 3.3. *For $X = \mathbb{C}$, if we define $\mathscr{O}_b(U)$ to be the set of bounded holomorphic functions on $U \subset X$, then $\mathscr{O}_b$ is a presheaf over $\mathbb{C}$, but not a sheaf.*

Example 3.4. *Let G be a given abelian group, we define the constant presheaf over X to be $\underline{G}_{pre}(U) := G$ for any non-empty open set $U \subset X$, and $r^U_V = id$ for any non-empty pair $V \subset U$. Then it is in general not a sheaf.*

Example 3.5. *Let $\pi : Y \to X$ be a continuous surjective map between topological spaces. We define the sheaf of continuous sections of π as follows: for any open $U \subset X$, define $C_\pi(U) := \{\sigma : U \to Y | \, \pi \circ \sigma = id_U\}$. Then it is a sheaf* **of sets** *over X.*[4] *This example is in fact very general.*

Proposition 3.6. *For any presheaf $\mathscr{F}$ over X, there is a unique (up to isomorphism) sheaf $\mathscr{F}^+$ and a morphism $\theta : \mathscr{F} \to \mathscr{F}^+$ satisfying the following "universal property": for any sheaf $\mathscr{G}$ over X and any morphism of presheaves $f : \mathscr{F} \to \mathscr{G}$, there is a unique morphism of sheaves $f^+ : \mathscr{F}^+ \to \mathscr{G}$ such that $f = f^+ \circ \theta$.*

If $\mathscr{F}$ is already a sheaf, then θ is an isomorphism. $\mathscr{F}^+$ is called the "sheafification" of $\mathscr{F}$.

Outline of proof. I just outline one way of proof. From $\mathscr{F}$, we define a topological space, called the "étalé space" associated to $\mathscr{F}$:

$$\tilde{\mathscr{F}} := \coprod_{p \in X} \mathscr{F}_p.$$

We have a natural surjective projection map $\pi : \tilde{\mathscr{F}} \to X$. The topology on $\tilde{\mathscr{F}}$ is given as follows: If $s \in \mathscr{F}(U)$, then we have a natural map $\tilde{s} : U \to \tilde{\mathscr{F}}$, sending p to the germ of s at p, which is an element of $\mathscr{F}_p$. Then we require $\{\tilde{s}(U) | \, s \in \mathscr{F}(U), \forall U\}$ to be a topological basis for $\tilde{\mathscr{F}}$.

Now we can use the construction of Example3.5 to get a sheaf $\mathscr{F}^+$. The morphism θ is defined by $\theta_U : \mathscr{F}(U) \to \mathscr{F}^+(U)$, $\theta_U(s) := \tilde{s}$. □

Exercise: Check that we have the following concrete description of $\mathscr{F}^+$: a map $\tilde{s} : U \to \coprod_{p \in U} \mathscr{F}_p$ is an element of $\mathscr{F}^+(U)$ if and only if:

1. $\pi \circ \tilde{s} = id_U$;
2. For any $p \in U$, there is an open neighborhood $p \in V \subset U$ and a $s \in \mathscr{F}(V)$ such that for any $q \in V$, $\tilde{s}(q)$ equals the germ of s at q.

[4] For the general definition of presheaves and sheaves of sets, one need to generalize our previous definition properly. For example, all "homomorphisms between Abelian groups" need to be replaced by "maps between sets". The last sentence of "sheaf axiom" (S1) should read "For any section $s, t \in \mathscr{F}(U)$, if $r^U_{U_i}(s) = r^U_{U_i}(t)$ for any i, then $s = t$."

3.2 Sheaf cohomology (Čech's theory)

Sheaf is a useful tool to describe the obstructions to solve global problems when we can always solve a local one.

To illustrate this point, we come back to the Mittag-Leffler problem on a Riemann surface M. Suppose we are given finitely many points $p_1, \ldots, p_m \in M$, and for each p_i we are given a Laurant polynomial $\sum_{k=1}^{n_i} \frac{c_k^{(i)}}{z^k}$. We can view this as an element of $\mathscr{M}_p/\mathscr{O}_p$. We want to find a meromorphic function on M whose poles are precisely those p_i's with the given Laurant polynomial as its principal part at p_i.

This problem is always solvable locally: we can find a locally finite open covering $\mathcal{U} = \{U_i | \, i \in \Lambda\}$ of M such that each U_i contains at most one of the p_i's, and $f_i \in \mathscr{M}(U_i)$ such that the only poles of f_i are those of $\{p_i\}$ contained in U_i with principal part equals the given Laurent polynomial. The problem is that we can not patch them together: if $U_i \cap U_j$, there is no reason to have $f_i = f_j$. We have to define $f_{ij} := f_i - f_j$ and view the totality of these f_{ij}'s as the obstruction to solve the problem. Now by our choice of f_i, $f_{ij} \in \mathscr{O}(U_i \cap U_j)$. Note that we have $f_{ij} + f_{ji} = 0$ on $U_i \cap U_j$ and whenever $U_i \cap U_j \cap U_k \neq \emptyset$, we have on $U_i \cap U_j \cap U_k$: $f_{ij} + f_{jk} + f_{ki} = 0$. We call this the "cocycle" condition and $\{f_{ij}\}$ is a "Čech cocycle" for the sheaf $\mathscr{O}$ with respect to the cover $\mathcal{U}$.

When can we solve the Mittag-Leffler problem on M? We can solve it if we can modify the f_i by a holomorphic function $h_i \in \mathscr{O}(U_i)$ such that $\tilde{f}_i := f_i - h_i$ with patch together. This means that $f_{ij} = h_i - h_j$ on $U_i \cap U_j$. We call a cocycle of the form $\{h_i - h_j\}$ where each h_i is holomorphic a Čech coboundary. We get the conclusion that we can solve the Mittag-Leffler problem if the Čech cocycle $\{f_{ij}\}$ is a coboundary.

This motivates the introduction of the following Čech cohomology of a sheaf $\mathscr{F}$ with respect to a locally finite cover $\mathcal{U}$ of X: We first define the chain groups:

$$\begin{aligned}
&C^0(\mathcal{U}, \mathscr{F}) := \Pi_{i\in\Lambda} \mathscr{F}(U_i) \\
&C^1(\mathcal{U}, \mathscr{F}) \subset \Pi_{(i,j)\in\Lambda^2} \mathscr{F}(U_i \cap U_j) \\
&\qquad\qquad \ldots \\
&C^p(\mathcal{U}, \mathscr{F}) \subset \Pi_{(i_0,i_1,\ldots,i_p)\in\Lambda^{p+1}} \mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_p}) \\
&\qquad\qquad \ldots
\end{aligned}$$

where $\{\sigma_{i_0,\ldots,i_p}\}$ is in $C^p(\mathcal{U}, \mathscr{F})$ if and only if:

(1) Whenever $i_k = i_l$ for some $k \neq l$, we have $\sigma_{i_0,\ldots,i_p} = 0$;

(2) For any permutation $\tau \in S_{p+1}$, we have $\sigma_{i_{\tau(0)},\ldots,i_{\tau(p)}} = (-1)^\tau \sigma_{i_0,\ldots,i_p}$.

Note that we always define $\mathscr{F}(U) = \{0\}$ if $U = \emptyset$.

We define the coboundary operator $\delta : C^p(\mathcal{U}, \mathscr{F}) \to C^{p+1}(\mathcal{U}, \mathscr{F})$ to be:

$$(\delta\sigma)_{i_0,\ldots,i_{p+1}} := \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0,\ldots,\hat{i}_j,\ldots,i_{p+1}}|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}}.$$

Here we use $\ldots|_{\ldots}$ to denote the restriction homomorphism of $\mathscr{F}$. It is direct to check that $\delta \circ \delta = 0$. So we have a chain complex

$$0 \to C^0(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} \ldots \xrightarrow{\delta} C^p(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} \ldots$$

We can define

$$Z^p(\mathcal{U}, \mathscr{F}) = Ker\delta \subset C^p(\mathcal{U}, \mathscr{F}),$$

whose elements are called Čech p-cocycles. Also define

$$B^p(\mathcal{U}, \mathscr{F}) = \delta C^{p-1}(\mathcal{U}, \mathscr{F}) \subset Z^p(\mathcal{U}, \mathscr{F}),$$

whose elements are called Čech p-coboundaries. Then we define the Čech cohomology of $\mathscr{F}$ with respect to $\mathcal{U}$:

$$H^p(\mathcal{U}, \mathscr{F}) := Z^p(\mathcal{U}, \mathscr{F})/B^p(\mathcal{U}, \mathscr{F}).$$

For example, an element of $H^0(\mathcal{U}, \mathscr{F})$ is given by a family of sections $f_i \in \mathscr{F}(U_i)$ such that $\delta\{f_i\} = 0$. This means precisely

$$r^{U_i}_{U_i \cap U_j}(f_i) = r^{U_j}_{U_i \cap U_j}(f_j)$$

whenever $U_i \cap U_j \neq \emptyset$. By sheaf axiom (S2), we get a global section of $\mathscr{F}$ over X. So $H^0(\mathcal{U}, \mathscr{F})$ is in fact independent of $\mathcal{U}$ and we have a canonical isomorphism

$$H^0(\mathcal{U}, \mathscr{F}) \cong \mathscr{F}(X).$$

When $p = 1$, $\{f_{ij}\} \in C^p(\mathcal{U}, \mathscr{F})$ is a cocycle if $f_{ij} + f_{ji} = 0$ and $f_{jk} - f_{ik} + f_{ij} = f_{ij} + f_{jk} + f_{ki} = 0$. This is precisely the "cocycle condition" we met before. However, this time the cohomology may depend on the cover.

Let $\mathcal{V} = \{V_\alpha\}_{\alpha\in\Gamma}$ be a locally finite refinement of $\mathcal{U}$. This means we have a map $\tau : \Gamma \to \Lambda$ (not unique) such that $V_\alpha \subset U_{\tau(\alpha)}$. Then we have a homomorphism $\Phi^{\mathcal{U}}_{\mathcal{V}} : H^p(\mathcal{U}, \mathscr{F}) \to H^p(\mathcal{V}, \mathscr{F})$ induced by

$$\{\sigma_{i_0,\ldots,i_p}\} \mapsto \{\sigma_{\tau(\alpha_0),\ldots,\tau(\alpha_p)}|_{V_{\alpha_0}\cap\cdots\cap V_{\alpha_p}}\}.$$

One can prove that $\Phi^{\mathcal{U}}_{\mathcal{V}}$ is in fact independent of the choice of the map τ. Then the cohomology of X with coefficients sheaf $\mathscr{F}$ is defined to be the direct limit:

$$H^p(X, \mathscr{F}) := \varinjlim H^p(\mathcal{U}, \mathscr{F}) = \coprod_{\mathcal{U}} H^p(\mathcal{U}, \mathscr{F})/\sim$$

where two cohomology classes $[\{\sigma_{i_0,\ldots,i_p}\}] \in H^p(\mathcal{U}, \mathscr{F})$ and $[\{\eta_{j_0,\ldots,j_p}\}] \in H^p(\mathcal{V}, \mathscr{F})$ are equivalent if we can find a common refinement $\mathcal{W}$ of $\mathcal{U}, \mathcal{V}$ such that

$$\Phi^{\mathcal{U}}_{\mathcal{W}}([\{\sigma_{i_0,\ldots,i_p}\}]) = \Phi^{\mathcal{V}}_{\mathcal{W}}([\{\eta_{j_0,\ldots,j_p}\}]).$$

Thus an element of $H^p(X, \mathscr{F})$ is an equivalent class of Čech cohomology classes, represented by an element of $H^p(\mathcal{U}, \mathscr{F})$, for some cover $\mathcal{U}$. But in many cases, in particular all the sheaves we use in this course, there exists sufficiently fine cover $\mathcal{U}$ such that $H^p(\mathcal{U}, \mathscr{F}) \cong H^p(X, \mathscr{F})$.

3.3 Useful results for sheaf cohomology

We present two useful results for sheaf cohomology. In many cases, it is safe to know only these results and forget the definition details.

Recall that a morphism $f : \mathscr{F} \to \mathscr{G}$ of sheaves over X induces for each point $p \in X$ a homomorphism of stalks: $f_p : \mathscr{F}_p \to \mathscr{G}_p$. We call a sequence of morphisms of sheaves an "exact sequence" if the induced sequence on stalks is so for each pint p.

The first result saying that a short exact sequence for morphisms of sheaves gives rise to a long exact sequence for sheaf cohomology:

Theorem 3.7. *If we have a short exact sequence for sheaves of abelian groups over X*

$$0 \to \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \to 0,$$

then we have a long exact sequence for cohomologies

$$\begin{aligned} 0 \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{G}) \to H^0(X, \mathscr{H}) \to H^1(X, \mathscr{F}) \to \dots \\ \dots \to H^p(X, \mathscr{H}) \to H^{p+1}(X, \mathscr{F}) \to H^{p+1}(X, \mathscr{G}) \to \dots \end{aligned}$$

We won't prove this, but will explain the meaning of this theorem.

For the given short exact sequence, we always get an exact sequence

$$0 \to \mathscr{F}(X) \to \mathscr{G}(X) \to \mathscr{H}(X),$$

(**Exercise:** Show that for any open set U, the sequence $0 \to \mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{H}(U)$ is always exact.)but the last homomorphism is in general not surjective. Let's explain why. Given an element $\sigma \in \mathscr{H}(X)$, we'd like to know whether we can find $\eta \in \mathscr{G}(X)$ such that $g_X(\eta) = \sigma$. But we already know that $0 \to \mathscr{F}_p \xrightarrow{f_p} \mathscr{G}_p \xrightarrow{g_p} \mathscr{H}_p \to 0$ is exact, so we can always find a germ $\eta_p \in \mathscr{G}_p$ such that $g_p(\eta_p) = \sigma_p$. This actually means that we can find a cover $\mathcal{U} = \{U_i\}$ of X and a sequence $\eta_i \in \mathscr{G}(U_i)$ such that $g_{U_i}(\eta_i) = \sigma|_{U_i}$. If all the $\eta_{ij} := \eta_j - \eta_i = 0$ on $U_i \cap U_j$, then we can patch these η_i's together, then we finish the problem. We'd like to modify η_i. Note that since $g_{U_i \cap U_j}(\eta_{ij}) = 0$, we can find $\mu_{ij} \in \mathscr{F}(U_i \cap U_j)$ such that $f_{U_i \cap U_j}(\mu_{ij}) = \eta_{ij}$. By the injectivity of f, we in fact get a cocycle $\{\mu_{ij}\} \in C^1(\mathcal{U}, \mathscr{F})$. So we get a homomorphism $\mathscr{H}(X) \to H^1(X, \mathscr{F})$. It is fairly easy to check that if σ goes to 0 in $H^1(X, \mathscr{F})$, then we can modify η_i properly (on a refinement of $\mathcal{U}$) such that they patch together to get an element of $\mathscr{G}(X)$.

A corollary of Theorem3.7 is the following "abstract de Rham theorem":

Theorem 3.8. *Suppose we have an exact sequence of the form:*

$$0 \to \mathscr{F} \to \mathscr{S}_0 \to \mathscr{S}_1 \to \cdots \to \mathscr{S}_r \to \dots$$

where each $\mathscr{S}_r$ satisfies $H^p(X, \mathscr{S}_r) = 0, \forall p \geq 1$. (This is called an "acyclic resolution of $\mathscr{F}$".) Then $H^*(X, \mathscr{F})$ is isomorphic to the cohomology of the chain complex

$$0 \to \mathscr{S}_0(X) \to \mathscr{S}_1(X) \to \cdots \to \mathscr{S}_r(X) \to \dots$$

i.e., $H^*(X, \mathscr{F}) \cong H^*(\Gamma(X, \mathscr{S}^*))$.

Proof. We break the sheaf sequence into a sequence of short exact sequences for $p \geq 1$: $0 \to \mathscr{K}_{p-1} \to \mathscr{S}_{p-1} \to \mathscr{K}_p \to 0$, where $\mathscr{K}_p = Ker(\mathscr{S}_p \to \mathscr{S}_{p+1}) = Im(\mathscr{S}_{p-1} \to \mathscr{S}_p)$. Note that $\mathscr{K}_0 \cong \mathscr{F}$. By the above theorem and the assumption for $\mathscr{S}_p$, we have an exact sequence

$$0 \to \mathscr{K}_{p-1}(X) \to \mathscr{S}_{p-1}(X) \to \mathscr{K}_p(X) \to H^1(X, \mathscr{K}_{p-1}) \to 0.$$

Also note that $\mathscr{K}_p(X) \cong Ker(\mathscr{S}_p(X) \to \mathscr{S}_{p+1}(X))$, so we get

$$H^1(X, \mathscr{K}_{p-1}) \cong Ker(\mathscr{S}_p(X) \to \mathscr{S}_{p+1}(X))/Im(\mathscr{S}_{p-1}(X) \to \mathscr{K}_p(X)) = H^p(\Gamma(X, \mathscr{S}^*)).$$

We need to prove $H^1(X, \mathscr{K}_{p-1}) \cong H^p(X, \mathscr{F}) = H^p(X, \mathscr{K}_0)$. For this, we only need to show for $2 \leq r \leq p$

$$H^{r-1}(X, \mathscr{K}_{p-r+1}) \cong H^r(X, \mathscr{K}_{p-r}).$$

But this again follows from the segment of long exact sequence:

$$\cdots \to H^{r-1}(X, \mathscr{S}_{p-r}) \to H^{r-1}(X, \mathscr{K}_{p-r+1}) \to H^r(X, \mathscr{K}_{p-r}) \to H^r(X, \mathscr{S}_{p-r}) \to \ldots.$$

□

When can we get an acyclic resolution? In particular, how can we find a lot of sheaves $\mathscr{S}_r$ such that $H^p(X, \mathscr{S}_r) = 0, \forall p \geq 1$?

Definition 3.9. *A sheaf $\mathscr{F}$ over X is called a "fine sheaf", if for any locally finite open cover $\mathcal{U} = \{U_i\}$, we can find a family of morphisms $\eta_i : \mathscr{F} \to \mathscr{F}$ such that:*

(1) For each i, $\eta_i(p) : \mathscr{F}_p \to \mathscr{F}_p$ equals 0 for p outside a compact set $W_i \subset U_i$;

(2) $\sum_i \eta_i = id_{\mathscr{F}}$.

It is obvious that in case we can use a smooth function to multiply the sections of $\mathscr{F}$, then a usual partition of unity will make $\mathscr{F}$ a fine sheaf.

Proposition 3.10. *If $\mathscr{F}$ is a fine sheaf, then $H^p(X, \mathscr{F}) = 0, \forall p \geq 1$.*

Proof. For any p-cocycle $\{\sigma_{i_0,\ldots,i_p}\} \in C^p(\mathcal{U}, \mathscr{F})$ for a locally finite cover $\mathcal{U} = \{U_i\}_{i\in\Lambda}$. Let η_i be the above morphisms in the definition. We define a $p-1$ cochain $\{\psi_{i_0,\ldots,i_{p-1}}\}$ as follows:

$$\psi_{i_0,\ldots,i_{p-1}} := \sum_i \eta_i(\sigma_{i,i_0,\ldots,i_{p-1}}).$$

Then (using the fact that $\delta\{\sigma_{\ldots}\} = 0$)

$$\begin{aligned}
(\delta\psi)_{i_0,\ldots,i_p} &= \sum_{j=0}^{p}(-1)^j \psi_{i_0,\ldots,\hat{i_j},\ldots,i_p} \\
&= \sum_j \sum_i (-1)^j \eta_i(\sigma_{i,i_0,\ldots,\hat{i_j},\ldots,i_p}) \\
&= \sum_i \eta_i(\sigma_{i_0,\ldots,i_p}) = \sigma_{i_0,\ldots,i_p}.
\end{aligned}$$

□

3.4 Applications of sheaf cohomology

Cohomology of constant sheaf

Let G be a given abelian group, we can define the constant sheaf $\underline{G}$ over X by $\underline{G}(U) = \{\text{locally constant maps } U \to G\}$, then we usually denote $H^p(X, \underline{G})$ by $H^p(X, G)$. One can show that when X is a manifold, this is isomorphic to the singular cohomology or simplicial cohomology. But we won't prove this. For the isomorphism to simplicial cohomology when $G = \mathbb{Z}$, one can read Chapter 0 of Griffiths-Harris.

Picard group

Recall that when X is a complex manifold, then a holomorphic line bundle can be described by a family of "transition functions" $f_{ij} \in \mathscr{O}^*(U_i \cap U_j)$, satisfying the "cocycle" condition. So any holomorphic line bundle L determines an element of $H^1(X, \mathscr{O}^*)$. And on the other hand, given an element of $H^1(\mathcal{U}, \mathscr{O}^*)$, we can construct a holomorphic line bundle. In fact, one can show that $[\{f_{ij}\}] \in H^1(\mathcal{U}, \mathscr{O}^*)$ and $[\{h_{\alpha\beta}\}] \in H^1(\mathcal{V}, \mathscr{O}^*)$ determines isomorphic line bundles if and only if they define the same class in $H^1(X, \mathscr{O}^*)$. So we can in fact identify $H^1(X, \mathscr{O}^*)$ with the Picard group of X.

de Rham and Dolbeault theorem

We use the de Rham resolution of $\underline{\mathbb{C}}$:

$$0 \to \underline{\mathbb{C}} \to \mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \xrightarrow{d} \ldots \xrightarrow{d} \mathscr{A}^{2n} \to 0$$

to get de Rham isomorphism:

$$H^p(X, \mathbb{C}) \cong H^p_{dR}(X, \mathbb{C}), \quad p = 0, \ldots, 2n.$$

The reason for this to be a resolution is Poincaré's Lemma.

Similarly, we have a Dolbeault-Grothendieck Lemma, which says that a $\bar{\partial}$-closed form is locally $\bar{\partial}$-exact. So we get a fine resolution for any $0 \le p \le n$:

$$0 \to \Omega^p \to \mathscr{A}^{p,0} \xrightarrow{\bar{\partial}} \mathscr{A}^{p,1} \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} \mathscr{A}^{p,n} \to 0,$$

so we get

$$H^q(X, \Omega^p) \cong H^{p,q}_{\bar{\partial}}(X).$$

Also for a holomorphic vector bundle E, we have

$$H^q(X, \Omega^p(E)) \cong H^{p,q}_{\bar{\partial}}(X, E).$$

Divisor and line bundle

We define the sheaf of meromorphic functions $\mathscr{M}$ on X, where X is a compact complex manifold, to be the sheafification of the presheaf $U \mapsto$ quotient field of $\mathscr{O}(U)$. We define $\mathscr{M}^*$ to be the sheaf of meromorphic functions that are not identically 0, and let $\mathscr{O}^*$ be the subsheaf of $\mathscr{M}^*$, consisting of no-where vanishing holomorphic functions. The short exact sequence

$$1 \to \mathscr{O}^* \to \mathscr{M}^* \to \mathscr{M}^*/\mathscr{O}^* \to 1$$

gives us a long exact sequence, starting with

$$\{1\} \to \mathbb{C}^* \to \mathscr{M}^*(X) \to \mathscr{M}^*/\mathscr{O}^*(X) \to H^1(X, \mathscr{O}^*) \to \dots .$$

The global section of $\mathscr{M}^*/\mathscr{O}^*(X)$ can be equivalently described as a finite formal sum $\sum_i a_i D_i$, where $a_i \in \mathbb{Z}$ and D_i is codimension 1 irreducible analytic hypersurface of X. This is called a "divisor". We define the groups of divisor classes by

$$Div(X) := \left(\mathscr{M}^*/\mathscr{O}^*(X)\right)/\mathscr{M}^*(X).$$

Two divisors are called linearly equivalent, if their difference is a divisor of a global meromorphic function.

The map $\mathscr{M}^*/\mathscr{O}^*(X) \to H^1(X, \mathscr{O}^*)$ is given as follows: locally we can cover X by $\{U_i\}$ such that an element of $\mathscr{M}^*/\mathscr{O}^*(X)$ is given by $f_i \in \mathscr{M}^*(U_i)$. Then $g_{ij} := f_i/f_j$ defines a class in $H^1(X, \mathscr{O}^*)$.

<u>First Chern class</u>

A very useful exact sequence is the following

$$0 \to \underline{\mathbb{Z}} \to \mathscr{O} \xrightarrow{\exp(2\pi\sqrt{-1}\cdot)} \mathscr{O}^* \to 1.$$

We get the exact sequence

$$\dots \to H^1(X, \mathscr{O}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to \dots .$$

We call $c_1 : H^1(X, \mathscr{O}^*) \to H^2(X, \mathbb{Z})$ the "first Chern class" map. We shall use differential forms to give another characterization of Chern classes in the next chapter.

4 Differential geometry of vector bundles

4.1 Metrics, connections and curvatures

Definition 4.1. *Let $E \to X$ be a complex vector bundle of rank r over a smooth manifold X. A smooth Hermitian metric on E is an assignment of Hermitian inner products $h_p(\cdot,\cdot) =< \cdot,\cdot >_p$ on each fiber E_p, such that if ξ, η are smooth sections of E over an open set U, then $h(\xi,\eta) \in C^\infty(U;\mathbb{C})$.*

If U is a local triviliazation neighborhood of E via $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$, then we can define r smooth sections of E over U:

$$e_\alpha(p) := \varphi_U^{-1}(p, 0, \ldots, 0, 1, 0 \ldots, 0).$$

Then at any point $p \in U$, $\{e_\alpha(p)\}_{\alpha=1}^r$ is a basis of E_p. We call $\{e_\alpha\}_{\alpha=1}^r$ a local frame of E over U. Note that when E is a holomorphic bundle and (U, φ_U) a holomorphic trivialization, then these e_α's are also holomorphic sections, and we call it a holomorphic frame.

If ξ is a smooth section over U, then we can write in a unique way $\xi = \xi^\alpha e_\alpha$, with $\xi_\alpha \in C^\infty(U;\mathbb{C}), \alpha = 1, \ldots, r$. If we define the (positive definite) Hermitian matrix-valued smooth functions: $h_{\alpha\bar{\beta}} := h(e_\alpha, e_\beta)$, then we have

$$h(\xi,\eta) = h(\xi^\alpha e_\alpha, \eta^\beta e_\beta) = h_{\alpha\bar{\beta}}\xi^\alpha\bar{\eta}^\beta.$$

Sometimes, we also denote the matrix-valued smooth function $(h_{\alpha\bar{\beta}})$ by h. Hopefully this will cause no confusion.

Notation: We shall denote the space of smooth sections of E over U by $C^\infty(U;E)$. When E is a holomorphic bundle, the set of holomorphic sections over U is denoted by $\Gamma(U;E)$ or $\mathscr{O}(E)(U)$.

Definition 4.2. *A connection on a smooth rank r complex vector bundle over a manifold X is a map $D : C^\infty(X;E) \to C^\infty(X, T^{*\mathbb{C}}X \otimes E)$ satisfying :*

1. *D is $\mathbb{C}$-linear;*
2. *(Leibniz rule) $D(f\xi) = df \otimes \xi + fD\xi$, $\forall f \in C^\infty(X;\mathbb{C}), \xi \in C^\infty(X;E)$.*

If $\{e_\alpha\}$ is a local frame, then we can define a family of local smooth 1-forms $\theta_\alpha^\beta \in A^1(U)$ satisfying:

$$De_\alpha = \theta_\alpha^\beta \otimes e_\beta.$$

Sometimes we just write $De_\alpha = \theta_\alpha^\beta e_\beta$ for short. We call these $\{\theta_\alpha^\beta\}$ "connection one-forms". For $\xi = \xi^\alpha e_\alpha \in C^\infty(U;E)$, we then have

$$D\xi = D(\xi^\alpha e_\alpha) = (d\xi^\alpha + \xi^\beta\theta_\beta^\alpha)e_\alpha.$$

Convention: We always regard ξ^α as a column vector, and for θ^α_β we always regard the upper index as line index and the lower index the column index.

So if we identify ξ with its coordinate representation with respect to the frame $\{e_\alpha\}$, then we can write $D\xi = d\xi + \theta\xi$, or $D = d + \theta$. Physicists always use this way to represent a connection.

We can extend the action of D to bundle-valued differential forms. We write $A^k(X, E) := C^\infty(X; \Lambda^k T^{*\mathbb{C}} X \otimes E)$. Then we define $D : A^k(X, E) \to A^{k+1}(X, E)$ by

$$D(\varphi\xi) := (d\varphi)\xi + (-1)^k \varphi \wedge D\xi,$$

where φ is a $\mathbb{C}$-valued k-form and ξ is a smooth section of E.

Definition 4.3. *We define the curvature of D to be* $\Theta := D^2 : A^0(X; E) \to A^2(X, E)$.

If f is a smooth function and $\xi \in A^0(X, E)$, we have

$$\begin{aligned} \Theta(f\xi) &= D(df\xi + fD\xi) \\ &= d(df)\xi - df \wedge D\xi + df \wedge D\xi + fD^2\xi \\ &= f\Theta(\xi). \end{aligned}$$

Locally if we define the 2-forms Θ^β_α by

$$\Theta(e_\alpha) = \Theta^\beta_\alpha e_\beta.$$

Then we have

$$\begin{aligned} \Theta(\xi) &= \Theta(\xi^\alpha e_\alpha) \\ &= \xi^\alpha \Theta(e_\alpha) \\ &= \Theta^\alpha_\beta \xi^\beta e_\alpha. \end{aligned}$$

From this, we conclude that $\Theta \in A^2(X, End(E))$.

We can also represent Θ^α_β in terms of θ^α_β:

$$\begin{aligned} \Theta^\beta_\alpha e_\beta &= D(De_\alpha) = D(\theta^\gamma_\alpha e_\gamma) \\ &= d\theta^\gamma_\alpha e_\gamma - \theta^\gamma_\alpha \wedge De_\gamma \\ &= d\theta^\beta_\alpha e_\beta - \theta^\gamma_\alpha \wedge \theta^\beta_\gamma e_\beta \\ &= (d\theta^\beta_\alpha + \theta^\beta_\gamma \wedge \theta^\gamma_\alpha) e_\beta. \end{aligned}$$

So we get

$$\Theta^\alpha_\beta = d\theta^\alpha_\beta + \theta^\alpha_\gamma \wedge \theta^\gamma_\beta,$$

or $\Theta = d\theta + \theta \wedge \theta$ for short. Note that our sign convention is different from Griffiths-Harris, since they regard the upper index as the column index.

We now study the change of connection forms and curvature forms under the change of frames.

Suppose $\{\tilde{e}_\alpha\}$ is another local frame on U, then we can write $\tilde{e}_\alpha = a_\alpha^\beta e_\beta$, where (a_α^β) is a $GL(r, \mathbb{C})$-valued smooth function on U. (When both frames are local holomorphic frames of a holomorphic bundle, then (a_α^β) is a $GL(r, \mathbb{C})$-valued holomorphic function on U.) The new connection forms and curvature forms are denoted by $\tilde{\theta}$ and $\tilde{\Theta}$. We have

$$\begin{aligned} \tilde{\theta}_\alpha^\gamma \tilde{e}_\gamma = D\tilde{e}_\alpha &= D(a_\alpha^\beta e_\beta) \\ &= da_\alpha^\beta e_\beta + a_\alpha^\beta \theta_\beta^\gamma e_\gamma \\ &= (da_\alpha^\beta + \theta_\gamma^\beta a_\alpha^\gamma) e_\beta. \end{aligned}$$

On the other hand, the left equals

$$\tilde{\theta}_\alpha^\gamma a_\gamma^\beta e_\beta.$$

So we get

$$a\tilde{\theta} = da + \theta a,$$

or

$$\tilde{\theta} = a^{-1} da + a^{-1} \theta a. \tag{4.1}$$

From this, we get

$$\begin{aligned} \tilde{\Theta} &= d\tilde{\theta} + \tilde{\theta} \wedge \tilde{\theta} \\ &= d(a^{-1}da + a^{-1}\theta a) + (a^{-1}da + a^{-1}\theta a) \wedge (a^{-1}da + a^{-1}\theta a) \\ &= -a^{-1}da \wedge a^{-1}da - a^{-1}da \wedge a^{-1}\theta a + a^{-1}d\theta a - a^{-1}\theta \wedge da \\ &\quad + a^{-1}da \wedge a^{-1}da + a^{-1}da \wedge a^{-1}\theta a + a^{-1}\theta \wedge da + a^{-1}\theta \wedge \theta a \\ &= a^{-1}(d\theta + \theta \wedge \theta)a. \end{aligned}$$

So we conclude

$$\tilde{\Theta} = a^{-1} \Theta a. \tag{4.2}$$

From this, we can construct a family of globally defined differential forms:

$$\det\Big(I_r + \frac{\sqrt{-1}}{2\pi}\Theta\Big) := 1 + c_1(E, D) + \cdots + c_r(E, D),$$

where $c_k(E, D) \in A^{2k}(X)$ is called the "k-th" Chern form of E associated to the connection D.

In physicists' language, a connection is a "field", the curvature is the "strength" of the field, and choosing a local frame is called "fixing the gauge". The reason for these names comes from H. Weyl's work, rewriting Maxwell's equations. The "vector potential" and "scalar potential" together form the connection 1-form, and the curvature 2-form has 6 components, consisting the components of the electric field and the magnetic field.

4.2 Chern connection on holomorphic vector bundles

In general, there is no "canonical connections" on a given vector bundle with a smooth Hermitian metric. However, if the bundle is a holomorphic vector bundle, there is indeed a canonical connection, called the "Chern connection":

Theorem 4.4. *On a given holomorphic vector bundle E with a smooth Hermitian metric h, there is a unique connection D, called the "Chern connection" satisfying the following two additional conditions:*

1. **(Compatibility with the metric)** *If ξ, η are two smooth sections, then we have*
$$dh(\xi, \eta) = h(D\xi, \eta) + h(\xi, D\eta).$$
2. **(Compatibility with the complex structure)** *If ξ is a holomorphic section of E, then $D\xi$ is a E-valued $(1, 0)$-form.*

Proof. We first prove the uniqueness part. Let $\{e_\alpha\}_{\alpha=1}^r$ be a local holomorphic frame, and the connection 1-form with respect to this frame is $(\theta_\alpha^\beta)_{1\le\alpha,\beta\le r}$, satisfying $De_\alpha = \theta_\alpha^\beta e_\beta$. By the compatibility with complex structure, each θ_α^β is a smooth $(1, 0)$-form. Now we use the compatibility with metric to get

$$\begin{aligned} dh_{\alpha\bar\beta} &= h(De_\alpha, e_\beta) + h(e_\alpha, De_\beta) \\ &= \theta_\alpha^\gamma h_{\gamma\bar\beta} + \bar\theta_\beta^\gamma h_{\alpha\bar\gamma}. \end{aligned}$$

On the other hand, we have $dh_{\alpha\bar\beta} = \partial h_{\alpha\bar\beta} + \bar\partial h_{\alpha\bar\beta}$. Comparing types, we get $\partial h = \theta^t h$, so $\theta^t = \partial h \cdot h^{-1}$. Denote $h^{-1} = (h^{\bar\beta\alpha})$, then we can rewrite this as

$$\theta_\alpha^\beta = h^{\bar\nu\beta}\partial h_{\alpha\bar\nu}.$$

Also, since $\bar h^t = h$, the $(0, 1)$-part gives the same equation. This proves the uniqueness.

For existence, we simply set locally $\theta_\alpha^\beta := h^{\bar\nu\beta}\partial h_{\alpha\bar\nu}$, and define for $s = f^\alpha e_\alpha$:

$$Ds := (df^\alpha + f^\beta\theta_\beta^\alpha)e_\alpha.$$

We need to check that this is globally well-defined. For this, if $\tilde e_\alpha = a_\alpha^\beta e_\beta$ is another holomorphic frame on V with $U \cap V \neq \emptyset$. Then a is a holomorphic matrix. We have $\tilde h = a^t h\bar a$, so we have $\tilde\theta := (\tilde h^t)^{-1}\partial\tilde h^t = a^{-1}\partial a + a^{-1}\theta a$. Since $s = \tilde f^\alpha\tilde e_\alpha = f^\alpha e_\alpha$, we have $\tilde f = a^{-1}f$, so

$$\begin{aligned} \tilde e(d\tilde f + \tilde\theta\tilde f) &= ea(-a^{-1}daa^{-1}f + a^{-1}df + a^{-1}\partial aa^{-1}f + a^{-1}\theta aa^{-1}f) \\ &= e(df + \theta f). \end{aligned}$$

So D is globally defined. It is direct to check that D is compatible with both the metric and the complex structure of the bundle. □

It is worth pointing out that the line bundle case is particularly simple: if e is a local holomorphic frame and we set $h = h(e, e) > 0$. Then the connection 1-form is $\theta = h^{-1}\partial h = \partial\log h$. Then the curvature is $\Theta = d\theta + \theta\wedge\theta = d\theta = d\partial\log h = \bar\partial\partial\log h$. It is already a globally defined closed $(1, 1)$-form.

4.3 Chern classes

We give a very elementary introduction to Chern-Weil theory in this section, following Professor Weiping Zhang's book [9].

We first define a trace map $tr : A^k(X, EndE) \to A^k(X)$. For a $EndE$-valued form $\eta \in A^k(X, EndE)$, the trace of η is the k-form $tr(\eta)$ obtained by tracing out the $EndE$ factor. Locally, we can write η as a matrix of k-forms, and $tr(\eta)$ is just the trace of this matrix. Or equivalently, we can write η as $\sum_i \omega_i \otimes A_i$ with ω_i a family of k-forms and A_i a family of local sections of $EndE$, and then $tr(\eta) = \sum_i tr(A_i)\omega_i$.

Another tool we shall use is the (super)-commutator, defined by $[\omega \otimes A, \eta \otimes B] := (\omega \wedge \eta) \otimes [A, B]$, where ω, η are locally defined forms and A, B are local sections of $EndE$. It is easy to see that

$$[\omega \otimes A, \eta \otimes B] = \omega A \wedge \eta B - (-1)^{deg(\omega)deg(\eta)} \eta B \wedge \omega A.$$

The appearance of the extra factor $(-1)^{deg(\omega)deg(\eta)}$ is the reason why sometimes it is called a "super"-commutator. We sometimes extend the definition: we define for the connection D and $\omega \otimes A$: $[D, \omega \otimes A]s := D(\omega \otimes As) - (-1)^{deg(\omega)} \omega \otimes A \wedge Ds$.

We state two useful lemmas, whose proofs are left as exercises.

Lemma 4.5. *If $\tilde{D}$ is another connection on E, then $\tilde{D} - D \in A^1(X, EndE)$.*

Lemma 4.6. *If P, Q are both $EndE$-valued differential forms, then $tr[P, Q] = 0$.*

The first nontrivial lemma is:

Lemma 4.7 (Bianchi identity)**.** *We have $[D, \Theta^k] = 0$, for any $k \in \mathbb{N}$.*

Proof. Simply note that $\Theta = D^2$, so $[D, \Theta^k] = [D, D^{2k}] = 0$. $\square$

Exercise: Check that under local frames $[D, \Theta] = 0$ means $d\Theta = [\Theta, \theta]$.

The next lemma is one of our key tool:

Lemma 4.8. *For $A \in A^k(X, EndE)$, we have*

$$d\, tr(A) = tr[D, A].$$

Proof. First note that the left hand side is obviously independent of the connection. For the right hand side, if we use another connection $\tilde{D}$, by Lemma4.5 and Lemma4.6, we have $tr[\tilde{D}, A] = tr[\tilde{D} - D, A] + tr[D, A] = tr[D, A]$. So the right hand side is also independent of the connection.

So we can in fact choose a trivial connection locally to carry out the computation: Let $D_0 = d$ be a trivial connection on $E|_U \to U$, then

$$\begin{aligned}
[D_0, A]s &= D_0(As) - (-1)^{deg(A)} A \wedge D_0 s \\
&= d(A^\beta_\alpha f^\alpha)e_\beta - (-1)^{deg(A)} A^\beta_\alpha \wedge df^\alpha e_\beta \\
&= dA^\beta_\alpha f^\alpha e_\beta.
\end{aligned}$$

Hence $tr[D_0, A] = d\, tr(A)$. $\square$

For any formal power series in one variable $f(x) = a_0 + a_1 x + \dots$, we define $f(\Theta) := a_0 + a_1\Theta + \cdots + a_n\Theta^n \in A^*(X)$.

Theorem 4.9 (Chern-Weil)**.** *For f as above, we have:*

1. $d\, tr f(\Theta) = 0$;
2. *If $\tilde{D}$ is another connection with curvature $\tilde{\Theta}$, there is a differential form $\eta \in A^*(X)$ such that $tr\ f(\tilde{\Theta}) - tr\ f(\Theta) = d\eta$.*

So the cohomology class of $tr\ f(\Theta)$ is independent of the connection. We call it the "characteristic class" of E associated to f, and $tr\ f(\Theta)$ the corresponding "characteristic form" of E associated to f and D.

Example 4.10. *Since $det(I_r + \frac{\sqrt{-1}}{2\pi}\Theta) = \exp\Big(tr \log(I_r + \frac{\sqrt{-1}}{2\pi}\Theta)\Big)$. So $c_i(E, D) \in A^{2i}(X)$ are all closed forms, whose cohomology classes are all independent of D. These are called "Chern classes". For example we have*

$$c_1(E, D) = \frac{\sqrt{-1}}{2\pi} tr\Theta, \quad c_2(E, D) = \frac{1}{8\pi^2}\Big(tr(\Theta^2) - (tr\Theta)^2\Big).$$

Proof of Theorem 4.9: For the first conclusion, by Lemma4.8, we have

$$\begin{aligned} d\, tr f(\Theta) &= tr[D, f(\Theta)] \\ &= \sum_k a_k tr[D, \Theta^k] = 0, \end{aligned}$$

where we used Lemma4.7 in the last step.

For the second one, we choose a family of connections $D_t := t\tilde{D} + (1-t)D$. Then

$$\dot{D}_t := \frac{dD_t}{dt} = \tilde{D} - D \in A^1(X, EndE),$$

and

$$\dot{\Theta}_t := \frac{d\Theta_t}{dt} = \frac{dD_t}{dt} D_t + D_t \frac{dD_t}{dt} = [D_t, \frac{dD_t}{dt}] = [D_t, \dot{D}_t].$$

So we have (by Lemma4.6, we can change the positions of Θ and $\dot{\Theta}$)

$$\begin{aligned} \frac{d}{dt} tr\ f(\Theta_t) &= tr(\dot{\Theta}_t f'(\Theta_t)) \\ &= tr([D_t, \dot{D}_t] f'(\Theta_t)) \\ &\overset{\text{Bianchi}}{=} tr[D_t, \dot{D}_t f'(\Theta_t)] \\ &= d\ tr\Big(\dot{D}_t f'(\Theta_t)\Big). \end{aligned}$$

So we conclude that $tr\ f(\tilde{\Theta}) - tr\ f(\Theta) = d\int_0^1 tr\Big(\dot{D}_t f'(\Theta_t)\Big) dt$. □

4.4 Comparing two definitions of first Chern classes

Let X be a complex manifold, using the short exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathscr{O} \xrightarrow{\exp(2\pi\sqrt{-1}\cdot)} \mathscr{O}^* \to 1$$

we get the exact sequence

$$\cdots \to H^1(X, \mathscr{O}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to \ldots.$$

We call $\delta : H^1(X, \mathscr{O}^*) \to H^2(X, \mathbb{Z})$ the "first Chern class" map.

Instead of holomorphic line bundles, we can consider C^∞ line bundles. These bundles are classified by $H^1(X, \mathcal{E}^*)$. Similarly, we have short exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{E} \xrightarrow{\exp(2\pi\sqrt{-1}\cdot)} \mathcal{E}^* \to 1,$$

and consequently a short exact sequence:

$$\cdots \to H^1(X, \mathcal{E}) \cdots \to H^1(X, \mathcal{E}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{E}) \to \ldots.$$

Since $\mathcal{E}$ is a fine sheaf, we have $H^p(X, \mathcal{E}) = 0$ whenever $p \geq 1$. So $\delta : H^1(X, \mathcal{E}^*) \to H^2(X, \mathbb{Z})$ is an isomorphism (also called "first Chern class map"). This means that *complex line bundles are determined up to C^∞ isomorphisms by their first Chern class.*

On the other hand, we can use a connection on a given C^∞ complex line bundle L, and use the curvature form Θ to define

$$c_1(L) := \Big[\frac{\sqrt{-1}}{2\pi}\Theta\Big] \in H^2_{dR}(X; \mathbb{R}) \cong H^2(X, \mathbb{R}).$$

Since we have a natural homomorphism $\Phi : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$ using the sheaf morphism $\underline{\mathbb{Z}} \to \underline{\mathbb{R}}$. We shall explore the relation between $\Phi(\delta([L])) \in H^2(X, \mathbb{R})$ and $c_1(L) \in H^2_{dR}(X, \mathbb{R})$.

For simplicity, in the following we assume L is a holomorphic line bundle with Hermitian metric h. We leave the necessary modification in the general complex line bundle case as an exercise. (hint: you need to replace the Chern connection by any connection on the bundle, use the transformation formula for connection 1-forms when you change a frame.)

First recall the construction of $\delta : H^1(X, \mathscr{O}^*) \to H^2(X, \mathbb{Z})$. Let L be a complex line bundle. We use sufficiently fine locally finite trivializations $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ such that each U_α is simply connected and $H^*(X, \mathscr{O}^*)$ is isomorphic to $H^*(\mathcal{U}, \mathscr{O}^*)$. Then $[L] \in H^1(X, \mathscr{O}^*)$ is determined by the Čech cocycle $\{\psi_{\alpha\beta}\}$, $\psi_{\alpha\beta} \in \mathscr{O}^*(U_\alpha \cap U_\beta)$. We define $\phi_{\alpha\beta} := \frac{1}{2\pi\sqrt{-1}} \log \psi_{\alpha\beta}$. Note that this is not a well-defined Čech cochain: *log* is a multi-valued function!

However, since $\psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$, we get

$$z_{\alpha\beta\gamma} := \phi_{\alpha\beta} + \phi_{\beta\gamma} - \phi_{\alpha\gamma} \in \underline{\mathbb{Z}}(U_\alpha \cap U_\beta \cap U_\gamma).$$

This defines a Čech cocycle, whose cohomology class defines $\delta([L])$. Then $\Phi(\delta([L]))$ is also defined by $\{z_{\alpha\beta\gamma}\}$, just viewing $\underline{\mathbb{Z}}$ as a subsheaf of $\underline{\mathbb{R}}$.

To compare it with $c_1(L)$, we need a closer look at the de Rham isomorphism. We first break the resolution

$$0 \to \underline{\mathbb{R}} \to \mathscr{A}^0 \to \mathscr{A}^1 \to \dots$$

into short exact sequences:

$$0 \to \underline{\mathbb{R}} \to \mathscr{A}^0 \to \mathcal{K}_1 \to 0, \quad 0 \to \mathcal{K}_1 \to \mathscr{A}^1 \to \mathcal{K}_2 \to 0, \quad \dots$$

where $\mathcal{K}_i$ is the sheaf of closed i-forms. We get exact sequence for cohomology:

$$0 \to H^1(X, \mathcal{K}_1) \to H^2(X, \mathbb{R}) \to 0, \quad A^1(X) \to \mathcal{K}_2(X) \to H^1(X, \mathcal{K}_1) \to 0.$$

The first one gives $\delta_2 : H^1(X, \mathcal{K}_1) \cong H^2(X, \mathbb{R})$ and the second gives $\delta_1 : H^2_{dR}(X) \cong H^1(X, \mathcal{K}_1)$.

First we study δ_1: Our de Rham class is given by $\frac{\sqrt{-1}}{2\pi}\Theta(h) \in \mathcal{K}_2(X)$. Locally, we have $\Theta = d\theta_\alpha$, where $\theta_\alpha = \partial \log h_\alpha$, $h_\alpha = h(e_\alpha, e_\alpha)$, $e_\alpha(p) = \varphi_\alpha^{-1}(p, 1)$. Then $\delta_1\Big([\frac{\sqrt{-1}}{2\pi}\Theta(h)]\Big)$ is given by $[\{\frac{\sqrt{-1}}{2\pi}(\theta_\beta - \theta_\alpha)\}]$.

Now

$$e_\beta(p) = \varphi_\beta^{-1}(p, 1) = \varphi_\alpha^{-1} \circ (\varphi_\alpha \circ \varphi_\beta^{-1})(p, 1) = \varphi_\alpha^{-1}(p, \psi_{\alpha\beta}(p)) = \psi_{\alpha\beta}(p) e_\alpha(p).$$

So we get $h_\beta = h_\alpha |\psi_{\alpha\beta}|^2$, and hence $\log h_\beta = \log h_\alpha + \log|\psi_{\alpha\beta}|^2$. So on $U_\alpha \cap U_\beta$, we have

$$\frac{\sqrt{-1}}{2\pi}(\theta_\beta - \theta_\alpha) = \frac{\sqrt{-1}}{2\pi}\partial \log|\psi_{\alpha\beta}|^2 = \frac{\sqrt{-1}}{2\pi}\partial \log \psi_{\alpha\beta} = \frac{\sqrt{-1}}{2\pi} d \log \psi_{\alpha\beta}.$$

Then $\delta_2([\{\frac{\sqrt{-1}}{2\pi}(\theta_\beta - \theta_\alpha)\}])$ is represented by

$$\Big\{\frac{\sqrt{-1}}{2\pi}\Big(\log \psi_{\beta\gamma} - \log \psi_{\alpha\gamma} + \log \psi_{\alpha\beta}\Big)\Big\}.$$

This is precisely our $\{z_{\alpha\beta\gamma}\}$.

4.5 Hermitian metrics and Kähler metrics

Let X be a complex manifold of dimension n. We denote the canonical almost complex structure by J. A Riemannian metric g on X is called "Hermitian", if g is J-invariant, i.e.

$$g(Ju, Jv) = g(u, v), \quad \forall u, v \in T_x^{\mathbb{R}}X, \forall x \in X.$$

As before, we extend g to $T^{\mathbb{C}}X$ as a complex bilinear form. For simplicity, we also denote this bilinear form by g. Then we have

$$g(T^{1,0}, T^{1,0}) = 0 = g(T^{0,1}, T^{0,1})$$

and $\langle Z, W\rangle := g(Z, \bar{W})$ defines an Hermitian metric on the rank n holomorphic vector bundle $T^{1,0}X$. Conversely, any Hermitian metric on $T^{1,0}X$ determines uniquely a J-invariant Riemannian metric on X.

For an Hermitian metric g on (X, J), we define the associated Kähler form ω_g by

$$\omega_g(u, v) := g(Ju, v).$$

It is direct to check that ω_g is a real 2-form on X.

Definition 4.11. *An Hermitian metric g on X is called a Kähler metric, if $d\omega_g = 0$. Its cohomology class in $H^2_{dR}(X)$ is call the "Kähler class" of g. If a (compact) complex manifold admits a Kähler metric, we call it a "Kähler manifold".*

Locally, if $(z_1, \dots, z_n)$ is a holomorphic coordinate system, then g is determined by

$$g_{i\bar{j}} := g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}),$$

since $g_{ij} = g_{\bar{i}\bar{j}} = 0$. Then we have

$$\omega_g = \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

where Einstein's summation convention is always used. Now we have

$$\begin{aligned}
0 = d\omega_g &= \sqrt{-1} dg_{i\bar{j}} dz_i \wedge d\bar{z}_j \\
&= \sqrt{-1}\frac{\partial g_{i\bar{j}}}{\partial z_k} dz_k \wedge dz_i \wedge d\bar{z}_j - \sqrt{-1}\frac{\partial g_{i\bar{j}}}{\partial \bar{z}_l} dz_i \wedge d\bar{z}_l \wedge d\bar{z}_j \\
&= \sqrt{-1}\sum_j \sum_{k<i} \Big(\frac{\partial g_{i\bar{j}}}{\partial z_k} - \frac{\partial g_{k\bar{j}}}{\partial z_i}\Big) dz_k \wedge dz_i \wedge d\bar{z}_j \\
&\quad + \sqrt{-1}\sum_i \sum_{j<l} \Big(\frac{\partial g_{i\bar{j}}}{\partial \bar{z}_l} - \frac{\partial g_{i\bar{l}}}{\partial \bar{z}_j}\Big) dz_i \wedge d\bar{z}_j \wedge d\bar{z}_l.
\end{aligned}$$

So being Kähler mean that $g_{i\bar{j}}$ have the additional symmetries:

$$\frac{\partial g_{i\bar{j}}}{\partial z_k} = \frac{\partial g_{k\bar{j}}}{\partial z_i}, \quad \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_l} = \frac{\partial g_{i\bar{l}}}{\partial \bar{z}_j}, \quad \forall i, j, k, l.$$

Example 4.12. *The Euclidean metric* $g = \sum_{i=1}^{n}(dx_i \otimes dx_i + dy_i \otimes dy_i)$ *is a Kähler metric, since we have*

$$\omega_g = \frac{\sqrt{-1}}{2}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i.$$

To give more examples, note that to define a Kähler metrics, it suffices to define its associated Kähler form, since we have $g(u, v) = g(Ju, Jv) = \omega_g(u, Jv)$. So sometimes we will also say "Let ω_g be a Kähler metric..."

Example 4.13. *Let* $X = B(1) \subset \mathbb{C}^n$ *be the unit ball in* $\mathbb{C}^n$*. We define a Kähler metric:*

$$\omega_g := \sqrt{-1}\partial\bar{\partial}\log\frac{1}{1-|z|^2}.$$

This is called the "complex hyperbolic metric".

Example 4.14. *Let* $X = \mathbb{C}P^n$ *with homogeneous coordinates* $[Z_0, \ldots, Z_n]$*, we define a Kähler metric:*

$$\omega_g := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(|Z_0|^2 + \cdots + |Z_n|^2).$$

It is easy to check that this is well-defined. It is called the "Fubini-Study metric".

Not every compact complex manifold is Kähler, since, for example, $H^2_{dR}(X)$ must be non-trivial[5]. So Calabi-Eckman manifolds are never Kähler. However, we have the following:

Lemma 4.15. *If X is Kähler and Y is a complex analytic submanifold of X, then Y is also Kähler.*

Proof. (Outline) Let g be a Kähler metric on X and $\iota : Y \to X$ be the embedding map, then $\iota^* g$ is a Kähler metric on Y and the associated Kähler form is just $\iota^*\omega_g$. □

By this lemma, all projective algebraic manifolds are Kähler.

In Riemannian geometry, normal coordinates are very useful in tensor calculations. The next lemma shows that being Kähler is both necessary and sufficient for the existence of complex analogue of normal coordinates.

Lemma 4.16. *For an Hermitian metric g on X, the follows two properties are equivalent:*

(1) *g is Kähler;*

(2) *For any point* $p \in X$*, there are local holomorphic coordinates* $(z_1, \ldots, z_n)$ *such that* $z_i(p) = 0$*,* $g_{i\bar{j}}(p) = \delta_{ij}$ *and* $dg_{i\bar{j}}(p) = 0$*.*

[5]If not, ω_g will be exact, so $\int_X \omega_g^n = 0$ by Stokes theorem. But this is impossible since $\int_X \omega_g^n > 0$.

Proof. $\underline{(2) \Longrightarrow (1)}$: For any given point p, we choose the coordinate in (2), then since first order derivatives of $g_{i\bar{j}}$ at p vanish, we will have $d\omega_g(p) = 0$. This implies $d\omega_g = 0$, i.e., g is Kähler.

$\underline{(1) \Longrightarrow (2)}$: Suppose g is Kähler. Given any point $p \in X$, we can first choose local holomorphic coordinates $(w_1, \ldots, w_n)$ such that $w_i(p) = 0$ and $g(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial \bar{w}_j})(p) = \delta_{ij}$. We want to find holomorphic coordinate transformation of the form $w_i = z_i + \frac{1}{2}a_{ijk}z_j z_k$ with $a_{ijk} = a_{ikj}$ such that

$$\omega_g = \sqrt{-1}(\delta_{ij} + O(|z|^2))dz_i \wedge d\bar{z}_j.$$

Direct computation shows that

$$\begin{aligned}
\omega_g &= \sqrt{-1}\Big(\delta_{ij} + g_{i\bar{j},k}(0)w_k + g_{i\bar{j},\bar{l}}(0)\bar{w}_l + O(|w|^2)\Big)dw_i \wedge d\bar{w}_j \\
&= \sqrt{-1}\Big(\delta_{ij} + g_{i\bar{j},k}(0)z_k + g_{i\bar{j},\bar{l}}(0)\bar{z}_l + O(|z|^2)\Big)(dz_i + a_{ipq}z_p dz_q) \wedge (d\bar{z}_j + \bar{a}_{jst}\bar{z}_s d\bar{z}_t) \\
&= \sqrt{-1}\Big(\delta_{ij}dz_i \wedge d\bar{z}_j + \bar{a}_{ilj}\bar{z}_l dz_i \wedge d\bar{z}_j + a_{jki}z_k dz_i \wedge d\bar{z}_j \\
&\quad + (g_{i\bar{j},k}(0)z_k + g_{i\bar{j},\bar{l}}(0)\bar{z}_l)dz_i \wedge d\bar{z}_j + O(|z|^2)\Big).
\end{aligned}$$

So the condition we need is $a_{jki} + g_{i\bar{j},k}(0) = 0$ and $\bar{a}_{ilj} + g_{i\bar{j},\bar{l}}(0) = 0$. So we simply take

$$a_{jki} := -\frac{\partial g_{i\bar{j}}}{\partial w_k}(0).$$

The Kähler condition makes sure that this is well-defined. $\square$

Remark 4.17. *We shall call such a holomorphic coordinate system a "Kähler normal coordinate system".*

Recall that for a connection ∇ on a vector bundle E, we can define the covariant derivative of a section s with respect to a tangent vector $v \in T_pX$ by setting $\nabla_v s := \nabla s(v)$. If e_α is a local frame of E, then we have $\nabla e_\alpha = \omega_\alpha^\beta e_\beta$, and $\nabla_v e_\alpha = \omega_\alpha^\beta(v)e_\beta$. Another good feature of the Kähler condition is that if we complexify the usual Levi-Civita connection, we will automatically get the Chern connection on $T^{1,0}X$.

Proposition 4.18. *Let (X, J, g) be a Kähler manifold. Then the complexification of the Levi-Civita connection restricts to the Chern connection on $T^{1,0}X$.*

Proof. We also denote the complexified Levi-Civita connection by ∇. Recall that ∇ is characterized as the only connection on $T^{\mathbb{R}}X$ that is both torsion free and compatible with g. For short, we write $\partial_i := \frac{\partial}{\partial z_i}$ and $\partial_{\bar{j}} := \frac{\partial}{\partial \bar{z}_j}$. By definition, we can assume $\nabla_{\partial_i}\partial_j := \Gamma_{ij}^k\partial_k + \Gamma_{ij}^{\bar{k}}\partial_{\bar{k}}$, $\nabla_{\partial_{\bar{i}}}\partial_j := \Gamma_{\bar{i}j}^k\partial_k + \Gamma_{\bar{i}j}^{\bar{k}}\partial_{\bar{k}}$. Since ∇ is a real operator, we also have $\nabla_{\partial_{\bar{i}}}\partial_{\bar{j}} := \overline{\Gamma_{ij}^k}\partial_{\bar{k}} + \overline{\Gamma_{ij}^{\bar{k}}}\partial_k$, $\nabla_{\partial_i}\partial_{\bar{j}} := \overline{\Gamma_{\bar{i}j}^k}\partial_{\bar{k}} + \overline{\Gamma_{\bar{i}j}^{\bar{k}}}\partial_k$. Since ∇ is torsion free, we have $\Gamma_{ij}^k = \Gamma_{ji}^k$, $\Gamma_{ij}^{\bar{k}} = \Gamma_{ji}^{\bar{k}}$, and $\Gamma_{\bar{i}j}^k = \overline{\Gamma_{\bar{j}i}^{\bar{k}}}$, $\Gamma_{\bar{i}j}^{\bar{k}} = \overline{\Gamma_{\bar{j}i}^k}$. Now we use the metric compatibility:

$$\begin{aligned}
0 &= \partial_i g(\partial_k, \partial_l) = g(\nabla_{\partial_i}\partial_k, \partial_l) + g(\partial_k, \nabla_{\partial_i}\partial_l) \\
&= \Gamma_{ik}^{\bar{q}} g_{l\bar{q}} + \Gamma_{il}^{\bar{q}} g_{k\bar{q}},
\end{aligned}$$

Exchange i and k, we get $0 = \Gamma^{\bar{q}}_{ki} g_{l\bar{q}} + \Gamma^{\bar{q}}_{kl} g_{i\bar{q}}$, and hence $\Gamma^{\bar{q}}_{kl} g_{i\bar{q}} = \Gamma^{\bar{q}}_{il} g_{k\bar{q}}$. So

$$\Gamma^{\bar{q}}_{ik} g_{l\bar{q}} = \Gamma^{\bar{q}}_{lk} g_{i\bar{q}} = \Gamma^{\bar{q}}_{kl} g_{i\bar{q}} = \Gamma^{\bar{q}}_{il} g_{k\bar{q}}.$$

This implies $\Gamma^{\bar{q}}_{ik} g_{l\bar{q}} = 0$ and hence $\Gamma^{\bar{q}}_{ik} = 0$. This means

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k, \quad \nabla_{\partial_{\bar{i}}} \partial_{\bar{j}} = \overline{\Gamma^k_{ij}} \partial_{\bar{k}}. \tag{4.3}$$

On the other hand,

$$\begin{aligned} \partial_i g(\partial_k, \partial_{\bar{l}}) &= g(\nabla_{\partial_i} \partial_k, \partial_{\bar{l}}) + g(\partial_k, \nabla_{\partial_i} \partial_{\bar{l}}) \\ &= \Gamma^p_{ik} g_{p\bar{l}} + \Gamma^{\bar{q}}_{\bar{l}i} g_{k\bar{q}}. \end{aligned}$$

By Kähler condition, the last quantity also equals

$$\partial_k g(\partial_i, \partial_{\bar{l}}) = \Gamma^p_{ki} g_{p\bar{l}} + \Gamma^{\bar{q}}_{\bar{l}k} g_{i\bar{q}},$$

so we get $\Gamma^{\bar{q}}_{\bar{l}k} g_{i\bar{q}} = \Gamma^{\bar{q}}_{\bar{l}i} g_{k\bar{q}}$. But the sum of these two quantity equals $\partial_{\bar{l}} g(\partial_i, \partial_k) = 0$, we get $\Gamma^{\bar{q}}_{\bar{l}i} g_{k\bar{q}} = 0$ and hence $\Gamma^{\bar{q}}_{\bar{l}i} = 0$. This also implies $\Gamma^{q}_{\bar{i}l} = 0$. So we get

$$\nabla_{\partial_i} \partial_{\bar{j}} = 0 = \nabla_{\partial_{\bar{j}}} \partial_i, \tag{4.4}$$

and also

$$\partial_i g_{k\bar{l}} = \Gamma^p_{ik} g_{p\bar{l}},$$

equivalently,

$$\Gamma^k_{ij} = g^{\bar{l}k} \frac{\partial g_{i\bar{l}}}{\partial z_j}. \tag{4.5}$$

This is precisely the formula for the Chern connection. □

For curvature, we also extend the curvature tensor $\mathbb{C}$-linearly to the complexified tangent bundle. Then this curvature tensor automatically satisfies the Bianchi identities. The Kähler condition also implies that the curvature tensor has more symmetries, and hence has much simpler formula. We leave this to later sections. Here we only add one warming exercise:

Exercise: Let (E, ∇) be a vector bundle with connection. We define for $u, v \in \Gamma(TX)$ and $s \in \Gamma(E)$, $R(u, v)s := (\nabla_u \nabla_v - \nabla_u \nabla_v - \nabla_{[u,v]})s$. Show that $R(u, v)s = \Omega(u, v)s$, where $\Omega \in \Gamma(\Lambda^2 T^* X \otimes End(E))$ is the curvature form of ∇.

Let (X, J, g) be a Kähler manifold. We know from Proposition 4.18 that the connection of g has very special properties. We now explore its implication for the curvature.

Lemma 4.19. *For a Kähler manifold (X, J, g), we always have $\nabla J = 0$.*

Proof. For any given point $p \in X$, we compute using Kähler normal coordinates in Lemma 4.16. Now in complex coordinates, J has constant coefficients, this implies ∇J vanishes at p. Since p is arbitrary, we have $\nabla J = 0$. □

By definition, this implies that $\nabla(JX) = J\nabla X$, so for the curvature $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, we have $R(X,Y)JZ = JR(X,Y)Z$. Also, by symmetry of curvature tensor, we have

$$\langle R(JX,JY)Z,W\rangle = \langle R(Z,W)JX,JY\rangle = \langle R(Z,W)X,Y\rangle = \langle R(X,Y)Z,W\rangle.$$

Since W is arbitrary, we also have $R(JX,JY)Z = R(X,Y)Z$. Moreover, we have:

Proposition 4.20. *We $\mathbb{C}$-linearly extend the curvature tensor of the Kähler metric g, then $\langle R(\partial_i,\partial_j)\cdot,\cdot\rangle = 0 = \langle R(\partial_{\bar{i}},\partial_{\bar{j}})\cdot,\cdot\rangle$, and the only essentially non-trivial term is*

$$R_{i\bar{j}k\bar{l}} := \langle R(\partial_i,\partial_{\bar{j}})\partial_k,\partial_{\bar{l}}\rangle = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k\partial\bar{z}_l} + g^{\bar{q}p}\frac{\partial g_{i\bar{q}}}{\partial z_k}\frac{\partial g_{p\bar{j}}}{\partial\bar{z}_l}.$$

In particular, besides Bianchi identities, we have an extra symmetry: $R_{i\bar{j}k\bar{l}} = R_{i\bar{l}k\bar{j}} = R_{k\bar{j}i\bar{l}}$. The Ricci curvature Rc is also J-invariant, and the 2-form $Ric(\omega_g) := Rc(J\cdot,\cdot)$ is called the Ricci form, and we have $Ric(\omega_g) = \sqrt{-1}R_{i\bar{j}}dz_i\wedge d\bar{z}_j$, with

$$R_{i\bar{j}} = Rc(\partial_i,\partial_{\bar{j}}) = g^{\bar{l}k}R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z_i\partial\bar{z}_j}\log\det(g_{p\bar{q}}).$$

Proof. We compute by definition:

$$\begin{aligned}
R_{i\bar{j}k\bar{l}} &= \langle(\nabla_i\nabla_{\bar{j}} - \nabla_{\bar{j}}\nabla_i)\partial_k,\partial_{\bar{l}}\rangle = -\langle\nabla_{\bar{j}}(\Gamma^p_{ik}\partial_p),\partial_{\bar{l}}\rangle\\
&= -\partial_{\bar{j}}\Gamma^p_{ik}g_{p\bar{l}} = -\partial_{\bar{j}}(g^{\bar{q}p}\frac{\partial g_{k\bar{q}}}{\partial z_i})g_{p\bar{l}}\\
&= -g^{\bar{q}p}\frac{\partial^2 g_{k\bar{q}}}{\partial z_i\partial z_{\bar{j}}}g_{p\bar{l}} + g^{\bar{q}s}g^{\bar{t}p}\frac{\partial g_{s\bar{t}}}{\partial z_{\bar{j}}}\frac{\partial g_{k\bar{q}}}{\partial z_i}g_{p\bar{l}}\\
&= -\frac{\partial^2 g_{k\bar{l}}}{\partial z_i\partial z_{\bar{j}}} + g^{\bar{q}s}\frac{\partial g_{s\bar{l}}}{\partial z_{\bar{j}}}\frac{\partial g_{k\bar{q}}}{\partial z_i}.
\end{aligned}$$

The first conclusion follows by Kähler metric's special symmetry.

For Ricci curvature, we choose a local orthonormal frame $\{e_i\}_{i=1}^{2n}$ to compute:

$$\begin{aligned}
Rc(JX,JY) &= \sum_{i=1}^{2n}\langle R(JX,e_i)e_i,JY\rangle = \sum_{i=1}^{2n}\langle JR(JX,e_i)e_i,J^2Y\rangle\\
&= -\sum_{i=1}^{2n}\langle R(JX,e_i)Je_i,Y\rangle = -\sum_{i=1}^{2n}\langle R(J^2X,Je_i)Je_i,Y\rangle\\
&= \sum_{i=1}^{2n}\langle R(X,Je_i)Je_i,Y\rangle = Rc(X,Y),
\end{aligned}$$

since $\{Je_i\}_{i=1}^{2n}$ is also an orthonormal frame. As the computation for ω_g, we easily get the formula

$$Ric(\omega_g) = \sqrt{-1}R_{i\bar{j}}dz_i \wedge d\bar{z}_j.$$

Finally, we calculate $R_{i\bar{j}}$: Choose a local orthonormal frame of the form $\{e_\alpha, Je_\alpha\}_{\alpha=1}^n$ at one point, and write $Z_\alpha := e_\alpha - \sqrt{-1}Je_\alpha$. Then we have

$$\begin{aligned}
R_{i\bar{j}} = Rc(\partial_i, \partial_{\bar{j}}) &= \sum_\alpha \langle R(\partial_i, e_\alpha)e_\alpha, \partial_{\bar{j}}\rangle + \sum_\alpha \langle R(\partial_i, Je_\alpha)Je_\alpha, \partial_{\bar{j}}\rangle \\
&= \sum_\alpha \langle R(\partial_i, e_\alpha)e_\alpha, \partial_{\bar{j}}\rangle + \sqrt{-1}\sum_\alpha \langle R(\partial_i, Je_\alpha)e_\alpha, \partial_{\bar{j}}\rangle \\
&= \sum_\alpha \langle R(\partial_i, \bar{Z}_\alpha)e_\alpha, \partial_{\bar{j}}\rangle \\
&= \frac{1}{2}\sum_\alpha \langle R(\partial_i, \bar{Z}_\alpha)e_\alpha, \partial_{\bar{j}}\rangle - \frac{\sqrt{-1}}{2}\sum_\alpha \langle R(\partial_i, \bar{Z}_\alpha)Je_\alpha, \partial_{\bar{j}}\rangle \\
&= \frac{1}{2}\sum_\alpha \langle R(\partial_i, \bar{Z}_\alpha)Z_\alpha, \partial_{\bar{j}}\rangle.
\end{aligned}$$

On the other hand, we have $Z_\alpha = a_\alpha^\mu \partial_\mu$ and $\partial_\mu = b_\mu^\alpha Z_\alpha$, with $a_\alpha^\mu b_\mu^\beta = \delta_\alpha^\beta$, so at the given point, we have

$$2\delta_{\alpha\beta} = g(Z_\alpha, \bar{Z}_\beta) = a_\alpha^\mu \bar{a}_\beta^\nu g_{\mu\bar{\nu}},$$

which implies that $g^{\bar{\beta}\alpha} = \frac{1}{2}\bar{a}_\mu^\beta a_\mu^\alpha$, and so

$$\begin{aligned}
R_{i\bar{j}} &= \frac{1}{2}\bar{a}_\alpha^\nu a_\alpha^\mu R_{i\bar{\nu}\mu\bar{j}} = g^{\bar{l}k}R_{i\bar{l}k\bar{j}} = g^{\bar{l}k}R_{i\bar{j}k\bar{l}} \\
&= -g^{\bar{l}k}\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + g^{\bar{l}k}g^{\bar{q}p}\frac{\partial g_{k\bar{q}}}{\partial z_i}\frac{\partial g_{p\bar{l}}}{\partial \bar{z}_j} = \frac{\partial}{\partial z_i}\Big(-g^{\bar{l}k}\frac{\partial g_{k\bar{l}}}{\partial \bar{z}_j}\Big) \\
&= -\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\log\det(g_{p\bar{q}}).
\end{aligned}$$

□

5 Hodge theorem

5.1 Hodge theorem on compact Riemannian manifolds

Let (M^m, g) be a compact oriented Riemannian manifold. Then we can define inner product on the space of real differential forms: for $\omega, \eta \in A^p(M)$

$$(\omega, \eta) := \int_M \langle \omega, \eta \rangle_g dV_g.$$

The idea of Hodge theorem is to represent a de Rham cohomology class by a "best" closed form. Since we can define norm of a differential form, a natural idea is to find a closed form of minimal norm within its cohomology class.

To be precise, start with a closed p-form $\eta \in A^p(M)$, we want to minimize the functional:

$$\Phi(\xi) := \|\eta + d\xi\|^2, \quad \xi \in A^{p-1}(M).$$

We can solve this variational problem by considering the corresponding Euler-Lagrange equation, which is an elliptic system.

Suppose $\eta_0 = \eta + d\xi_0$ achieves the minimum of $\|\eta + d\xi\|^2$, then for any $\xi \in A^{p-1}(M)$,

$$\|\eta_0 + td\xi\|^2 = (\eta_0 + td\xi, \eta_0 + td\xi) = \|\eta_0\|^2 + 2t(\eta_0, d\xi) + t^2\|d\xi\|^2$$

achieves its minimum at $t = 0$. This happens if and only if $(\eta_0, d\xi) = 0$ for any $\xi \in A^{p-1}(M)$. We can define an operator d^*, the "formal adjoint" of d, such that $(\alpha, d\beta) = (d^*\alpha, \beta)$ for any $\alpha \in A^p(M)$ and $\beta \in A^{p-1}(M)$. Then $(\eta_0, d\xi) = 0$ for any $\xi \in A^{p-1}(M)$ if and only if $(d^*\eta_0, \xi) = 0$ for any $\xi \in A^{p-1}(M)$, which implies $d^*\eta_0 = 0$.

Definition 5.1. *Let (M^m, g) be a compact oriented Riemannian manifold. A smooth differential form $\omega \in A^p(M)$ is called a "harmonic p-form" if $d\omega = 0, d^*\omega = 0$.*

If we define the Laplacian operator to be $\Delta_d : A^p(M) \to A^p(M)$, $\Delta_d := dd^* + d^*d$, then for any smooth p-form $\omega \in A^p(M)$, we have

$$(\omega, \Delta_d\omega) = (\omega, dd^*\omega) + (\omega, d^*d\omega) = \|d^*\omega\|^2 + \|d\omega\|^2.$$

So we conclude that $\omega \in A^p(M)$ is harmonic if and only if $\Delta_d\omega = 0$.

To write down a precise formula for d^*, we introduce Hodge's "star"-operator: $* : A^p(M) \to A^{m-p}(M)$. If $\omega_1, \dots, \omega_m$ is an orthonormal basis of T^*_xM, such that $\omega_1 \wedge \cdots \wedge \omega_m = dV_g$ gives the positive orientation, then we define

$$*\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} = \delta^{i_1,\dots,i_p,j_1,\dots,j_{m-p}}_{1,2,\dots,m} \omega_{j_1} \wedge \cdots \wedge \omega_{j_{m-p}}.$$

(Note that this implies $\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \wedge *\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} = \omega_1 \wedge \cdots \wedge \omega_m$.) Then we extend $*$ linearly. It is direct to check that this is well-defined.

Moreover, if $\alpha = \sum_{i_1<\cdots<i_p} a_{i_1,\ldots,i_p}\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}$, $\beta = \sum_{i_1<\cdots<i_p} b_{i_1,\ldots,i_p}\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}$, we have

$$\begin{aligned}
\alpha \wedge *\beta &= \sum_{k_1<\cdots<k_p}\sum_{i_1<\cdots<i_p} a_{k_1,\ldots,k_p} b_{i_1,\ldots,i_p}\omega_{k_1} \wedge \cdots \wedge \omega_{k_p} \wedge *(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}) \\
&= \sum_{i_1<\cdots<i_p} a_{i_1,\ldots,i_p} b_{i_1,\ldots,i_p}\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \wedge *(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}) \\
&= \langle\alpha,\beta\rangle_g dV_g = \beta \wedge *\alpha.
\end{aligned}$$

From the definition, it is easy to check that $** = (-1)^{p(m-p)} = (-1)^{pm+p}$ on $A^p(M)$. Also, we have

$$\langle *\alpha, *\beta\rangle_g dV_g = *\alpha \wedge * * \beta = (-1)^{p(m-p)} * \alpha \wedge \beta = \beta \wedge *\alpha = \langle\beta,\alpha\rangle_g dV_g = \langle\alpha,\beta\rangle_g dV_g.$$

So $*$ is a point-wise isometry. Using $*$, we can also express d^* as:

Lemma 5.2. *We have $d^* = (-1)^{mp+m+1} * d*$ on $A^p(M)$.*

Proof. Let $\alpha \in A^p(M)$, $\beta \in A^{p-1}(M)$, then we have

$$\begin{aligned}
(d^*\alpha,\beta) = (\alpha, d\beta) &= \int_M \langle\alpha, d\beta\rangle_g dV_g = \int_M d\beta \wedge *\alpha \\
&= \int_M d(\beta \wedge *\alpha) + (-1)^p \beta \wedge d(*\alpha) \\
&= \int_M (-1)^p \beta \wedge (-1)^{(m-p+1)(p-1)} * *d(*\alpha) \\
&= (-1)^{mp+m+1} \int_M \langle\beta, *d(*\alpha)\rangle_g dV_g \\
&= \Big((-1)^{mp+m+1} * d * \alpha, \beta\Big).
\end{aligned}$$

□

From this concrete formula, we have on $A^p(M)$:

$$\Delta_d = dd^* + d^*d = (-1)^{mp+m+1} d*d* + (-1)^{m(p+1)+m+1} *d*d = (-1)^{mp+m+1} d*d* + (-1)^{mp+1} *d*d.$$

We have

$$\begin{aligned}
\Delta_d * &= (-1)^{m(m-p)+m+1} d * d * * + (-1)^{m(m-p)+1} * d * d * \\
&= (-1)^{mp+1}(-1)^{mp+p} d * d + (-1)^{mp+m+1} * d * d * \\
&= (-1)^{p+1} d * d + (-1)^{mp+m+1} * d * d * .
\end{aligned}$$

And

$$\begin{aligned}
*\Delta_d &= (-1)^{mp+m+1} * d * d * + (-1)^{mp+1} * * d * d \\
&= (-1)^{mp+m+1} * d * d * + (-1)^{mp+1+p(m-p)} d * d \\
&= (-1)^{mp+m+1} * d * d * + (-1)^{p+1} d * d.
\end{aligned}$$

So we get $*\Delta_d = \Delta_d *$. Similarly,

$$d\Delta_d = (-1)^{mp+1} d * d * d,$$

$$\Delta_d d = (-1)^{m(p+1)+m+1} d * d * d = (-1)^{mp+1} d * d * d = d\Delta_d.$$

Example 5.3. *In case of $(\mathbb{R}^n, g_{Euc})$, we can define d^* by the same formula, then we still have $(\xi, d\eta) = (d^*\xi, \eta)$ when one of them has compact support. Then we have*

$$\begin{aligned}
d^* &\Big(\sum_{1 \le i_1 < \cdots < i_p \le m} f_{i_1 \ldots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \Big) \\
= &\ (-1)^{mp+m+1} * d\Big(\sum_{1 \le i_1 < \cdots < i_p \le m} f_{i_1 \ldots i_p} * dx_{i_1} \wedge \cdots \wedge dx_{i_p} \Big) \\
= &\ (-1)^{mp+m+1} * \Big(\sum_{1 \le i_1 < \cdots < i_p \le m} \sum_{i \in \{i_1, \ldots, i_p\}} \frac{\partial f_{i_1 \ldots i_p}}{\partial x_i} dx_i \wedge * dx_{i_1} \wedge \cdots \wedge dx_{i_p} \Big) \\
= &\ (-1)^{mp+m+1} \sum_{1 \le i_1 < \cdots < i_p \le m} \sum_{k=1}^{p} \frac{\partial f_{i_1 \ldots i_p}}{\partial x_{i_k}} (-1)^{(p-1)(m-p)+k-1} dx_{i_1} \wedge \cdots \wedge d\hat{x}_{i_k} \wedge \cdots \wedge dx_{i_p} \\
= &\ \sum_{1 \le i_1 < \cdots < i_p \le m} \sum_{k=1}^{p} \frac{\partial f_{i_1 \ldots i_p}}{\partial x_{i_k}} (-1)^k dx_{i_1} \wedge \cdots \wedge d\hat{x}_{i_k} \wedge \cdots \wedge dx_{i_p}.
\end{aligned}$$

From this we get

$$\begin{aligned}
dd^* &\Big(\sum_{1 \le i_1 < \cdots < i_p \le m} f_{i_1 \ldots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \Big) \\
= &\ - \sum_{1 \le i_1 < \cdots < i_p \le m} \sum_{k=1}^{p} \frac{\partial^2 f_{i_1 \ldots i_p}}{\partial x_{i_k}^2} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \\
&\ + \sum_{1 \le i_1 < \cdots < i_p \le m} \sum_{k=1}^{p} \sum_{i \notin \{i_1, \ldots, i_p\}} \frac{\partial^2 f_{i_1 \ldots i_p}}{\partial x_{i_k} \partial x_i} (-1)^k dx_i \wedge dx_{i_1} \wedge \cdots \wedge d\hat{x}_{i_k} \wedge \cdots \wedge dx_{i_p},
\end{aligned}$$

and

$$\begin{aligned} d^*d & \Big(\sum_{1\le i_1<\cdots<i_p\le m} f_{i_1\dots i_p} dx_{i_1}\wedge\cdots\wedge dx_{i_p}\Big) \\ = \quad & d^*\Big(\sum_{1\le i_1<\cdots<i_p\le m}\sum_{i\notin\{i_1,\dots,i_p\}} \frac{\partial f_{i_1\dots i_p}}{\partial x_i} dx_i\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_p}\Big) \\ = \quad & -\sum_{1\le i_1<\cdots<i_p\le m}\sum_{i\notin\{i_1,\dots,i_p\}} \frac{\partial^2 f_{i_1\dots i_p}}{\partial x_i^2} dx_{i_1}\wedge\cdots\wedge dx_{i_p} \\ & -\sum_{1\le i_1<\cdots<i_p\le m}\sum_{i\notin\{i_1,\dots,i_p\}}\sum_{k=1}^{p} \frac{\partial^2 f_{i_1\dots i_p}}{\partial x_{i_k}\partial x_i}(-1)^k dx_i\wedge dx_{i_1}\wedge\cdots\wedge \widehat{dx_{i_k}}\wedge\cdots\wedge dx_{i_p}. \end{aligned}$$

So we have

$$\Delta_d\Big(\sum_{1\le i_1<\cdots<i_p\le m} f_{i_1\dots i_p} dx_{i_1}\wedge\cdots\wedge dx_{i_p}\Big) = -\sum_{1\le i_1<\cdots<i_p\le m}\Big(\sum_i \frac{\partial^2 f_{i_1\dots i_p}}{\partial x_i^2}\Big) dx_{i_1}\wedge\cdots\wedge dx_{i_p}.$$

The main result is that harmonic forms exists in each cohomology class:

Theorem 5.4 (Hodge)**.** *Let (M^m, g) be a compact oriented Riemannian manifold. Then each de Rham cohomology class has a unique harmonic representative, so we have a linear isomorphism*

$$\mathcal{H}^p(M) := \{\omega\in A^p(M) \mid \Delta_d\omega = 0\} \cong H^p_{dR}(M;\mathbb{R}),\quad p = 0,\dots,m.$$

Moreover, $\mathcal{H}^p(M)$ is always a finite dimensional vector space,[6] *and we have a linear operator $G : A^p(M)\to A^p(M)$ such that for any $\omega\in A^p(M)$, if we denote its orthogonal projection to $\mathcal{H}^p(M)$ by ω_h, then we have the decomposition:*

$$\omega = \omega_h + \Delta_d G\omega = \omega_h + d(d^*G\omega) + d^*(dG\omega).$$

In fact, we have a orthogonal direct sum decomposition $A^p(M) = \mathcal{H}^p(M)\oplus Im\, d\oplus Im\, d^*$.

Remark 5.5. *G is usually called the "Green operator". It is constructed in the following way: suppose the eigenvalues of Δ_d on $A^p(M)$ are $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$. The corresponding eigenspaces are $\mathcal{H}^p(M)$ and $E_1, E_2, \dots$. Then we define $G|_{\mathcal{H}^p(M)}\equiv 0$ and $G|_{E_i} := \frac{1}{\lambda_i} id_{E_i}$.*

Proof of parts of the results: Uniqueness: Suppose ω_1 and ω_2 are both harmonic p-forms and $\omega_2 = \omega_1 + d\eta$ for some $\eta\in A^{p-1}(M)$. Then

$$(d\eta, d\eta) = (\omega_2-\omega_1, d\eta) = (d^*(\omega_2-\omega_1),\eta) = 0.$$

[6] We can prove directly that $H^p_{dR}(M;\mathbb{R})$ is a finite dimensional vector space via the Mayer-Vietoris argument as in Bott-Tu's book.

So we necessarily have $d\eta = 0$ and $\omega_2 = \omega_1$.

$\mathcal{H}^p(M), Im\, d, Im\, d^*$ are orthogonal to each other: Let $\omega_h \in \mathcal{H}^p(M)$, $\xi \in A^{p+1}(M), \eta \in A^{p-1}(M)$, then

$$\begin{aligned}(\omega_h, d^*\xi) = &\quad (d\omega_h, \xi) = 0\\ (\omega_h, d\eta) = &\quad (d^*\omega_h, \eta) = 0\\ (d^*\xi, d\eta) = &\quad (\xi, dd\eta) = 0.\end{aligned}$$

Rough idea about existence: One can show that Δ_d is a 2nd order elliptic operator, and we have a "basic estimate" of the form

$$\|\omega\|^2_{W^{1,2}} \leq C(\Delta_d\omega + \omega, \omega) = C(\|\omega\|^2 + \|d\omega\|^2 + \|d^*\omega\|^2).$$

(For general elliptic operator, this kind of estimates still hold, known as "Gårding's inequality".) We consider the quadratic form on $W^{1,2}(M, \Lambda^p T^*M)$:

$$\mathcal{D}(\xi, \eta) := (\xi, \eta) + (d\xi, d\eta) + (d^*\xi, d^*\eta).$$

Gårding's inequality implies that $\mathcal{D}(\omega)$ is an equivalent norm on $W^{1,2}(M, \Lambda^p T^*M)$. Given $\eta \in L^2(M, \Lambda^p T^*M), \xi \mapsto (\xi, \eta)$ is a bounded linear functional on $A^p(M) \subset W^{1,2}(M, \Lambda^p T^*M)$:

$$|(\xi, \eta)| \leq \|\xi\| \cdot \|\eta\| \leq \|\eta\| \cdot \|\xi\|_{W^{1,2}} \leq C\sqrt{\mathcal{D}(\xi, \xi)}.$$

This extends to a bounded linear functional on $W^{1,2}(M, \Lambda^p T^*M)$, and we can use Riesz representation theorem to get a unique $\varphi \in W^{1,2}(M, \Lambda^p T^*M)$ such that for all $\xi \in A^p(M)$:

$$(\xi, \eta) = \mathcal{D}(\xi, \varphi).$$

Using this to define a linear map $T(\eta) := \varphi$. It is a bounded linear operator from $L^2(M, \Lambda^p T^*M)$ to $W^{1,2}(M, \Lambda^p T^*M)$. Its composition with the compact embedding $W^{1,2} \to L^2$ (also denoted by T) gives us a compact self-adjoint operator on $L^2(M, \Lambda^p T^*M)$. Intuitively, $T = (id + \Delta_d)^{-1}$.

By spectrum theorem and elliptic regularity, we have a Hilbert space direct sum decomposition $L^2(M, \Lambda^p T^*M) = \oplus_{m=0}^{\infty} E_m$, where each E_m is a finite dimensional space of smooth p-forms, satisfying $T\varphi = \rho_m\varphi, \forall\varphi \in E_m$, with $\rho_0 = 1 > \rho_1 > \rho_2 \ldots$ and $\rho_m \to 0$. Then $E_0 = \mathcal{H}^p(M)$ and for $\varphi \in E_m$, we have $\Delta_d\varphi = \left(\frac{1}{\rho_m} - 1\right)\varphi =: \lambda_m\varphi$, $\lambda_m \nearrow \infty$. □

5.2 The Hermitian case

Now let X^n be a n-dimensional compact complex manifold, with almost complex structure J and Hermitian metric g. As before, we define $\omega_g := \sqrt{-1}\sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$. It is a real $(1,1)$-form. A direct computation shows that we always have

$$dV_g = \frac{\omega_g^n}{n!}.$$

In fact, we can choose coordinates around a given point p such that at p, $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\}_{i,j=1}^n$ is an orthonormal basis with $z_i = x_i + \sqrt{-1}y_i$ the complex coordinate function. [7] Then at p the left equals $dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ while on the other hand, we have at p: $\omega_g = \frac{\sqrt{-1}}{2}\sum_i dz_i \wedge d\bar{z}_i = \sum_i dx_i \wedge dy_i$ and hence $\frac{\omega_g^n}{n!} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n = dV_g$.

Exercise: *Show that under local coordinates, we have*

$$\frac{\omega_g^n}{n!} = \det(g_{i\bar{j}})(\sqrt{-1})^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.$$

In this case, we also extend Hodge's star operator complex linearly to complex differential forms. Then we also have $** = (-1)^{p(2n-p)} = (-1)^p$ on $A^p(X)$ and

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle_{\mathbb{C}} dV_g.$$

On the space of smooth complex differential forms, the correct Hermitian inner product should be

$$(\alpha, \beta) := \int_X \alpha \wedge *\bar{\beta}.$$

Lemma 5.6. *The $*$ operator maps $A^{p,q}(X)$ to $A^{n-q,n-p}(X)$.*

Proof. We compute at a given point x, and we choose complex coordinates such that $g_{i\bar{j}}(x) = \frac{1}{2}\delta_{ij}$. Then $dx_1, dy_1, \ldots, dx_n, dy_n$ is a positively oriented orthonormal basis of $T_x^{*\mathbb{R}}X$. For multi-index $I = (\mu_1, \ldots, \mu_p)$, we shall write

$$dz_I := dz_{\mu_1} \wedge \cdots \wedge dz_{\mu_p}, \quad dx_I := dx_{\mu_1} \wedge \cdots \wedge dx_{\mu_p}, \quad \ldots$$

Also for multi-index M, we define

$$w_M := \Pi_{\mu \in M} dz_\mu \wedge d\bar{z}_\mu = (-2\sqrt{-1})^{|M|}\Pi_{\mu \in M} dx_\mu \wedge dy_\mu.$$

A direct computation shows that for mutually disjoint increasing multi-indices A, B, M, we have

$$*(dz_A \wedge d\bar{z}_B \wedge w_M) = \gamma(a, b, m) dz_A \wedge d\bar{z}_B \wedge w_{M'},$$

[7] What we need to do is to use a complex linear coordinate transformation such that $g_p(\frac{\partial}{\partial z_i}|_p, \frac{\partial}{\partial \bar{z}_j}|_p) = \frac{1}{2}\delta_{ij}$.

where $a = |A|, b = |B|, m = |M|$, $M' = (1, 2, , \ldots, n) - (A \cup B \cup M)$, and $\gamma(a, b, m)$ is a non-vanishing constant. In fact, one can show that

$$\gamma(a, b, m) = (\sqrt{-1})^{a-b}(-1)^{\frac{k(k+1)}{2}+m}(-2\sqrt{-1})^{k-n},$$

where $k = a + b + 2m$ is the total degree.

If we write $p = a + m, q = b + m$, then all (p, q)-form is locally a linear combination of forms of the type $dz_A \wedge d\bar{z}_B \wedge w_M$. Since $dz_A \wedge d\bar{z}_B \wedge w_{M'}$ is a $(a + m', b + m') = (a+n-a-b-m, b+n-a-b-m) = (n-q, n-p)$-form, we get $*A^{p,q}(X) \subset A^{n-q,n-p}(X)$. □

As in the real case, we consider the Hermitian inner product on $A^{p,q}(X)$, and define an operator $\bar{\partial}^*$ by

$$(\xi, \bar{\partial}\eta) = (\bar{\partial}^*\xi, \eta), \quad \forall \xi \in A^{p,q}(X), \eta \in A^{p,q-1}(X).$$

Then we get

$$\begin{aligned}
(\bar{\partial}^*\xi, \eta) &= \int_X \bar{\partial}^*\xi \wedge *\bar{\eta} \\
&= (\xi, \bar{\partial}\eta) = \overline{(\bar{\partial}\eta, \xi)} = \overline{\int_X \bar{\partial}\eta \wedge *\bar{\xi}} = \int_X \partial\bar{\eta} \wedge *\xi \\
&= \int_X \partial\big(\bar{\eta} \wedge *\xi\big) - (-1)^{p+q-1}\bar{\eta} \wedge \partial(*\xi) = (-1)^{p+q} \int_X \bar{\eta} \wedge \partial(*\xi) \\
&= -\int_X \partial(*\xi) \wedge \bar{\eta} = -\int_X *\partial(*\xi) \wedge *\bar{\eta}.
\end{aligned}$$

So we get:

Lemma 5.7. *On $A^{p,q}(X)$, we always have $\bar{\partial}^* = - * \partial *$.*

Exercise: Show that on the space of complex valued p-forms $A^p(X)$, we have $d^* = - * d *$.

We define the $\bar{\partial}$-Laplacian $\Delta_{\bar{\partial}} : A^{p,q}(X) \to A^{p,q}(X)$ by

$$\Delta_{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

We look for $\bar{\partial}$-closed form of minimal norm within a given Dolbeault cohomology class. Suppose $\xi \in A^{p,q}(X)$ is such a $\bar{\partial}$-closed form, then for any $\eta \in A^{p,q-1}(X)$, the quadratic function of $t \in \mathbb{R}$:

$$||\xi + t\bar{\partial}\eta||^2 = (\xi + t\bar{\partial}\eta, \xi + t\bar{\partial}\eta) = ||\xi||^2 + 2tRe(\xi, \bar{\partial}\eta) + t^2||\bar{\partial}\eta||^2$$

takes its minimum at $t = 0$. We get $Re(\xi, \bar{\partial}\eta) = 0$ for all $\eta \in A^{p,q-1}(X)$. Using $||\xi + t\sqrt{-1}\bar{\partial}\eta||^2$ instead, we get $Im(\xi, \bar{\partial}\eta) = 0$ for all $\eta \in A^{p,q-1}(X)$. So we get $(\xi, \bar{\partial}\eta) = (\bar{\partial}^*\xi, \eta) = 0$ for all $\eta \in A^{p,q-1}(X)$. This implies $\bar{\partial}^*\xi = 0$.

Definition 5.8. *If $\omega \in A^{p,q}(X)$ satisfies $\bar{\partial}\omega = 0$ and $\bar{\partial}^*\omega = 0$ (equivalently, $\Delta_{\bar{\partial}}\omega = 0$), then ω is called a "$\bar{\partial}$-harmonic (p,q)-form".*

The counterpart of Hodge theorem for Dolbeault cohomology is the following:

Theorem 5.9 (Hodge)**.** *Let (X^n, J, g) be a compact Hermitian manifold. Then each Dolbeault cohomology class has a unique $\bar{\partial}$-harmonic representative, so we have a complex linear isomorphism*

$$\mathcal{H}^{p,q}(X) := \{\omega \in A^{p,q}(X) \mid \Delta_{\bar{\partial}}\omega = 0\} \cong H^{p,q}_{\bar{\partial}}(X), \quad p, q = 0, \ldots, n.$$

Moreover, $\mathcal{H}^{p,q}(X)$ is always a finite dimensional complex vector space, and we have a complex linear operator $G : A^{p,q}(X) \to A^{p,q}(X)$ such that for any $\omega \in A^{p,q}(X)$, if we denote its orthogonal projection to $\mathcal{H}^{p,q}(X)$ by ω_h, then we have the decomposition:

$$\omega = \omega_h + \Delta_{\bar{\partial}} G\omega = \omega_h + \bar{\partial}(\bar{\partial}^* G\omega) + \bar{\partial}^*(\bar{\partial} G\omega).$$

In fact, we have an orthogonal direct sum decomposition $A^{p,q}(X) = \mathcal{H}^{p,q}(X) \oplus Im\, \bar{\partial} \oplus Im\, \bar{\partial}^*$.

Generalization: Assume also that we have a holomorphic vector bundle $E \to X$ of rank r, with Hermitian metric h. X is compact. We define an Hermitian inner product on $C^\infty(X, \Lambda^{p,q}(X) \otimes E)$ by

$$(s, t) := \int_X \langle s, t \rangle_{g,h} dV_g,$$

where the pointwise Hermitian inner product $\langle , \rangle_{g,h}$ is induced from the Hermitian metric g on X and bundle metric h on E. We can define a $\bar{\partial}$-operator on $A^{p,q}(X, E)$, which we shall write $\bar{\partial}_E : A^{p,q}(X, E) \to A^{p,q+1}(X, E)$. We can also define a formal adjoint operator $\bar{\partial}^*_E : A^{p,q}(X, E) \to A^{p,q-1}(X, E)$ by requiring that

$$(s, \bar{\partial}_E t) = (\bar{\partial}^*_E s, t), \quad \forall s \in A^{p,q}(X, E), t \in A^{p,q-1}(X, E).$$

Then we define $\Delta_{\bar{\partial}_E} := \bar{\partial}^*_E \bar{\partial}_E + \bar{\partial}_E \bar{\partial}^*_E : A^{p,q}(X, E) \to A^{p,q}(X, E)$, and $\mathcal{H}^{p,q}(X, E) := Ker\Big(\Delta_{\bar{\partial}_E}|_{A^{p,q}(X,E)}\Big)$. The elements of $\mathcal{H}^{p,q}(X, E)$ are called "E-valued harmonic (p,q)-forms". In this case, we also have:

Theorem 5.10. *Let (X^n, J, g) be a compact Hermitian manifold. $E \to X$ be a holomorphic vector bundle of rank r, with Hermitian metric h. Then each cohomology class in $H^{p,q}_{\bar{\partial}}(X, E)$ has a unique harmonic representative, so we have a complex linear isomorphism*

$$\mathcal{H}^{p,q}(X, E) \cong H^{p,q}_{\bar{\partial}}(X, E), \quad p, q = 0, \ldots, n.$$

Moreover, $\mathcal{H}^{p,q}(X, E)$ is always a finite dimensional complex vector space, and we have a complex linear operator $G_E : A^{p,q}(X, E) \to A^{p,q}(X, E)$ such that for any $\omega \in A^{p,q}(X, E)$, if we denote its orthogonal projection to $\mathcal{H}^{p,q}(X, E)$ by ω_h, then we have the decomposition:

$$\omega = \omega_h + \Delta_{\bar{\partial}_E} G_E\omega = \omega_h + \bar{\partial}_E(\bar{\partial}^*_E G_E\omega) + \bar{\partial}^*_E(\bar{\partial}_E G_E\omega).$$

In fact, we have an orthogonal direct sum decomposition $A^{p,q}(X, E) = \mathcal{H}^{p,q}(X, E) \oplus Im\, \bar{\partial}_E \oplus Im\, \bar{\partial}^*_E$.

5.3 Applications

Theorem 5.11 (Poincaré duality for de Rham cohomology)**.** *Let M^m be a compact oriented differentiable manifold. Then*

$$H^p_{dR}(M,\mathbb{R}) \cong H^{m-p}_{dR}(M,\mathbb{R}).$$

In particular, $b_p(M) = b_{m-p}(M)$.

Proof. Since $*$ commutes with Δ_d, and $** = \pm 1$, we conclude that $*$ induces a linear isomorphism between $\mathcal{H}^p(M)$ and $\mathcal{H}^{m-p}(M)$. Then the result follows from Hodge theorem. □

Theorem 5.12 (Kodaira-Serre duality)**.** *Let $E \to X$ be a holomorphic vector bundle over a compact complex manifold X of complex dimension n. Then we have a conjugate-linear isomorphism*

$$\sigma : H^r(X, \Omega^p(E)) \xrightarrow{\cong} H^{n-r}(X, \Omega^{n-p}(E^*)).$$

Proof. (Sketch) We introduce a conjugate-linear operator $\bar{*}_E$, constructing from $* : A^{p,q} \to A^{n-q,n-p}$ and the conjugate-linear isomorphism $\tau : E \to E^*$ via bundle metric h. To make everything conjugate-linear, we also define $\bar{*} : A^{p,q}(X) \to A^{n-p,n-q}(X)$ by $\bar{*}(\eta) := *\bar{\eta}$. Then $\bar{*}_E : A^{p,q}(X,E) \to A^{n-p,n-q}(X,E^*)$ is defined by

$$\bar{*}_E(\eta \otimes s) := \bar{*}(\eta) \otimes \tau(s).$$

Then we have $\bar{\partial}^*_E = -\bar{*}_{E^*} \circ \bar{\partial}_{E^*} \circ \bar{*}_E$ and hence $\bar{*}_E \Delta_{\bar{\partial}_E} = \Delta_{\bar{\partial}_{E^*}} \bar{*}_E$.

By Hodge theorem, we have

$$H^r(X, \Omega^p(E)) \cong H^{p,r}_{\bar{\partial}}(X, E), \quad H^{n-r}(X, \Omega^{n-p}(E^*)) \cong H^{n-p,n-r}_{\bar{\partial}}(X, E^*).$$

Then $\bar{*}_E$ induces a conjugate-linear map $\sigma : H^r(X, \Omega^p(E)) \to H^{n-r}(X, \Omega^{n-p}(E^*))$, and the Kodaira-Serre duality follows from the fact $\bar{*}_E \circ \bar{*}_{E^*} = \pm 1$. □

5.4 The Kähler case

Now we assume (X^n, J, g) is a compact Kähler manifold. Then we will have a better understanding of harmonic forms and Dolbeault cohomology. We shall begin by exploring the relation between Δ_d and $\Delta_{\bar\partial}$.

5.4.1 Hodge identities for Kähler metrics

We introduce some operators that will be useful in our discussion:

$$d^c := \sqrt{-1}(\bar\partial - \partial).$$

Here my notation is the same as Wells, but differs from Griffiths-Harris by a factor 4π. Then $dd^c = \sqrt{-1}(\partial + \bar\partial)(\bar\partial - \partial) = 2\sqrt{-1}\partial\bar\partial$. We define the *"Lefschetz operator"* $L : A^{p,q}(X) \to A^{p+1,q+1}(X)$ by:

$$L(\eta) := \omega_g \wedge \eta =: L\eta.$$

Its adjoint will be denoted by $\Lambda : A^{p+1,q+1}(X) \to A^{p,q}(X)$. We have

$$(\xi, L\eta) = (\Lambda\xi, \eta), \quad \forall \xi \in A^{p+1,q+1}(X), \eta \in A^{p,q}(X).$$

The basic equality in the Kähler case is:

Lemma 5.13. *On $A^{p,q}(X)$, we have $[\Lambda, \partial] = \sqrt{-1}\bar\partial^*$.*

Given this, since L is a real operator, so is Λ, and we have

$$[\Lambda, \bar\partial] = -\sqrt{-1}\partial^*.$$

Combining these two identities, we further get

$$[\Lambda, d] = -d^{c*}, \quad [\Lambda, d^c] = d^*.$$

Proof of Lemma 5.13 . We first prove the identity in $\mathbb{C}^n$. Let $\omega = \frac{\sqrt{-1}}{2}\sum_i dz_i \wedge d\bar z_i$ be the standard Kähler form on $\mathbb{C}^n$. Let $A_c^{p,q}(\mathbb{C}^n)$ be the space of smooth (p,q)-forms on $\mathbb{C}^n$ with compact support. Then $L : A_c^{p,q}(\mathbb{C}^n) \to A_c^{p+1,q+1}(\mathbb{C}^n), L\eta := \omega \wedge \eta$.

To derive a formula for $\Lambda = L^*$, we introduce operators $e_k, \bar e_k$ by

$$e_k(\eta) := dz_k \wedge \eta, \quad \bar e_k(\eta) := d\bar z_k \wedge \eta.$$

Their adjoints are denoted by i_k and $\bar i_k$ respectively. Recall that $|dz_k|^2 = |dx|^2 + |dy|^2 = 2$, so we conclude that $i_k = 2\iota_{\frac{\partial}{\partial z_k}}$, where $\iota_{\frac{\partial}{\partial z_k}}$ is the "interior product" operator, defined by $\iota_{\frac{\partial}{\partial z_k}}\eta = \eta(\frac{\partial}{\partial z_k}, \cdot, \dots, \cdot)$. Similarly, $\bar i_k = 2\iota_{\frac{\partial}{\partial \bar z_k}}$. It is easy to check that

$$i_k e_k + e_k i_k = 2, \quad \bar i_k \bar e_k + \bar e_k \bar i_k = 2.$$

And for $k \neq l$,

$$e_k i_l + i_l e_k = 0, \quad \bar{e}_k \bar{i}_l + \bar{i}_l \bar{e}_k = 0.$$

We also define the degree-preserving linear maps $\partial_k, \bar{\partial}_k$ by

$$\partial_k\Big(\sum_{I,J} \eta_{I\bar{J}} dz_I \wedge d\bar{z}_J\Big) := \sum_{I,J} \frac{\eta_{I\bar{J}}}{\partial z_k} dz_I \wedge d\bar{z}_J,$$

$$\bar{\partial}_k\Big(\sum_{I,J} \eta_{I\bar{J}} dz_I \wedge d\bar{z}_J\Big) := \sum_{I,J} \frac{\eta_{I\bar{J}}}{\partial \bar{z}_k} dz_I \wedge d\bar{z}_J.$$

Note that ∂_k and $\bar{\partial}_k$ commute with $e_k, \bar{e}_k$ and hence also $i_k, \bar{i}_k$. Also an "integration by part" trick gives us the relation $\partial_k^* = -\bar{\partial}_k$, $\bar{\partial}_k^* = -\partial_k$.

Now we can express all the operators we care by $e_k, \bar{e}_k, i_k, \bar{i}_k$ and $\partial_k, \bar{\partial}_k$:

$$\partial = \sum_k \partial_k e_k = \sum_k e_k \partial_k, \quad \bar{\partial} = \sum_k \bar{\partial}_k \bar{e}_k = \sum_k \bar{e}_k \bar{\partial}_k.$$

Taking adjoints, we get

$$\partial^* = -\sum_k \bar{\partial}_k i_k = \sum_k i_k \bar{\partial}_k, \quad \bar{\partial}^* = -\sum_k \partial_k \bar{i}_k = -\sum_k \bar{i}_k \partial_k.$$

Also

$$L = \frac{\sqrt{-1}}{2} \sum_k e_k \bar{e}_k, \quad \Lambda = -\frac{\sqrt{-1}}{2} \sum_k \bar{i}_k i_k.$$

So we can compute

$$\begin{aligned} \Lambda\partial &= -\frac{\sqrt{-1}}{2} \sum_{k,l} \bar{i}_k i_k \partial_l e_l = -\frac{\sqrt{-1}}{2} \sum_{k,l} \partial_l \bar{i}_k i_k e_l \\ &= -\frac{\sqrt{-1}}{2} \Big(\sum_k \partial_k \bar{i}_k i_k e_k + \sum_{k \neq l} \partial_l \bar{i}_k i_k e_l \Big). \end{aligned}$$

We compute the last two summands seperately.

$$\begin{aligned} -\frac{\sqrt{-1}}{2} \sum_k \partial_k \bar{i}_k i_k e_k &= -\frac{\sqrt{-1}}{2} \sum_k \partial_k \bar{i}_k (2 - e_k i_k) \\ &= -\sqrt{-1} \sum_k \partial_k \bar{i}_k - \frac{\sqrt{-1}}{2} \sum_k \partial_k e_k \bar{i}_k i_k \\ &= \sqrt{-1} \bar{\partial}^* - \frac{\sqrt{-1}}{2} \sum_k \partial_k e_k \bar{i}_k i_k, \end{aligned}$$

and

$$-\frac{\sqrt{-1}}{2}\sum_{k\neq l}\partial_l\bar{i}_k i_k e_l \quad = \quad \frac{\sqrt{-1}}{2}\sum_{k\neq l}\partial_l\bar{i}_k e_l i_k = -\frac{\sqrt{-1}}{2}\sum_{k\neq l}\partial_l e_l\bar{i}_k i_k.$$

So we get

$$\Lambda\partial = \sqrt{-1}\bar{\partial}^* - \frac{\sqrt{-1}}{2}\sum_{k}\partial_k e_k\bar{i}_k i_k - \frac{\sqrt{-1}}{2}\sum_{k\neq l}\partial_l e_l\bar{i}_k i_k = \sqrt{-1}\bar{\partial}^* + \partial\Lambda.$$

For the general compact Kähler case, one can use Kähler normal coordinates to reduce the computations to our $\mathbb{C}^n$ case. The key point is that only first order derivatives are involved. □

5.4.2 Hodge decomposition for compact Kähler manifolds

A direct consequence of Hodge identities is that Δ_d commutes with both L and Λ: Since ω_g is closed, we have $dL(\eta) = d(\omega_g\wedge\eta) = \omega_g\wedge d\eta$, so $[L, d] = 0$. Taking adjoints, we get $[\Lambda, d^*] = 0$. So using $[\Lambda, d] = -d^{c*}$, we get

$$\begin{aligned}
\Lambda\Delta_d &= \Lambda(dd^* + d^*d) = [\Lambda, d]d^* + d\Lambda d^* + d^*\Lambda d\\
&= -d^{c*}d^* + dd^*\Lambda + d^*[\Lambda, d] + d^*d\Lambda\\
&= -d^{c*}d^* - d^*d^{c*} + \Delta_d\Lambda = \Delta_d\Lambda.
\end{aligned}$$

Taking adjoints, we also get $[L, \Delta_d] = 0$.

Besides Δ_d and $\Delta_{\bar{\partial}}$, we can similarly define Δ_∂. For compact Kähler manifolds, we have the following:

Proposition 5.14. *In the Kähler case, we always have $\Delta_{\bar{\partial}} = \Delta_\partial = \frac{1}{2}\Delta_d$.*

Proof. Use $d = \partial + \bar{\partial}$ and $d^* = \partial^* + \bar{\partial}^*$ to compute:

$$\begin{aligned}
\Delta_d &= dd^* + d^*d = (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial})\\
&= (\partial\partial^* + \partial^*\partial) + (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) + \partial\bar{\partial}^* + \bar{\partial}\partial^* + \partial^*\bar{\partial} + \bar{\partial}^*\partial\\
&= \Delta_\partial + \Delta_{\bar{\partial}} + (\partial\bar{\partial}^* + \bar{\partial}^*\partial) + (\bar{\partial}\partial^* + \partial^*\bar{\partial}).
\end{aligned}$$

We need to prove:

- $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$, $\bar{\partial}\partial^* + \partial^*\bar{\partial} = 0$ (these two identities are equivalent by conjugation);
- $\Delta_\partial = \Delta_{\bar{\partial}}$.

To prove $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$, we use the Hodge identity $[\Lambda, \partial] = \sqrt{-1}\bar{\partial}^*$:

$$\begin{aligned}\sqrt{-1}(\partial\bar{\partial}^* + \bar{\partial}^*\partial) &= \partial[\Lambda, \partial] + [\Lambda, \partial]\partial \\ &= \partial\Lambda\partial - \partial^2\Lambda + \Lambda\partial^2 - \partial\Lambda\partial \\ &= 0.\end{aligned}$$

Now we compute Δ_∂ and $\Delta_{\bar{\partial}}$ separately, both using Hodge identities:

$$\begin{aligned}-\sqrt{-1}\Delta_\partial &= \partial[\Lambda, \bar{\partial}] + [\Lambda, \bar{\partial}]\partial \\ &= \partial\Lambda\bar{\partial} - \partial\bar{\partial}\Lambda + \Lambda\bar{\partial}\partial - \bar{\partial}\Lambda\partial.\end{aligned}$$

$$\begin{aligned}\sqrt{-1}\Delta_{\bar{\partial}} &= \bar{\partial}[\Lambda, \partial] + [\Lambda, \partial]\bar{\partial} \\ &= \bar{\partial}\Lambda\partial - \bar{\partial}\partial\Lambda + \Lambda\partial\bar{\partial} - \partial\Lambda\bar{\partial} \\ &= \bar{\partial}\Lambda\partial + \partial\bar{\partial}\Lambda - \Lambda\bar{\partial}\partial - \partial\Lambda\bar{\partial} \\ &= \sqrt{-1}\Delta_\partial.\end{aligned}$$

From the above computations, we conclude that $\Delta_d = \Delta_\partial + \Delta_{\bar{\partial}} = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$. □

From this we conclude that $\Delta_d : A^{p,q}(X) \to A^{p,q}(X)$, and

$$\mathcal{H}_d^{p+q}(X, \mathbb{C}) \cap A^{p,q}(X) = \mathcal{H}_{\bar{\partial}}^{p,q}(X).$$

Since $\mathcal{H}_d^r(X, \mathbb{C}) = \oplus_{p+q=r}\left(\mathcal{H}_d^{p+q}(X, \mathbb{C}) \cap A^{p,q}(X)\right) = \oplus_{p+q=r}\mathcal{H}_{\bar{\partial}}^{p,q}(X)$. Also note that $\overline{\mathcal{H}_{\bar{\partial}}^{p,q}(X)} = \mathcal{H}_{\bar{\partial}}^{q,p}(X)$. Applying Hodge theorem for compact Hermitian manifolds, we get:

Theorem 5.15 (Hodge decomposition for compact Kähler manifolds)**.** *Let (X^n, J, g) be a compact Kähler manifold, then we have isomorphisms*

$$H_{dR}^r(X, \mathbb{C}) \cong \oplus_{p+q=r}H_{\bar{\partial}}^{p,q}(X) \cong \oplus_{p+q=r}H^q(X, \Omega^p), \quad r = 0, 1, \ldots, 2n,$$

and

$$\overline{H_{\bar{\partial}}^{p,q}(X)} \cong H_{\bar{\partial}}^{q,p}(X).$$

In particular, we have

$$b_r = \sum_{p+q=r} h^{p,q}, \quad h^{p,q} = h^{q,p}.$$

For example, we always have

$$\mathcal{H}_{\bar{\partial}}^{p,0}(X) = H^0(X, \Omega^p),$$

since any $(p, 0)$-form is $\bar{\partial}^*$-closed and it is $\bar{\partial}$-closed if and only if it is holomorphic. Then we conclude that *any holomorphic p-form on a compact Kähler manifold is also d-closed and even d-harmonic.*

Exercise: Show that any holomorphic 1-form on a compact complex surface (not necessarily Kähler) is always d-closed. (Kodaira)

Corollary 5.16. *The odd Betti number b_{2k+1} of a compact Kähler manifold X^n is always even.*

Proof. We have

$$\begin{aligned} b_{2k+1} &= \sum_{0\le p,q\le n, p+q=2k+1} h^{p,q} \\ &= \sum_{p<q,p+q=2k+1} h^{p,q} + \sum_{p>q,p+q=2k+1} h^{p,q} \\ &= \sum_{p<q,p+q=2k+1} h^{p,q} + \sum_{p>q,p+q=2k+1} h^{q,p} \\ &= 2 \sum_{p<q,p+q=2k+1} h^{p,q} \equiv 0 \mod 2. \end{aligned}$$

□

As a concrete application, let's compute the cohomologies of $\mathbb{C}P^n$: The topological structure is rather simple: we have $\mathbb{C}P^n = U_0 \cup \{z_0 = 0\}$, with $U_0 \cong \mathbb{C}^n$ and $\{z_0 = 0\} \cong \mathbb{C}P^{n-1}$. So we can construct $\mathbb{C}P^n$ in the following way: start with a point (a "0-cell"), glue a $\mathbb{C}^1$ (a "2-cell") to get $\mathbb{C}P^1$, then glue a $\mathbb{C}^2$ (a "4-cell") to get $\mathbb{C}P^2$, ... So the cellular cohomologies of $\mathbb{C}P^n$ are:

$$H^{2k+1}(\mathbb{C}P^n, \mathbb{Z}) = 0, \quad H^{2k}(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}, k = 0, \dots, n.$$

Now ω_{FS} is a Kähler forms on $\mathbb{C}P^n$. Since $\omega_{FS}^k = L^k 1$ and $\Delta_d L = L\Delta_d$, each ω_{FS}^k is a harmonic (k,k)-form. So we conclude that $h^{p,p} \ge 1, p = 0, \dots, n$. On the other hand, $1 = b_{2p} \ge h^{p,p}$, we must have $b_{2p} = h^{p,p}$. Also, $h^{p,q} = 0$ when $p+q$ is odd. So the only non-zero Dolbeault cohomologies of $\mathbb{C}P^n$ are $H^{p,p}_{\bar\partial}(X) \cong \mathbb{C}, p = 0, \dots, n$. In particular, there are no non-zero holomorphic forms on $\mathbb{C}P^n$.

For another application, we state the so called "$\partial\bar\partial$-lemma", which is very useful in Kähler geometry:

Lemma 5.17. *If η is any d-closed (p,q)-form on a compact Kähler manifold X^n, and η is d- or ∂- or $\bar\partial$-exact, then*

$$\eta = \partial\bar\partial\gamma$$

for some $(p-1, q-1)$-form γ. When $p = q$ and η is real, then we can take $\gamma = \sqrt{-1}\xi$ for a real $(p-1, q-1)$-form ξ.

Proof. Recall that in the Kähler case we have $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$, they share the same kernel: harmonic forms. Since η is d- or ∂- or $\bar\partial$-exact, its harmonic projection must be zero. So we have

$$\eta = \Delta_{\bar\partial} G_{\bar\partial}\eta = \bar\partial\bar\partial^* G_{\bar\partial}\eta.$$

Here we use the fact that $\bar\partial$ commutes with $G_{\bar\partial}$ and that $d\eta = 0 \Rightarrow \bar\partial\eta = 0$.

Now we look at the form $\bar\partial^* G_{\bar\partial}\eta$, it is also orthogonal to harmonic forms. Also since $G_\partial = G_{\bar\partial}$, we have $\partial\bar\partial^* G_{\bar\partial}\eta = -\bar\partial^*\partial G_\partial\eta = -\bar\partial^* G_\partial\partial\eta = 0$. Then we can use Hodge decomposition for Δ_∂:

$$\bar\partial^* G_{\bar\partial}\eta = \Delta_\partial G_\partial \bar\partial^* G_{\bar\partial}\eta = \partial\partial^* G_\partial\bar\partial^* G_{\bar\partial}\eta.$$

So we get

$$\eta = \bar\partial\partial\partial^* G_\partial\bar\partial^* G_{\bar\partial}\eta = \partial\bar\partial\Big(-\partial^* G_\partial\bar\partial^* G_{\bar\partial}\eta\Big) = \partial\bar\partial\Big(-\partial^*\bar\partial^* G_{\bar\partial}^2\eta\Big).$$

□

The most often used case is about $(1,1)$-class. Let ω and $\tilde\omega$ be two Kähler forms on X such that $[\omega] = [\tilde\omega] \in H^2_{dR}(X)$. Then $\tilde\omega - \omega$ is a $d-$exact form, so by the $\partial\bar\partial$-lemma, we can find a smooth function $\varphi \in C^\infty(X;\mathbb{R})$ such that

$$\tilde\omega = \omega + \sqrt{-1}\partial\bar\partial\varphi.$$

φ is unique up to a constant. On the other hand, if $\varphi \in C^\infty(X;\mathbb{R})$ satisfies $\omega + \sqrt{-1}\partial\bar\partial\varphi > 0$, then it defines a Kähler metric with the same Kähler class. So we conclude that the space of Kähler metrics within the same cohomology class $[\omega]$ is isomorphic to

$$\{\varphi \in C^\infty(X;\mathbb{R}) \mid \omega + \sqrt{-1}\partial\bar\partial\varphi > 0\}\Big/\mathbb{R}.$$

One of the most important problem in Kähler geometry is the existence of canonical metrics in a given Kähler class. Through the $\partial\bar\partial$-lemma, we can reduce the problem to a (usually non-linear) partial differential equation for φ. This is the starting point of using non-linear PDEs to solve problems in Kähler geometry.

Remark 5.18. *If we further introduce the operator $h : A^*(X) \to A^*(X)$ by $h = \sum_{p=0}^{2n}(n-p)\Pi_p$, then we will have*

$$[\Lambda, L] = h, \quad [h,\Lambda] = 2\Lambda, \quad [h, L] = -2L.$$

Recall the 3-dimensional complex Lie algebra sl_2, generated by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

They satisfy

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

So $H \mapsto h, X \mapsto \Lambda, Y \mapsto L$ gives a representation of sl_2 on $\mathcal{H}^*(X,\mathbb{C})$. Using elementary representation theory, we can get a finer decomposition result, due to S. Lefschetz.

6 Kodaira's vanishing theorem and its applications

6.1 Kodaira vanishing theorem

Using Hodge theorem, we can prove an important cohomology vanishing theorem of Kodaira. To state the theorem, we recall the following positivity notions for real $(1,1)$-forms and for line bundles: We say a real $(1,1)$-form ω is "positive" if locally it can be written as $\omega = \sqrt{-1}\sum_{i,j} a_{i\bar{j}} dz_i \wedge d\bar{z}_j$ where $(a_{i\bar{j}})$ is positive definite everywhere. A line bundle L is called "positive" if there exists an Hermitian metric h on L such that $\sqrt{-1}\Theta(h)$ is positive.

Theorem 6.1 (Kodaira-Nakano)**.** *If $L \to X$ is a positive holomorphic line bundle on a compact Kähler manifold,*[8] *then we have*

$$H^q(X, \Omega^p(L)) = 0, \quad \textit{for } p + q > n.$$

In particular, $H^q(X, \mathscr{O}(K_X \otimes L)) = 0$ for $q > 0$.

Proof. (due to Akizuki-Nakano) We use $\omega := \sqrt{-1}\Theta(h)$ as our reference Kähler metric. [9] The Hodge theorem ensures that $H^q(X, \Omega^p(L)) \cong \mathcal{H}^{p,q}(X, L)$. So we need to show that when $p + q > n$ each L-valued harmonic (p,q)-form must be zero.

We need the following lemma, whose proof is almost identical to the "un-twisted case" we proved before:

Lemma 6.2. *Let E be a holomorphic vector bundle over a compact Kähler manifold (X, ω) with Hermitian metric h. Introduce the operator $L : A^{p,q}(X, E) \to A^{p+1,q+1}(X, E)$ as before and define $\Lambda := L^*$. If we denote the $(1,0)$ and $(0,1)$ components of the Chern connection D by D' and $D''(= \bar{\partial})$, then we have*

$$[\Lambda, \bar{\partial}] = -\sqrt{-1}D'^*, \quad [\Lambda, D'] = \sqrt{-1}\bar{\partial}^*.$$

Assuming the lemma at present. Then the proof of Kodaira vanishing theorem essentially follows from the comparison of two "Laplacians", the so called "Bochner's technique":

$$\Delta_{\bar{\partial},E} - \Delta_{D',E} = [\sqrt{-1}\Theta(h), \Lambda],$$

where $\Delta_{D',E} := D'D'^* + D'^*D'$. The reason for this equality is:

$$\begin{aligned} -\sqrt{-1}\Delta_{D',E} &= D'[\Lambda, \bar{\partial}] + [\Lambda, \bar{\partial}]D' \\ &= D'\Lambda\bar{\partial} - D'\bar{\partial}\Lambda + \Lambda\bar{\partial}D' - \bar{\partial}\Lambda D', \end{aligned}$$

[8] We can just assume X is compact complex manifold. Then if $\sqrt{-1}\Theta(h) > 0$, then it is a Kähler form on X and so X is in fact Kähler. Later, by Kodaira's embedding theorem, X is in fact projective algebraic.

[9] In this case, $[\omega] = 2\pi c_1(L)$. In general, if we have a compact Kähler manifold (X, J, g) such that $[\omega_g] = 2\pi c_1(L)$ (or $c_1(L)$) for some holomorphic line bundle L, then we call the triple (X, L, g) a *"polarized manifold"*. L is called *"the polarizing line bundle"* or *"the polarization"*.

while

$$\begin{aligned}\sqrt{-1}\Delta_{\bar\partial,E} &= \bar\partial[\Lambda, D'] + [\Lambda, D']\bar\partial \\ &= \bar\partial\Lambda D' - \bar\partial D'\Lambda + \Lambda D'\bar\partial - D'\Lambda\bar\partial.\end{aligned}$$

So we get

$$\sqrt{-1}\Delta_{\bar\partial,E} - \sqrt{-1}\Delta_{D',E} = \Lambda(\bar\partial D' + D'\bar\partial) - (\bar\partial D' + D'\bar\partial)\Lambda.$$

Note that $\Theta(h)$ is of type $(1,1)$, we get $D'D' = 0$, $\bar\partial\bar\partial = 0$, so

$$\Theta(h) = D^2 = (D' + \bar\partial)(D' + \bar\partial) = D'\bar\partial + \bar\partial D'.$$

So we get

$$\Delta_{\bar\partial,E} - \Delta_{D',E} = -\sqrt{-1}[\Lambda, \Theta(h)] = [\sqrt{-1}\Theta(h), \Lambda].$$

Now back to the proof of Kodaira's vanishing theorem. We have $\sqrt{-1}\Theta(h) = \omega$, so the above Bochner formula reduces to

$$\Delta_{\bar\partial} - \Delta_{D'} = [L, \Lambda] = (p + q - n)id.$$

So if $s \in \mathcal{H}^{p,q}(X, L)$ is not identically zero, then we have

$$(\Delta_{\bar\partial}s - \Delta_{D'}s, s) = (p + q - n)\|s\|^2 > 0.$$

On the other hand,

$$(\Delta_{\bar\partial}s - \Delta_{D'}s, s) = -(\Delta_{D'}s, s) = -\|D's\|^2 - \|D'^*s\|^2 \leq 0.$$

This is a contradiction. $\square$

6.2 The embedding theorem

One important application of the Kodaira vanishing theorem is the following embedding theorem of Kodaira:

Theorem 6.3. *If a compact complex manifold X has a positive line bundle, then it is projective algebraic.*

The basic construction we shall use is the following: Let $L \to X$ be a holomorphic line bundle, such that $H^0(X, \mathcal{O}(L)) \neq 0$. Then we can take a basis of $H^0(X, \mathcal{O}(L))$, $s_0, \ldots, s_N$, and define a "map" from X to $\mathbb{C}P^N$:

$$x \mapsto [s_0(x), \ldots, s_N(x)].$$

This is defined using a local trivialization, so that we can identify each s_i as a locally defined holomorphic function. This map is independent of the trivialization we choose, but it is un-defined on the "base locus" of L: [10]

$$Bs(L) := \{x \in X \mid s(x) = 0, \quad \forall s \in H^0(X, \mathscr{O}(L))\}.$$

What Kodaira actually proved is the following: If $L \to X$ is a positive line bundle on a compact complex manifold, then *we can find a large integer $m_0 > 0$ such that for all $m > m_0$:*

1. *$L^{\otimes m}$ is "base point free", i.e. $Bs(L^{\otimes m}) = \emptyset$;*
2. *Choose a basis of $H^0(X, \mathscr{O}(L^{\otimes m}))$, $s_0, \ldots, s_{N_m}$, then the "Kodaira map" $\iota_{L^m} : X \to \mathbb{C}P^{N_m}$ defined by*
$$x \mapsto [s_0(x), \ldots, s_{N_m}(x)]$$
is a holomorphic embedding.

Definition 6.4. *Let $L \to X$ be a holomorphic line bundle on a compact complex manifold.*

- *If there is an integer $m_0 > 0$ such that for all $m > m_0$, $L^{\otimes m}$ is base point free, then we say L is* semi-ample*;*
- *If L is base point free and the Kodaira map ι_L is a holomorphic embedding, then we say L is* very ample*;*
- *If there is an integer $m_0 > 0$ such that for all $m > m_0$, $L^{\otimes m}$ is very ample, then we say L is* ample*.*

A corollary of Kodaira's theorem is that on a compact complex manifold, a holomorphic line bundle is ample if and only if it is positive.

In fact, if L is positive, then it is ample by Kodaira's theorem. On the other hand, if L is ample, we can find $m \in \mathbb{N}$ such that ι_{L^m} is a holomorphic embedding. Then the pulling back of the hyperplane bundle is isomorphic to $L^{\otimes m}$, and the induced metric has positive curvature. The corresponding metric on L also has positive curvature.

Outline of the proof of Kodaira embedding theorem: For simplicity, we only prove that there is a sufficiently large m such that ι_{L^m} is an embedding. We need to prove the following 3 properties:

1. Prove that $L^{\otimes m}$ is base point free when m large enough. We only need to show that for any point $p \in X$, we can find a $m_p \in \mathbb{N}$ such that for all $m \geq m_p$, we can find a $s \in H^0(X, \mathscr{O}(L^{\otimes m}))$ such that $s(p) \neq 0$. That is, the linear map $r_p : H^0(X, \mathscr{O}(L^{\otimes m})) \to L_p^{\otimes m}$ is surjective.

[10] In fact, one can suitably extend the map to codimension one part of $Bs(L)$.

2. Prove that for m large, global sections of $L^{\otimes m}$ separate points. For this, we need to prove that for any two points $p \neq q$ in X, the linear map $r_{p,q} : H^0(X, \mathscr{O}(L^{\otimes m})) \to L_p^{\otimes m} \oplus L_q^{\otimes m}$ is surjective for m sufficiently large.

3. Prove that for m large, ι_{L^m} is an immersion. That is, for any point $p \in X$, global sections of $L^{\otimes m}$ separate tangent directions at p. We only need to show the linear map $r_{p,p} : H^0(X, \mathscr{O}(L^{\otimes m})) \to L_p^{\otimes m} \otimes (\mathscr{O}_p/\mathfrak{m}_p^2)$ is surjective for m sufficiently large.

Note that property 2 is stronger than property 1. So we only need to prove 2 and 3. Also note that if we denote by $\mathfrak{m}_p$ the ideal sheaf of holomorphic germs vanishing at p and $\mathfrak{m}_{p,q}$ the ideal sheaf of holomorphic germs vanishing at p and q, then what we need prove is that

$$H^0(X, \mathscr{O}(L^{\otimes m})) \to H^0(X, \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_{p,q})$$

and

$$H^0(X, \mathscr{O}(L^{\otimes m})) \to H^0(X, \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_p^2)$$

are both surjective when m is large enough.

For this, we use short exact sequences of sheaves:

$$0 \to \mathfrak{m}_{p,q} \to \mathscr{O} \to \mathscr{O}/\mathfrak{m}_{p,q} \to 0, \qquad 0 \to \mathfrak{m}_p^2 \to \mathscr{O} \to \mathscr{O}/\mathfrak{m}_p^2 \to 0.$$

Tensor with the locally free sheaf $\mathscr{O}(L^{\otimes m})$, we get exact sequences

$$0 \to \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_{p,q} \to \mathscr{O}(L^{\otimes m}) \to \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_{p,q} \to 0$$

and

$$0 \to \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_p^2 \to \mathscr{O}(L^{\otimes m}) \to \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_p^2 \to 0.$$

The induced long exact sequences give us:

$$H^0(X, \mathscr{O}(L^{\otimes m})) \to H^0(X, \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_{p,q}) \to H^1(X, \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_{p,q})$$

and

$$H^0(X, \mathscr{O}(L^{\otimes m})) \to H^0(X, \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_p^2) \to H^1(X, \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_p^2).$$

We need to prove the vanishing of $H^1(X, \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_{p,q})$ and $H^1(X, \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_p^2)$.

Comparing with Kodaira's vanishing theorem, we found that the main problem is that $\mathfrak{m}_{p,q}$ and $\mathfrak{m}_p^2$ are not sheaves of germs of holomorphic line bundles. They are examples of "coherent analytic sheaves". This "generalized Kodaira vanishing theorem" for coherent analytic sheaves is indeed true, but harder to prove. Kodaira's method (as appeared in Griffiths-Harris and Wells) is to replace X by its blown-up $\tilde{X}$ at p and q. Pulling everything back to $\tilde{X}$ we can work purely with line bundles, and then Kodaira's vanishing theorem works. Then one need to show that vanishing upstairs implies vanishing downstairs.

Finally, since both property 2 and 3 are "open" properties, we can use a "finite covering trick" to find a uniform m, independent of $p, q \in X$. $\square$

In short, the proof says that positivity of a line bundle L implies $L^{\otimes m}$ has so many global sections that they can separate points and tangent directions. Here we use Kodaira's cohomology vanishing to prove the existence of global sections satisfying special properties. This is typical when applying vanishing theorems. Also, to prove the existence of global sections separating points and tangent directions, one can directly construct sections by solving $\bar{\partial}$-equations using Hörmander's L^2-method. It turns out that we also need a certain type of Bochner type identity, and the positivity of the line bundle is also crucial.

7 Calabi-Yau theorem

7.1 Calabi's problem and Aubin-Yau, Calabi-Yau theorem

Recall that $\Lambda^n T^{1,0}X =: K_X^{-1}$ is the anticanonical line bundle, and g induced an Hermitian metric on K_X^{-1}, with $|\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}|_g^2 := \det(g_{i\bar{j}})$, its curvature form is exactly $\bar{\partial}\partial \log \det(g_{i\bar{j}})$. So we get

$$\sqrt{-1}\Theta(K_X^{-1}, \det g) = Ric(\omega_g),$$

and by Chern's theorem,

$$[Ric(\omega_g)] = 2\pi c_1(K_X^{-1}) =: 2\pi c_1(X).$$

Calabi asked the following questions:

1. Given a Kähler metric g and a closed (1,1)-form η such that its cohomology class in $H^2_{dR}(X)$ is $[\eta] = 2\pi c_1(X)$, can we find another Kähler metric g' within the same Kähler class $[\omega_g]$ such that $Ric(\omega_{g'}) = \eta$?

2. When can we find a Kähler metric which is at the same time an Einstein metric? That is, $Ric(\omega_g) = \lambda\omega_g$ for a constant $\lambda \in \mathbb{R}$. We call such a metric an Kähler-Einstein metric.

Recall that by $\partial\bar{\partial}$-lemma, different Kähler metrics in the same Kähler class differ by $\sqrt{-1}\partial\bar{\partial}\varphi$ for a $\mathbb{R}$-valued function φ. So Calabi's problems actually ask whether we can find smooth function φ satisfying a specific equation.

Also recall that for a real $(1,1)$-form $\eta = \sqrt{-1}\eta_{i\bar{j}} dz_i \wedge d\bar{z}_j$, we say it is positive (write $\eta > 0$), if the matrix $(\eta_{i\bar{j}})$ is positive definite everywhere. And we say a real (1,1)-class $\alpha \in H^2_{dR}(X)$ is positive if we can find a closed $\eta > 0$ such that $[\eta] = \alpha$.

First, observe that:

Lemma 7.1. *If the compact Kähler manifold (X, J, g) is Einstein, then either $c_1(X) > 0$ or $c_1(X) < 0$ or $c_1(X) = 0$.*

Also observe that the Ricci form is invariant under rescaling, so for the Kähler-Einstein problem, we can assume $\lambda = 1, -1$ or 0.

The results we discuss in this chapter are:

Theorem 7.2 (Aubin-Yau)**.** *If X is compact Kähler manifold with $c_1(X) < 0$, then there is a unique Kähler metric g satisfying*

$$Ric(\omega_g) = -\omega_g.$$

Theorem 7.3 (Calabi-Yau theorem). *If X is compact Kähler manifold with a given Kähler metric g_0, then given any closed $(1,1)$-form η such that $[\eta] = 2\pi c_1(X)$, there is a unique Kähler metric g with $[\omega_g] = [\omega_{g_0}]$ satisfying*

$$Ric(\omega_g) = \eta.$$

In particular, if $c_1(X) = 0$, then for any Kähler class α, there is a unique Ricci-flat Kähler metric in the class α. A Ricci-flat Kähler metric is usually called a "Calabi-Yau metric" in the literature.

However, when $c_1(X) > 0$ (then we say "X is a Fano manifold" in honor of the Italian algebraic geometer Fano), in general we can not find Kähler-Einstein metrics, due to various obstructions, like the vanishing of Futaki invariant and the reductiveness of the automorphism group of X. The ultimate result is:

Theorem 7.4 (Chen-Donaldson-Sun, Tian). *Let X be a compact Kähler manifold with $c_1(X) > 0$. Then X admits a Kähler-Einstein metric if and only if X is K-polystable.*

I won't explain the meaning of K-stability here. For the original definition, we refer the readers to Tian's 1997 Invent. Math. paper. The uniqueness problem of positive Kähler-Einstein metrics is also very difficult, and first solved by Bando-Mabuchi. There is a recent proof by B. Berndtsson, using ideas from complex Brunn-Minkowski inequalities.

Now we derive the equation and prove the uniqueness part.

For Aubin-Yau theorem, we start with a g_0 such that its Kähler form $\omega \in -2\pi c_1(X) = -[Ric(\omega)]$, so we can apply the $\partial\bar{\partial}$-lemma to get a smooth function h satisfying $Ric(\omega) + \omega = \sqrt{-1}\partial\bar{\partial}h$, and h is unique if we require $\int_X e^h\omega^n = \int_X \omega^n$. We want to find $\varphi \in C^2(X;\mathbb{R})$ s.t. $\omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0$ and $Ric(\omega_\varphi) + \omega_\varphi = 0$, i.e.,

$$0 = -\partial_i\partial_{\bar{j}} \log\det(g_{p\bar{q}} + \varphi_{p\bar{q}}) + g_{i\bar{j}} + \varphi_{i\bar{j}} = -\partial_i\partial_{\bar{j}}\Big(\log\frac{\det(g_{p\bar{q}} + \varphi_{p\bar{q}})}{\det(g_{p\bar{q}})} - h - \varphi\Big).$$

So we get the equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h+\varphi}\omega^n. \tag{7.1}$$

For Calabi-Yau theorem, we have a unique h satisfying $Ric(\omega) - \eta = \sqrt{-1}\partial\bar{\partial}h$ and $\int_X e^h\omega^n = \int_X \omega^n$. We want to find φ such that $\omega_\varphi > 0$ and $Ric(\omega_\varphi) = \eta$, i.e.

$$-\partial_i\partial_{\bar{j}} \log\det(g_{p\bar{q}} + \varphi_{p\bar{q}}) = -\partial_i\partial_{\bar{j}} \log\det(g_{p\bar{q}}) - h_{i\bar{j}}.$$

So the equation is

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^h\omega^n. \tag{7.2}$$

Lemma 7.5 (Calabi). *The solutions to* (7.1) *and* (7.2) *are both unique.*

Proof. If both φ_1 and φ_2 solve (7.1), set $\psi := \varphi_2 - \varphi_1$. Then ψ satisfies $(\omega_1 + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{\psi}\omega_1^n$. At the maximum point of ψ, we have $e^{\psi}\omega_1^n \leq \omega_1^n$, so $\psi \leq 0$. Similarly, we get $\psi \geq 0$, hence $\psi \equiv 0$.

If both φ_1 and φ_2 solve (7.2), set $\psi := \varphi_2 - \varphi_1$. Then ψ satisfies an elliptic equation of the form $L\psi = 0$, with $L = A^{i\bar{j}}(z, \partial^2\varphi_1, \partial^2\varphi_2)\partial_i\partial_{\bar{j}}$. Since ψ must achieve its maximum and minimum somewhere, by strong maximum principle, ψ is a constant, and the corresponding metrics are the same. □

7.2 Proof of (Aubin-)Calabi-Yau theorem

We start with the Aubin-Yau theorem. The idea of proof is to use the so called "continuity method", introduced in the first half of 20th century by H. Weyl.

We introduce an extra parameter t into (7.1):

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{th+\varphi}\omega^n. \tag{7.3}$$

Then we study the set $S := \{t \in I = [0,1] \mid (7.3) \text{ is solvable in } C^{k,\alpha}(X)\}$. Obviously $0 \in S$, since in this case $\varphi \equiv 0$ is a solution. Then we try to show S is both open and closed. By connectness of I, we will get $1 \in S$, i.e. (7.1) is solvable.

To show the openness, we shall use the implicit function theorem in Banach spaces. We consider the operator $\Psi : I \times C^{k,\alpha}(X) \to C^{k-2,\alpha}$, where

$$\Psi(t,\varphi) := \log \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega^n} - \varphi - th.$$

Then we have

$$D_\varphi\Psi(\psi) = g_\varphi^{\bar{j}i}\partial_i\partial_{\bar{j}}\psi - \psi = (\Delta_\varphi - 1)\psi.$$

This is invertible by Fredholm alternative, since we can easily prove its injectivity, either use maximum principle or integration by parts. So we get the openness of S.

To prove the closedness, we shall derive *a priori* estimates: if $t_i \in S$ with solution $\varphi_i \in C^{k,\alpha}(X)$ and $t_i \to t_0 \in I$, we need to show that $\|\varphi_i\|_{k,\alpha} \leq C$ with a uniform constant C. Then we can find a converging subsequence in $C^{k,\alpha}$. If $k \geq 2$, then we will get a solution for t_0 and S must be closed.

The C^0 estimate of φ is rather direct: if

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{th+\varphi}\omega^n$$

and φ achieves its maximum at $p \in X$. Then

$$e^{th(p)+\max\varphi}\omega^n(p) \leq \omega^n(p),$$

so $\varphi \leq \|h\|_\infty$. Similarly, we get $\varphi \geq -\|h\|_\infty$, so $\|\varphi\|_\infty \leq \|h\|_\infty$. This is already known to Calabi.

We shall not prove C^1 estimate directly, (which is not simple, and first proved directly by Blocki, more than 30 years later than Yau's work) but use C^2 estimates.

The C^2 estimate is due independently to Aubin and Yau, with slightly different calculations.

We denote by $\Delta := g^{\bar{j}i}\partial_i\partial_{\bar{j}}$ and $\Delta_\varphi := g_\varphi^{\bar{j}i}\partial_i\partial_{\bar{j}}$. Since $(g_{i\bar{j}}+\varphi_{i\bar{j}})$ is positive definite, taking trace with respect to ω, we have $0 < g^{\bar{j}i}(g_{i\bar{j}} + \varphi_{i\bar{j}}) =: tr_\omega\omega_\varphi = n + \Delta\varphi$. Now we compute $\Delta_\varphi tr_\omega\omega_\varphi$ at a point p, using Kähler normal coordinates of g at p. Note that at this point, we have $R_{i\bar{j}k\bar{l}} = -\partial_i\partial_{\bar{j}} g_{k\bar{l}}$, so we have

$$\begin{aligned}
\Delta_\varphi tr_\omega\omega_\varphi &= g_\varphi^{\bar{j}i}\partial_i\partial_{\bar{j}}(g^{\bar{l}k}g_{\varphi,k\bar{l}}) = g_\varphi^{\bar{j}i}\partial_i(g^{\bar{l}k}\frac{\partial g_{\varphi,k\bar{l}}}{\partial\bar{z}_j} - g^{\bar{l}p}g^{\bar{q}k}\frac{\partial g_{p\bar{q}}}{\partial\bar{z}_j}g_{\varphi,k\bar{l}}) \\
&= g_\varphi^{\bar{j}i}g^{\bar{l}k}\frac{\partial^2 g_{\varphi,k\bar{l}}}{\partial z_i\partial\bar{z}_j} + g_\varphi^{\bar{j}i}g^{\bar{l}p}g^{\bar{q}k}R_{i\bar{j}p\bar{q}}g_{\varphi,k\bar{l}} \\
&= g_\varphi^{\bar{j}i}g^{\bar{l}k}\big(-R(g_\varphi)_{i\bar{j}k\bar{l}} + g_\varphi^{\bar{q}p}\varphi_{p\bar{l}\bar{j}}\varphi_{k\bar{q}i}\big) + g_\varphi^{\bar{j}i}g^{\bar{l}p}g^{\bar{q}k}R_{i\bar{j}p\bar{q}}g_{\varphi,k\bar{l}} \\
&= -tr_\omega Ric(\omega_\varphi) + g_\varphi^{\bar{j}i}g^{\bar{l}k}g_\varphi^{\bar{q}p}\varphi_{p\bar{l}\bar{j}}\varphi_{k\bar{q}i} + g_\varphi^{\bar{j}i}g^{\bar{l}p}g^{\bar{q}k}R_{i\bar{j}p\bar{q}}g_{\varphi,k\bar{l}}.
\end{aligned}$$

So we get

$$\begin{aligned}
\Delta_\varphi \log tr_\omega\omega_\varphi &= g_\varphi^{\bar{j}i}\partial_i\frac{\partial_{\bar{j}}tr_\omega\omega_\varphi}{tr_\omega\omega_\varphi} = \frac{\Delta_\varphi tr_\omega\omega_\varphi}{tr_\omega\omega_\varphi} - \frac{|\partial tr_\omega\omega_\varphi|_\varphi^2}{(tr_\omega\omega_\varphi)^2} \\
&= \frac{1}{tr_\omega\omega_\varphi}\Big(-tr_\omega Ric(\omega_\varphi) + g_\varphi^{\bar{j}i}g^{\bar{l}p}g^{\bar{q}k}R_{i\bar{j}p\bar{q}}g_{\varphi,k\bar{l}}\Big) \\
&\quad + \frac{(tr_\omega\omega_\varphi)g_\varphi^{\bar{j}i}g^{\bar{l}k}g_\varphi^{\bar{q}p}\varphi_{p\bar{l}\bar{j}}\varphi_{k\bar{q}i} - |\partial tr_\omega\omega_\varphi|_\varphi^2}{(tr_\omega\omega_\varphi)^2}.
\end{aligned}$$

Claim: We always have $(tr_\omega\omega_\varphi)g_\varphi^{\bar{j}i}g^{\bar{l}k}g_\varphi^{\bar{q}p}\varphi_{p\bar{l}\bar{j}}\varphi_{k\bar{q}i} - |\partial tr_\omega\omega_\varphi|_\varphi^2 \geq 0$.

To see this, recall that we work under a Kähler normal coordinate system. By an extra linear coordinate change, we can further assume that $\varphi_{i\bar{j}} = \lambda_i\delta_{ij}$, with $\lambda_i \in \mathbb{R}$ and $1+\lambda_i > 0$. So at this point, we have $g_{\varphi,i\bar{j}} = (1+\lambda_i)\delta_{ij}$ and $g_\varphi^{\bar{j}i} = \frac{\delta_{ij}}{1+\lambda_i}$, and so $tr_\omega\omega_\varphi = \sum_i(1+\lambda_i)$, and $g_\varphi^{\bar{j}i}g^{\bar{l}k}g_\varphi^{\bar{q}p}\varphi_{p\bar{l}\bar{j}}\varphi_{k\bar{q}i} = \sum_{i,p,k}\frac{1}{1+\lambda_i}\frac{1}{1+\lambda_p}|\varphi_{i\bar{p}k}|^2$. So we have

$$\begin{aligned}
|\partial tr_\omega\omega_\varphi|_\varphi^2 &= \sum_i \frac{1}{1+\lambda_i}|\partial_i(g^{\bar{l}k}g_{\varphi,k\bar{l}})|^2 = \sum_i\frac{1}{1+\lambda_i}|g^{\bar{l}k}\partial_i g_{\varphi,k\bar{l}}|^2 \\
&= \sum_i\frac{1}{1+\lambda_i}|\sum_k\varphi_{k\bar{k}i}|^2 = \sum_i\frac{1}{1+\lambda_i}|\sum_k\sqrt{1+\lambda_k}\frac{\varphi_{k\bar{k}i}}{\sqrt{1+\lambda_k}}|^2 \\
&\leq \sum_i\frac{1}{1+\lambda_i}\Big(\sum_k(1+\lambda_k)\Big)\Big(\sum_p\frac{|\varphi_{p\bar{p}i}|^2}{1+\lambda_p}\Big) = (tr_\omega\omega_\varphi)\sum_{i,p}\frac{1}{1+\lambda_i}\frac{|\varphi_{p\bar{p}i}|^2}{1+\lambda_p} \\
&\leq (tr_\omega\omega_\varphi)\sum_{i,p,k}\frac{1}{1+\lambda_i}\frac{1}{1+\lambda_p}|\varphi_{k\bar{p}i}|^2.
\end{aligned}$$

Lemma 7.6. *Let ω be a Kähler metric on a compact Kähler manifold X and $\varphi \in C^4(X;\mathbb{R})$ satisfies $\omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0$, then*

$$\Delta_\varphi \log tr_\omega\omega_\varphi \geq \frac{-tr_\omega Ric(\omega_\varphi)}{tr_\omega\omega_\varphi} - C tr_{\omega_\varphi}\omega. \tag{7.4}$$

Proof. By the above discussions, we have

$$\begin{aligned}
\Delta_\varphi \log tr_\omega\omega_\varphi &\geq \frac{1}{tr_\omega\omega_\varphi}\Big(- tr_\omega Ric(\omega_\varphi) + g_\varphi^{\bar{j}i} g^{\bar{l}p} g^{\bar{q}k} R_{i\bar{j}p\bar{q}} g_{\varphi,k\bar{l}}\Big) \\
&= \frac{-tr_\omega Ric(\omega_\varphi)}{tr_\omega\omega_\varphi} + \frac{1}{tr_\omega\omega_\varphi}\sum_{i,k}\frac{1+\lambda_k}{1+\lambda_i} R_{i\bar{i}k\bar{k}} \\
&\geq \frac{-tr_\omega Ric(\omega_\varphi)}{tr_\omega\omega_\varphi} + \frac{\inf_{i,k} R_{i\bar{i}k\bar{k}}}{tr_\omega\omega_\varphi}\sum_{i,k}\frac{1+\lambda_k}{1+\lambda_i} \\
&= \frac{-tr_\omega Ric(\omega_\varphi)}{tr_\omega\omega_\varphi} + \inf_{i,k} R_{i\bar{i}k\bar{k}} tr_{\omega_\varphi}\omega.
\end{aligned}$$

Since X is compact, we can find $C > 0$ such that $\inf_{i,k} R_{i\bar{i}k\bar{k}} \geq -C$. □

Note that we haven't use the equation! So the above computation applies to other situations.

Now we rewrite the equation (7.3) as

$$\begin{aligned}
Ric(\omega_\varphi) &= Ric(\omega) - t\sqrt{-1}\partial\bar{\partial}h - \sqrt{-1}\partial\bar{\partial}\varphi \\
&= Ric(\omega) - t(Ric(\omega) + \omega) - (\omega_\varphi - \omega) \\
&= (1-t)(Ric(\omega) + \omega) - \omega_\varphi.
\end{aligned}$$

So $-tr_\omega Ric(\omega_\varphi) \geq tr_\omega\omega_\varphi - C$. So we conclude that

$$\Delta_\varphi \log tr_\omega\omega_\varphi \geq 1 - C\Big(\frac{1}{tr_\omega\omega_\varphi} + tr_{\omega_\varphi}\omega\Big) \geq 1 - C' tr_{\omega_\varphi}\omega.$$

The last step used the fact $\frac{1}{tr_\omega\omega_\varphi} = \frac{1}{\sum_i(1+\lambda_i)} \leq \frac{1}{1+\lambda_1} \leq tr_{\omega_\varphi}\omega$.

On the other hand, we have

$$\Delta_\varphi\varphi = tr_{\omega_\varphi}(\omega_\varphi - \omega) = n - tr_{\omega_\varphi}\omega,$$

and so we get

$$\Delta_\varphi\big(\log tr_\omega\omega_\varphi - (C'+1)\varphi\big) \geq -C'' + tr_{\omega_\varphi}\omega.$$

At the maximum point of $\log tr_\omega\omega_\varphi - (C'+1)\varphi$, we have $tr_{\omega_\varphi}\omega \leq C''$. Use Kähler normal coordinates at that point and assume g_φ is diagonal as before, we get $\frac{1}{1+\lambda_i} \leq C''$ for each i. By (7.3), we have $\Pi_i(1+\lambda_i) = e^{th+\varphi} \leq C_0$, which implies $1+\lambda_i \leq C_0(C'')^{n-1}$. So $tr_\omega\omega_\varphi \leq nC_0(C'')^{n-1}$. This implies at this point $\log tr_\omega\omega_\varphi - (C'+1)\varphi$ is uniformly bounded from above (use $|\varphi| \leq ||h||_{C^0}$). This in turn implies $tr_\omega\omega_\varphi \leq C$ for a uniform constant C.

Since we have L^∞ control of $\Delta\varphi$, using L^p theory for linear elliptic equations, we get uniform control of C^1-norm for φ.

Also a direct consequence of the $\Delta\varphi$ estimate is that there is a uniform constant $C > 0$ such that $\frac{1}{C}\omega \leq \omega_\varphi \leq C\omega$.

After obtaining C^2 estimates, there are two ways to get higher order estimates. The original approach of Aubin and Yau used Calabi's 3rd order estimates, and then use Schauder estimates and then bootstrapping. Later, Evans and Krylov independently discovered that the $C^{2,\alpha}$ estimate follows directly from the C^2 estimate. The basic idea is that if we differentiate the equation in the tangent direction γ 2-times, we will get an elliptic equation for $u_{\gamma\gamma}$. The above estimate implies that we have uniform control for the ellipticity constants. Then we can get Harnack inequality for $u_{\gamma\gamma}$ by exploring the concavity structure of the complex Monge-Ampère operator. One can find the proof for real fully nonlinear equations in chapter 17 of Gilbarg-Trudinger's book. The adaptation to the complex Monge-Ampère equation has been carried out by Siu in his book [5].

After obtaining $C^{2,\alpha}$ control of φ, we can differentiate the equation once, then the coefficients have uniform Hölder norm, so we can use Schauder estimates and then bootstrapping. This finishes the proof to Theorem 7.2.

Now we study the Calabi-Yau equation.

First, we need a continuity path for the equation (7.2):

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{th+c_t}\omega^n, \tag{7.5}$$

where c_t is a constant defined by $\int_X e^{th+c_t}\omega^n = \int_X \omega^n$. Again let

$$S := \{t \in I \mid (7.5) \text{ is solvable in } C_0^{k,\alpha}\},$$

where we define $C_0^{k,\alpha} := \{\varphi \in C^{k,\alpha}(X) \mid \int_X \varphi\omega^n = 0\}$. When $t = 0$, $\varphi \equiv 0$ is the solution. So $S \neq \emptyset$. To show S is open, we use the implicit function theorem. However, there is additional difficulty caused by the change of c_t, so we modify the function spaces in Aubin-Yau's theorem.

We define the affine subspace of $C^{k-2,\alpha}$:

$$C_V^{k-2,\alpha} := \{f \in C^{k,\alpha}(X) \mid \int_X e^f\omega^n = \int_X \omega^n\}.$$

Then we define the operator $\Phi : C_0^{k,\alpha} \to C_V^{k-2,\alpha}$,

$$\Phi(\varphi) := \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega^n}.$$

The linearization at φ_{t_0} is $D\Phi_{\varphi_{t_0}} : C_0^{k,\alpha} \to C_0^{k-2,\alpha}$

$$D\Phi_{\varphi_{t_0}}(\psi) = \frac{\omega^n_{\varphi_{t_0}}}{\omega^n}\Delta_{\varphi_{t_0}}\psi.$$

This operator is invertible since $\Delta_{\varphi_{t_0}}\psi = f$ is solvable if and only if $\int_X f\omega^n_{\varphi_{t_0}} = 0$. This proves the openness.

For closedness, as before, we need to derive *a priori* estimates. Only the C^0 estimate is different, other parts are almost identical.

We will basically follow Yau's original proof using Moser iteration. Later there are other proofs, e.g. S. Kolodziej's approach using pluripotential theory and Z. Blocki's proof using Alexandrov's maximum principles. Our exposition follows [4].

Rewrite the equation as $(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = F\omega^n$ with $F = e^{th+c_t}$. Note that F has uniform positive upper and lower bounds, independent of t. Set $\psi := \sup_X \varphi - \varphi + 1 \geq 1$. Since

$$(F-1)\omega^n = (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n - \omega^n = \sqrt{-1}\partial\bar{\partial}\varphi \wedge \sum_{j=0}^{n-1} \omega_\varphi^{n-j-1} \wedge \omega^j,$$

we multiply $\psi^{\alpha+1}$ on both sides for some $\alpha \geq 0$, and integrate over X:

$$\begin{aligned}
\int_X \psi^{\alpha+1}(F-1)\omega^n &= (\alpha+1)\sum_{j=0}^{n-1}\int_X \psi^\alpha \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \wedge \omega_\varphi^{n-j-1} \wedge \omega^j \\
&\geq (\alpha+1)\int_X \psi^\alpha \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \wedge \omega^{n-1} \\
&= \frac{\alpha+1}{(\frac{\alpha}{2}+1)^2}\int_X \sqrt{-1}\partial\psi^{\frac{\alpha}{2}+1} \wedge \bar{\partial}\psi^{\frac{\alpha}{2}+1} \wedge \omega^{n-1} \\
&= \frac{\alpha+1}{(\frac{\alpha}{2}+1)^2}\|\nabla\psi^{\frac{\alpha}{2}+1}\|^2.
\end{aligned}$$

So we get

$$\|\nabla\psi^{\frac{\alpha}{2}+1}\|^2 \leq C_1\frac{(\frac{\alpha}{2}+1)^2}{\alpha+1}\int_X \psi^{\alpha+1}\omega^n,$$

where C_1 depends only on $\|F\|_{L^\infty}$.

On the other hand, we have Sobolev inequality

$$\|u\|^2_{L^{\frac{2n}{n-1}}} \leq C_2(\|\nabla u\|^2_{L^2} + \|u\|^2_{L^2}).$$

We apply this to $u := \psi^{\frac{p}{2}}$:

$$\|\psi\|^p_{L^{p\beta}} \leq C_2(\|\nabla\psi^{\frac{p}{2}}\|^2_{L^2} + \|\psi\|^p_{L^p}),$$

where $\beta := \frac{n}{n-1} > 1$. Then we choose $p = \alpha + 2$, to get

$$\|\psi\|_{L^{p\beta}} \leq (C_3 p)^{\frac{1}{p}}\|\psi\|_{L^p}, \quad p \geq 2.$$

Then we can iterate $p \to p\beta \to p\beta^2 \to \cdots \to p\beta^k \to \ldots$. Using the fact that $\lim_{k\to\infty}\|\psi\|_{L^{p\beta^k}} = \|\psi\|_{L^\infty}$, we conclude that once we have a uniform L^p bound for ψ for some $p \geq 2$, then we will have uniform L^∞ estimate for ψ.

To get this L^p bound, one can use, for example, the following result of G. Tian: Given a Kähler form ω, we can find a positive number $c > 0$, depending only on the Kähler class, such that we can find another uniform constant $C > 0$ such that

$$\int_X e^{-c(\varphi - \sup_X \varphi)} \omega^n \leq C,$$

$\forall \varphi \in C^\infty(X; \mathbb{R})$ such that $\omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0$. From this, we get uniform estimate of $\|\psi\|_{L^k}$ for any $k \in \mathbb{N}$.

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