Introduction to complex geometry
(Chapter 0 Preliminaries on complex analysis)

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Let Ω be a domain of $\mathbb{C}$ with piecewise C^1 boundary. $f = u + \sqrt{-1}v : \Omega \to \mathbb{C}$ be a C^1 map. We generally regard this as a complex-valued function. It is usually convenient to introduce the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\Big).$$

Then it is easy to see that

$$df := \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}.$$

Then it is direct to check that the Cauchy-Riemann equation can be expressed as

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Now we assume f is only C^1, not necessarily holomorphic. For any $z \in \Omega$, let $\Delta(z,\epsilon)$ be a small disc with center z and radius ϵ. By Green formula (written in the form of differential forms), we have

$$\begin{aligned}\int_{\partial(\Omega\setminus\Delta(z,\epsilon))} \frac{f(w)}{w-z}dw &= \int_{\Omega\setminus\Delta(z,\epsilon)} d\Big(\frac{f(w)}{w-z}dw\Big)\\ &= \int_{\Omega\setminus\Delta(z,\epsilon)} d\Big(\frac{f(w)}{w-z}\Big)\wedge dw\\ &= \int_{\Omega\setminus\Delta(z,\epsilon)} \frac{\frac{\partial f}{\partial \bar{w}}(w)}{w-z}d\bar{w}\wedge dw.\end{aligned}$$

In polar coordinates around z, the final integrand is in fact bounded, so we can let $\epsilon \to 0$ to get

$$\int_{\Omega} \frac{\frac{\partial f}{\partial \bar{w}}(w)}{w-z}d\bar{w}\wedge dw.$$

On the other hand, we have

$$\begin{aligned}\int_{\partial(\Omega\setminus\Delta(z,\epsilon))} \frac{f(w)}{w-z}dw &= \int_{\partial\Omega} \frac{f(w)}{w-z}dw - \int_{|w-z|=\epsilon} \frac{f(w)}{w-z}dw\\ &= \int_{\partial\Omega} \frac{f(w)}{w-z}dw - \sqrt{-1}\int_0^{2\pi} f(z+\epsilon e^{\sqrt{-1}\theta})d\theta\\ &\to \int_{\partial\Omega} \frac{f(w)}{w-z}dw - 2\pi\sqrt{-1}f(z).\end{aligned}$$

So we finally get

$$f(z) = \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Omega}\frac{f(w)}{w-z}dw + \frac{1}{2\pi\sqrt{-1}}\int_{\Omega}\frac{\frac{\partial f}{\partial \bar{z}}(w)}{w-z}dw\wedge d\bar{w}. \tag{0.1}$$

One direct corollary is the following solution formula for 1-dimensional $\bar{\partial}$-equation:

Lemma 0.1. *Let $f \in C_0^\infty(\mathbb{C})$ be a complex-valued function, then the function defined by*

$$u(z) := \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{C}}\frac{f(w)}{w-z}dw\wedge d\bar{w}$$

is a smooth function on $\mathbb{C}$ and satisfies the equation

$$\frac{\partial u}{\partial \bar{z}} = f.$$

Proof. Assume $\mathrm{supp} f \subset \Delta(0,R)$, then for any $z \in \Delta(0,R')$ we have

$$\begin{aligned} u(z) &= \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{C}}\frac{f(z+w)}{w}dw\wedge d\bar{w} \\ &= \frac{1}{2\pi\sqrt{-1}}\int_{\Delta(0,R+R')}\frac{f(z+w)}{w}dw\wedge d\bar{w}. \end{aligned}$$

We can taking derivative with respect to $\bar{z}$ under the integration sign to get

$$\begin{aligned} \frac{\partial u}{\partial \bar{z}}(z) &= \frac{1}{2\pi\sqrt{-1}}\int_{\Delta(0,R+R')}\frac{\frac{\partial f}{\partial \bar{z}}(z+w)}{w}dw\wedge d\bar{w} \\ &= \frac{1}{2\pi\sqrt{-1}}\int_{\Delta(0,R)}\frac{\frac{\partial f}{\partial \bar{z}}(w)}{w-z}dw\wedge d\bar{w}. \end{aligned}$$

By (0.1), this equals

$$f(z) - \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta(0,R)}\frac{f(w)}{w-z}dw = f(z),$$

since $\mathrm{supp} f \subset \Delta(0,R)$. □

Now we use the following conventions: $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, with $z_i = x_i + \sqrt{-1}y_i$, and

$$|z|^2 := |z_1|^2 + \cdots + |z_n|^2 = x_1^2 + y_1^2 + \cdots + x_n^2 + y_n^2.$$

For multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, we write

$$z^\alpha := z_1^{\alpha_1} \ldots z_n^{\alpha_n},$$

with $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\alpha! := \alpha_1! \cdots \alpha_n!$.

In one variable complex analysis, the unit disc plays a special role. The higher dimensional generalizations are balls and polydiscs:

- A *complex ball* with center $\mathbf{a} = (a_1, \ldots, a_n)$ and radius $r > 0$ is defined by

$$B(\mathbf{a}, r) := \{z \in \mathbb{C}^n \big| \, |z - \mathbf{a}| < r\}.$$

  This is nothing but the Euclidean ball in $\mathbb{R}^{2n}$.

- A *polydic* with center $\mathbf{a} = (a_1, \ldots, a_n)$ and multi-radius $\mathbf{r} = (r_1, \ldots, r_n)$ with $r_i > 0, \forall i = 1, \ldots, n$ is defined by

$$\Delta(\mathbf{a}, \mathbf{r}) := \{z \in \mathbb{C}^n \big| \, |z_i - a_i| < r_i, \forall i = 1, \ldots, n\}.$$

  This is the product of n 1-dimensional discs. When all the r_i equal $r > 0$, we usually abuse the notation to write it as $\Delta(\mathbf{a}, r)$.

Definition 0.2. *Let $\Omega \subset \mathbb{C}^n$ be a non-empty open set (we call it a "region"), $f = u + \sqrt{-1}v : \Omega \to \mathbb{C}$ a C^1 map. We call f a* holomorphic function*, denoted by $f \in \mathscr{O}(\Omega)$, if f satisfies the Cauchy-Riemann equation:*

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_i}, \quad \frac{\partial u}{\partial y_i} = -\frac{\partial v}{\partial x_i}, \quad i = 1, \ldots, n. \tag{0.2}$$

This is equivalent to say f is holomorphic in each of its complex variables.

Remark 0.3. *By a deep theorem of Hartogs, we can remove the C^1 assumption in the above definition. For a proof, see Hörmander's book.*

As in the one-variable case, we introduce

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\Big(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\Big), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\Big(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\Big).$$

We also define

$$\partial := \sum_{i=1}^n \frac{\partial}{\partial z_i} dz_i, \quad \bar{\partial} := \sum_{i=1}^n \frac{\partial}{\partial \bar{z}_i} d\bar{z}_i,$$

then it is direct to check that $df = \partial f + \bar{\partial} f$, and f is holomorphic if and only if $\bar{\partial} f = 0$.

A 1-form of the form

$$\varphi = \varphi_1 dz_1 + \cdots + \varphi_n dz_n$$

with φ_i functions on Ω is called a $(1,0)$-form on Ω, and a 1-form of the form

$$\eta = \eta_1 d\bar{z}_1 + \cdots + \eta_n d\bar{z}_n$$

with η_i functions on Ω is called a $(0,1)$-form on Ω. A central technique in the modern theory of complex analysis is to use the $\bar{\partial}$-equation $\bar{\partial} u = \eta$ with good estimates to construct holomorphic objects.

As in one variable case, Cauchy formula is very important in several complex variables:

Theorem 0.4 (Cauchy formula)**.** *If $f \in \mathscr{O}(\Delta(\mathbf{a},\mathbf{r})) \cap C^0(\overline{\Delta(\mathbf{a},\mathbf{r})})$, then we have*

$$f(z) = \Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n \int_{|\zeta_1|=r_1} \cdots \int_{|\zeta_n|=r_n} \frac{f(\zeta_1,\cdots,\zeta_n)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n, \quad \forall z \in \Delta(\mathbf{a},\mathbf{r}). \quad (0.3)$$

In particular, $f \in C^\infty(\Delta(\mathbf{a},\mathbf{r}))$.

Proof. If f is holomorphic in a neighborhood of $\overline{\Delta(\mathbf{a},\mathbf{r})}$, then (0.3) follows from repeating use of 1-dimensional Cauchy formula. In general, we work on $\Delta(\mathbf{a},\theta\mathbf{r})$ for $0 < \theta < 1$ and let $\theta \to 1$.

The last claim follows from Cauchy formula by taking derivatives with respect to z under the integration sign. □

Remark 0.5. *An interesting feature of this formula is that the interior value of f depends only on its value on a part of the boundary. We write*

$$\partial_0\Delta(\mathbf{a},\mathbf{r}) := \{z \mid |z_i - a_i| = r_i, i = 1,\ldots,n\}.$$

It is called "characteristic boundary" or "distinguished boundary" or "Shilov boundary" of $\Delta(\mathbf{a},\mathbf{r})$. If f is a given continuous function in a neighborhood of $\partial_0\Delta(\mathbf{a},\mathbf{r})$, then the integral (0.3) *defines a holomorphic function in $\Delta(\mathbf{a},\mathbf{r})$, since it is easy to see that the function is C^1 in z, and we can take derivatives under the integration sign.*

A direct corollary is the following useful derivative estimate:

Corollary 0.6 (Cauchy estimate)**.** *If $f \in \mathscr{O}(\Omega)$ and $\Delta(\mathbf{a},\mathbf{r}) \subset \Omega$, then we have*

$$|\partial^\alpha f(a)| \leq \frac{\alpha!}{\mathbf{r}^\alpha} \sup_{\Delta(\mathbf{a},\mathbf{r})} |f|. \quad (0.4)$$

Moreover, if $K \subset \Omega$ is compact, then for any relatively compact open neighborhood U, we have

$$\sup_K |\partial^\alpha f| \leq C_\alpha \sup_U |f|, \quad \forall f \in \mathscr{O}(\Omega),$$

where C_α is a constant depending only on α, K and U.

Proof. Again if $\overline{\Delta(\mathbf{a},\mathbf{r})} \subset \Omega$, by (0.3), we have for any $z \in \Delta(\mathbf{a},\mathbf{r})$:

$$\partial^\alpha f(z) = \frac{\alpha!}{(2\pi\sqrt{-1})^n} \int_{\partial_0\Delta(\mathbf{a},\mathbf{r})} \frac{f(\zeta_1,\cdots,\zeta_n)}{(\zeta_1-z_1)^{\alpha_1+1}\cdots(\zeta_n-z_n)^{\alpha_n+1}} d\zeta_1\cdots d\zeta_n. \tag{0.5}$$

This implies that

$$|\partial^\alpha f(a)| \le \frac{\alpha!}{(2\pi)^n} \sup_{\Delta(\mathbf{a},\mathbf{r})} |f| \Big(\Pi_i(2\pi r_i)\Big)\Big(\Pi_i \frac{1}{r_i^{\alpha_i+1}}\Big) = \frac{\alpha!}{\mathbf{r}^\alpha} \sup_{\Delta(\mathbf{a},\mathbf{r})} |f|.$$

Again if $\overline{\Delta(\mathbf{a},\mathbf{r})} \not\subset \Omega$, we work on $\Delta(\mathbf{a},\theta\mathbf{r})$ for $0<\theta<1$ and let $\theta \to 1$.

The second statement follows directly from (0.4) by a compactness argument. □

Remark 0.7. *By* (0.4)*, we can bound* $\frac{\partial f}{\partial z_i}$ *by* $\sup|f|$. *On the other hand, we always have* $\frac{\partial f}{\partial \bar{z}_i} = 0$*, so we can bound the real partial derivatives of f by* $\sup|f|$.

There are several interesting corollaries of Cauchy formula and Cauchy estimates:

We say a series of functions $\sum_i f_i$ converges normally to f in a domain Ω, if it converges uniformly and absolutely on any compact subset $K \subset \Omega$ to f. Then we have

Corollary 0.8. *If $f \in \mathscr{O}(\Delta(\mathbf{a},\mathbf{r}))$, then we can expand f into a power series, converging normally in* $\Delta(\mathbf{a},\mathbf{r})$*:*

$$f(z) = \sum_\alpha \frac{\partial^\alpha f(\mathbf{a})}{\alpha!}(z-\mathbf{a})^\alpha, \quad \forall z \in \Delta(\mathbf{a},\mathbf{r}).$$

Proof. For any compact subset $K \subset \Delta(\mathbf{a},\mathbf{r})$, we can find a $\theta \in (0,1)$ such that $K \subset \Delta(\mathbf{a},\theta\mathbf{r})$. So we can assume without loss of generality that f is holomorphic in a neighborhood of $\overline{\Delta(\mathbf{a},\mathbf{r})}$. Also we assume $\mathbf{a} = 0$. Note that we have

$$\frac{1}{\Pi_i(\zeta_i - z_i)} = \frac{1}{\Pi_i\zeta_i}\cdot\frac{1}{1-\frac{z_1}{\zeta_1}}\cdots\frac{1}{1-\frac{z_n}{\zeta_n}} = \frac{1}{\Pi_i\zeta_i}\sum_{\alpha\in\mathbb{Z}^n_{\ge 0}}\frac{z^\alpha}{\zeta^\alpha},$$

with the right hand side converging normally in $\Delta(0,\mathbf{r})$ when $\zeta \in \partial_0\Delta(0,\mathbf{r})$. So we have by Cauchy formula:

$$\begin{aligned} f(z) &= \Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n \int_{\partial_0\Delta(0,\mathbf{r})} \frac{f(\zeta_1,\cdots,\zeta_n)}{(\zeta_1-z_1)\cdots(\zeta_n-z_n)} d\zeta_1\cdots d\zeta_n \\ &= \Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n \int_{\partial_0\Delta(0,\mathbf{r})} \frac{f(\zeta_1,\cdots,\zeta_n)}{\zeta_1\cdots\zeta_n}\sum_\alpha \frac{z^\alpha}{\zeta^\alpha} d\zeta_1\cdots d\zeta_n \\ &= \sum_\alpha c_\alpha z^\alpha, \end{aligned}$$

where

$$c_\alpha = \Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n \int_{\partial_0\Delta(0,\mathbf{r})} \frac{f(\zeta_1,\cdots,\zeta_n)}{\zeta_1^{\alpha_1+1}\cdots\zeta_n^{\alpha_n+1}} d\zeta_1\cdots d\zeta_n.$$

Comparing with (0.5), we have $\partial^\alpha f(0) = \alpha! c_\alpha$. □

Remark 0.9. *It is a good exercise for the readers to develop a theory of power series in more than one variables. The corresponding Abel's lemma also holds. One can find it in Grauert and Fritzsche's book "Several Complex Variables" (GTM38).*

Corollary 0.10 (Weierstrass theorem)**.** *If $\{f_i\} \subset \mathscr{O}(\Omega)$, and f_i converges to a function f, uniformly on any compact subset of Ω , then $f \in \mathscr{O}(\Omega)$.*

Proof. For any $\overline{\Delta(\mathbf{a}, \mathbf{r})} \subset \Omega$, we have Cauchy formula for each f_i. By uniform convergence of f_i, we can take limit inside the integration to get

$$f(z) = \Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n \int_{\partial_0\Delta(\mathbf{a},\mathbf{r})} \frac{f(\zeta_1, \cdots, \zeta_n)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n, \quad \forall z \in \Delta(\mathbf{a}, \mathbf{r}).$$

Since f is continuous and the right hand side of the above formula is holomorphic in z, we conclude that $f \in \mathscr{O}(\Omega)$. □

Corollary 0.11 (Montel theorem)**.** *Let $\{f_\alpha\} \subset \mathscr{O}(\Omega)$. If they are uniformly bounded on any compact subset $K \subset \Omega$, then $\{f_\alpha\}$ is a normal family, i.e., any sequence of $\{f_\alpha\}$ contains a subsequence that converges to a holomorphic function $f \in \mathscr{O}(\Omega)$, uniformly on any compact subset of Ω.*

Proof. By (0.4), we can bound the first order derivatives of f_i uniformly on any compact set $K \subset \Omega$. So $\{f_i\}_{i=1}^\infty$ is equi-continuous. Then the corollary follows from Arzela-Ascoli theorem and Corollary 0.10. □

Corollary 0.12 (Uniqueness theorem)**.** *Let $\Omega \subset \mathbb{C}^n$ be a domain and $f \in \mathscr{O}(\Omega)$. If there is a non-empty open set $U \subset \Omega$ such that $f|_U \equiv 0$, then $f \equiv 0$ in Ω.*

Proof. Define the set

$$N := \{z \in \Omega \big| \, \partial^\alpha f(z) = 0, \forall \alpha \in \mathbb{Z}^n_{\geq 0}\}.$$

By definition it is a closed subset of Ω. By Cauchy formula, N is also open. Since by assumption $N \neq \emptyset$, the connectivity assumption of Ω implies that $N = \Omega$, so $f \equiv 0$. □

Corollary 0.13 (Maximum Principle)**.** *Let $\Omega \subset \mathbb{C}^n$ be a domain. If $f \in \mathscr{O}(\Omega) \cap C^0(\bar{\Omega})$, then*

$$\max_{\bar{\Omega}} |f| = \max_{\partial\Omega} |f|,$$

and $\max |f|$ *can not be achieved at an interior point unless f is a constant.*

Proof. Suppose $\max_{\bar{\Omega}} |f|$ is achieved at $\mathbf{a} \in \Omega$, choose $r > 0$ such that $\overline{\Delta(\mathbf{a}, r)} \subset \Omega$. Repeating the 1-dimensional maximum principle, we conclude that $f|_{\Delta(\mathbf{a},r)} \equiv f(\mathbf{a})$. By Corollary 0.12, $f \equiv f(\mathbf{a})$. □

One of the first examples showing that complex analysis in higher dimensions is drastically different form the 1-dimensional case is the following phenomenon discovered by Hartogs.

Example 0.14 (Hartogs phenomenon)**.** *Define a domain $H \subset \Delta(0,1) \subset \mathbb{C}^2$ by*

$$H := \{(z,w) \in \mathbb{C}^2 \mid |z| < \frac{1}{2}, |w| < 1\} \cup \{(z,w) \in \mathbb{C}^2 \mid |z| < 1, \frac{1}{2} < |w| < 1\}.$$

Then the restriction map $\mathscr{O}(\Delta(0,1)) \to \mathscr{O}(H)$ is always surjective, i.e. any holomorphic functions on H can be continued holomorphically to the larger domain $\Delta(0,1)$. In fact, for $f \in \mathscr{O}(H)$, we choose a $\frac{1}{2} < \beta < 1$, and define

$$\tilde{f}(z,w) := \frac{1}{2\pi\sqrt{-1}} \int_{|\xi|=\beta} \frac{f(z,\xi)}{\xi - w} d\xi, \quad |z| < 1, |w| < \beta.$$

Then by uniqueness, $\tilde{f}$ is independent of β, hence defines a function $\tilde{f} \in \mathscr{O}(\Delta(0,1))$. Again by uniqueness, we have $\tilde{f}|_H = f$.

Note that for a pair of domains $\Omega \subsetneq \Omega' \subset \mathbb{C}$, we can always find a $f \in \mathscr{O}(\Omega)$ such that f can not be continued holomorphically to Ω'. For example, choose any $a \in \partial\Omega \cap \Omega'$, then $\frac{1}{z-a}$ is what we want.

We have another extension theorem, also due to Hartogs:

Theorem 0.15 (Hartogs's extension theorem)**.** *Let K be a compact subset of the open set $\Omega \subset \mathbb{C}^n$. Assume $\Omega \setminus K$ is connected, then any $f \in \mathscr{O}(\Omega \setminus K)$ extends holomorphically to Ω.*

Proof. We need a lemma:

Lemma 0.16. *Let $\eta := \eta_1 d\bar{z}_1 + \dots \eta_n d\bar{z}_n$ be a smooth $(0,1)$-form with compact support on $\mathbb{C}^n$. If*

$$\frac{\partial \eta_i}{\partial \bar{z}_j} = \frac{\partial \eta_j}{\partial \bar{z}_i} \tag{0.6}$$

for any pair $i, j = 1, \dots, n$, then we can always find a smooth function $u \in C_0^\infty(\mathbb{C}^n)$ such that $\bar{\partial} u = \eta$.

Assuming the lemma at present. Choose a real-valued smooth function with compact support $\varphi \in C_0^\infty(\Omega)$ such that φ is identically 1 in a small neighborhood of K. Then $v := (1-\varphi)f$ can be viewed as a smooth function on Ω, vanishing near K and holomorphic outside suppφ. We define a smooth $(0,1)$-form with compact support on $\mathbb{C}^n$ by

$$\eta := \begin{cases} \bar{\partial} v = -f\bar{\partial}\varphi, & \text{on } \Omega, \\ 0, & \text{on } \mathbb{C}^n \setminus \Omega. \end{cases}$$

Then it is easy to see that η satisfies (0.6), so by Lemma0.16, we can find $u \in C_0^\infty(\mathbb{C}^n)$ such that $\bar{\partial} u = \eta$. We define a function F on Ω by

$$F(z) := v(z) - u(z), \quad \forall z \in \Omega.$$

Then we have $\bar{\partial} F = 0$, so $F \in \mathscr{O}(\Omega)$.

Finally we need to check that $F|_{\Omega \setminus K} = f$. Since $\Omega \setminus K$ is connected, by uniqueness theorem, we only need to show that they coincide on an open subset of $\Omega \setminus K$.

Note that u is in fact holomorphic on $\mathbb{C}^n \setminus \mathrm{supp}\varphi$ (which may not be connected). Since it also has compact support, it necessarily vanishes on the unbounded component of $\mathbb{C}^n \setminus \mathrm{supp}\varphi$ by the uniqueness theorem. But the boundary of this unbounded component must belong to $\Omega \setminus K$, so we can find open subset of $\Omega \setminus K$ on which $u = 0$ and $v = f$, thus $F = f$ there. □

Proof of Lemma 0.16: We define

$$u(z) := \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{\eta_1(w_1, z_2, \ldots, z_n)}{w_1 - z_1} dw_1 \wedge d\bar{w}_1.$$

Then it is easy to see that $u \in C^\infty(\mathbb{C}^n)$ and. since η_1 has compact support, vanishes when $|z_2| + \cdots + |z_n|$ is large enough. By Lemma0.1,

$$\frac{\partial u}{\partial \bar{z}_1} = \eta_1.$$

Also, for $k = 2, \ldots, n$, by (0.6)

$$\begin{aligned}
\frac{\partial u}{\partial \bar{z}_k}(z) &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{\partial_{\bar{k}} \eta_1(w_1, z_2, \ldots, z_n)}{w_1 - z_1} dw_1 \wedge d\bar{w}_1 \\
&= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{\partial_{\bar{1}} \eta_k(w_1, z_2, \ldots, z_n)}{w_1 - z_1} dw_1 \wedge d\bar{w}_1 \\
&= \eta_k(z).
\end{aligned}$$

The last equality also uses (0.1) as in the proof of Lemma0.1. So we proved that $\bar{\partial} u = \eta$.

Finally we prove that u has compact support. We already knew that u vanishes when one of $|z_2|, \ldots, |z_n|$ is large enough. Now choose $R > 0$ large enough and apply (0.1) to u as a function of z_2:

$$\begin{aligned}
u(z) &= \frac{1}{2\pi\sqrt{-1}} \int_{|w|<R} \frac{\frac{\partial u}{\partial \bar{z}_2}(z_1, w, z_3, \ldots, z_n)}{w - z_2} dw \wedge d\bar{w} \\
&= \frac{1}{2\pi\sqrt{-1}} \int_{|w|<R} \frac{\eta_2(z_1, w, z_3, \ldots, z_n)}{w - z_2} dw \wedge d\bar{w}.
\end{aligned}$$

From this expression, we conclude that u also vanishes when $|z_1|$ is large enough, hence u has compact support. □

Remark 0.17. *It is interesting to compare Lemma 0.16 with Lemma0.1. One could say that many of the "strange" properties in higher dimensional complex analysis are caused by the fact that we can solve the $\bar{\partial}$-equation with a solution also with* **compact** *support.*

As a direct corollary of Theorem 0.15, we see that all isolated singularities of holomorphic functions with more than one variable are always removable.

Definition 0.18. *Let $U \subset \mathbb{C}^n$ be a domain, then a map $f = (f_1, \ldots, f_m) : U \to \mathbb{C}^m$ is called holomorphic, if all its components are holomorphic, i.e. $f_i \in \mathscr{O}(U), \forall i = 1, \ldots, m$. If f is bijective onto its image and its inverse is also holomorphic, then we say it is biholomorphic*[1]*, and U is biholomorphic to $f(U)$.*

Example 0.19. *If $\Omega \subsetneq \mathbb{C}$ is a simply connected domain, then Ω is biholomorphic to $\Delta(1) \subset \mathbb{C}$. This is the famous "Riemann mapping theorem".*

Example 0.20.

1. *Any polydisc $\Delta(\mathbf{a}, \mathbf{r})$ is biholomorphic to $\Delta(0, 1)$: we can choose the biholomorphic map to be*

$$f(z_1, \ldots, z_n) = \Big(\frac{z_1 - a_1}{r_1}, \ldots, \frac{z_n - a_n}{r_n}\Big).$$

2. *The ball $B(0, 1) \subset \mathbb{C}^n$ is biholomorphic to the unbounded domain*

$$H := \{w \in \mathbb{C}^n \mid Im\, w_n > \sum_{i=1}^{n-1} |w_i|^2\}$$

by the map

$$w = f(z) = \Big(\frac{z_1}{1 + z_n}, \ldots, \frac{z_{n-1}}{1 + z_n}, \sqrt{-1}\frac{1 - z_n}{1 + z_n}\Big).$$

The boundary of H is called the "Heisenberg group", which plays important roles in CR-geometry and harmonic analysis.

Another example showing that complex analysis in higher dimensions is drastically different form the 1-dimensional case is the following theorem discovered by H. Poincaré.

Theorem 0.21 (H. Poincaré)**.** *Let $n \geq 2$, then $B(0, 1) \subset \mathbb{C}^n$ is not biholomorphic to $\Delta(0, 1) \subset \mathbb{C}^n$.*

Proof. I learnt the following proof from the book of R. Narasimhan , where the author attributes the idea to Remmert and Stein. Poincaré's original proof is to show that the groups of automorphisms (means biholomorphic maps onto itself) of these two domains are not isomorphic. For simplicity, we only prove the $n = 2$ case and left the general case to readers.

[1] The we necessarily have $m = n$

Step 1: Suppose we have a biholomorphic map $f(z,w) : \Delta(0,1) \to B(0,1)$. Then for any sequence $\{z_i\} \subset \Delta \subset \mathbb{C}$ with $|z_i| \to 1$, the sequence of one-variable holomorphic functions $g_i(w) = f(z_i, w) : \Delta \to B(0,1)$ is uniformly bounded. By Montel's theorem, we can assume that g_i converges uniformly on compact subsets of Δ to a holomorphic map $g(w) = (g_1(w), g_2(w)) : \Delta \to \overline{B(0,1)}$.

Step 2: We have $|g(w)| \equiv 1$ on Δ.

In fact, if there is a point $w_0 \in \Delta$ such that $g(w_0) \in B(0,1)$. Choose a small $\epsilon > 0$ such that $\overline{B(g(w_0),\epsilon)} \subset B(0,1)$. Since a biholomorphic map is necessarily proper (i.e. the preimage of any compact set is also compact), $f^{-1}(\overline{B(g(w_0),\epsilon)})$ is a compact subset of $\Delta(0,1)$. Since $(z_i, w_0) \to \partial\Delta(0,1)$ as $i \to \infty$, we have $(z_i, w_0) \notin f^{-1}(\overline{B(g(w_0),\epsilon)})$ when i is large enough. This means

$$f(z_i, w_0) \notin \overline{B(g(w_0),\epsilon)}$$

when i is large enough, contradicting the fact $f(z_i, w_0) \to g(w_0)$.

Step 3: From Step 2, we further conclude that $g(w)$ is a constant map, i.e. $g'(w) \equiv (0,0)$.

One way of seeing this is to use the fact that a non-constant holomorphic function in one variable is always an open map. Alternatively, we can compute the derivatives:

$$\begin{aligned} 0 &= \frac{\partial^2 |g(w)|^2}{\partial w \partial \bar{w}} \\ &= \frac{\partial}{\partial w}\Big(g_1(w)\frac{\partial \bar{g}_1}{\partial \bar{w}}(w)\Big) + \frac{\partial}{\partial w}\Big(g_2(w)\frac{\partial \bar{g}_2}{\partial \bar{w}}(w)\Big) \\ &= |g_1'(w)|^2 + |g_2'(w)|^2. \end{aligned}$$

It follows that

$$\lim_{i\to\infty} \frac{\partial f}{\partial w}(z_i, w) = g'(w) \equiv 0.$$

This implies that for each fixed $w \in \Delta$, $\frac{\partial f}{\partial w}(z,w)$, as a function of z, is holomorphic in $\Delta \subset \mathbb{C}$ and continuous on $\bar{\Delta}$ with boundary value 0. By maximum principle we get $\frac{\partial f}{\partial w}(z,w) \equiv 0$ on $\Delta(0,1)$. This implies f is independent of w, contradicts the fact that f is a biholomorphic map. $\square$

Many theorems in multi-variable calculus have "holomorphic" versions, for example, the inverse function theorem and implicit function theorem. Let Ω be a non-empty domain of $\mathbb{C}^n$ and $f : \Omega \to \mathbb{C}^m$ be a holomorphic map. Then we can define the holomorphic Jacobian of f at $z \in \Omega$ to be the $m \times n$ matrix:

$$J_f^{\mathbb{C}}(z) := \frac{\partial(f_1, \ldots, f_m)}{\partial(z_1, \ldots, z_n)} := \Big(\frac{\partial f_i}{\partial z_j}(z)\Big)_{1 \le i \le m, 1 \le j \le n}.$$

Theorem 0.22 (The inverse function theorem)**.** *Let $f : \Omega \to \mathbb{C}^n$ be a holomorphic map and $J_f^{\mathbb{C}}(z_0)$ is non-degenerate for some point $z_0 \in \Omega$, then f has a local holomorphic inverse g in a neighborhood of $f(z_0)$, and we have*

$$J_g^{\mathbb{C}}(f(z)) = J_f^{\mathbb{C}}(z)^{-1}.$$

Proof. We shall first apply the traditional inverse function theorem to get an inverse map. For this, we need to study the real Jacobian of f at z_0:

$$J_f^{\mathbb{R}}(z) := \begin{pmatrix} \Big(\frac{\partial u_i}{\partial x_j}\Big)_{1 \le i,j \le n} & \Big(\frac{\partial u_i}{\partial y_j}(z)\Big)_{1 \le i,j \le n} \\ \Big(\frac{\partial v_i}{\partial x_j}\Big)_{1 \le i,j \le n} & \Big(\frac{\partial v_i}{\partial y_j}(z)\Big)_{1 \le i,j \le n} \end{pmatrix},$$

where we write $z_i = x_i + \sqrt{-1}y_i$ and $f_i = u_i + \sqrt{-1}v_i$.

Claim: For holomorphic f, we have

$$\det J_f^{\mathbb{R}}(z) = |\det J_f^{\mathbb{C}}(z)|^2.$$

The reason is simple. For short, we write

$$J_f^{\mathbb{R}}(z) =: \begin{pmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} \end{pmatrix}.$$

Then the Cauchy-Riemann equation can be written as

$$\frac{\partial U}{\partial X} = \frac{\partial V}{\partial Y}, \quad \frac{\partial U}{\partial Y} = -\frac{\partial V}{\partial X},$$

and hence

$$J_f^{\mathbb{C}} = \frac{\partial f}{\partial Z} = \frac{\partial U}{\partial X} - \sqrt{-1}\frac{\partial U}{\partial Y}.$$

So we have

$$\begin{aligned} \det J_f^{\mathbb{R}} &= \det\begin{pmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} \end{pmatrix} = \det\begin{pmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} \\ -\frac{\partial U}{\partial Y} & \frac{\partial U}{\partial X} \end{pmatrix} \\ &= \det\begin{pmatrix} J_f^{\mathbb{C}} & \frac{\partial U}{\partial Y} \\ -\sqrt{-1}J_f^{\mathbb{C}} & \frac{\partial U}{\partial X} \end{pmatrix} = \det\begin{pmatrix} J_f^{\mathbb{C}} & \frac{\partial U}{\partial Y} \\ 0 & \overline{J_f^{\mathbb{C}}} \end{pmatrix} \\ &= |\det J_f^{\mathbb{C}}|^2. \end{aligned}$$

Now we have $\det J_f^{\mathbb{R}}(z_0) = |\det J_f^{\mathbb{C}}(z_0)|^2 \neq 0$. By classical inverse function theorem, we have a local C^1 inverse of f near $w_0 := f(z_0)$. We write it as $z = g(w)$. We shall prove that it is holomorphic.

In fact, from the identity $w_i = f_i(g_1(w), \ldots, g_n(w))$, we have, by the chain rule,

$$0 = \frac{\partial w_i}{\partial \bar{w}_j} = \sum_k \frac{\partial f_i}{\partial z_k}(g(w)) \frac{\partial g_k}{\partial \bar{w}_j}(w).$$

Since the matrix $J_f^{\mathbb{C}}$ is invertible near z_0, we conclude that $\frac{\partial g_k}{\partial \bar{w}_j} = 0$ for all k, j. So g is holomorphic. Again taking $\frac{\partial}{\partial w_j}$ on both sides of $w_i = f_i(g_1(w), \ldots, g_n(w))$, we get $I_n = J_f^{\mathbb{C}}(g(w)) J_g^{\mathbb{C}}(w)$. □

Theorem 0.23 (The implicit function theorem)**.** *Let $f : \Omega \to \mathbb{C}^m$ be a holomorphic map with $m < n$. Suppose $f(z_0, w_0) = 0$ with $z_0 \in \mathbb{C}^{n-m}, w_0 \in \mathbb{C}^m$ and $(z_0, w_0) \in \Omega$. If*

$$\det \frac{\partial(f_1, \ldots, f_m)}{\partial(z_{n-m+1}, \ldots, z_n)}(z_0, w_0) \neq 0,$$

then we can find a holomorphic map $g : \Delta(z_0, \epsilon) \to \Delta(w_0, \delta) \subset \mathbb{C}^m$ such that $g(z_0) = w_0$ and

$$f(z, g(z)) \equiv 0, \quad \forall z \in \Delta(z_0, \epsilon).$$

Moreover, we have

$$\{(z, w) \in \Delta(z_0, \epsilon) \times \Delta(w_0, \delta) |\, f(z, w) = 0\} = \{(z, w) |\, z \in \Delta(z_0, \epsilon), w = g(z)\}.$$

Proof. There are at least two ways of proof. For example, we can argue as in the inverse function theorem by reducing it to the classical implicit function theorem, or we can consider the map $\tilde{f}(z, w) = (z, f(z, w)) : \Omega \to \mathbb{C}^n$ and apply Theorem 0.22. We leave the detail as an exercise. □

Remark 0.24. *The implicit function theorem says that if a holomorphic map is non-degenerate at a given zero point, then its zero locus is locally a graph near that point. What happens if the Jacobian degenerates at a given point? For example, consider the $m = 1$ case. If a holomorphic function $f(z_1, \ldots, z_{n-1}, w)$ satisfies $\frac{\partial^k f}{\partial w^k}(z_0, w_0) \neq 0$ but $\frac{\partial^i f}{\partial w^i}(z_0, w_0) = 0, \forall i = 0, \ldots, k-1$. What can we say about the zero locus of f near (z_0, w_0)? Weierstrass's "preparation theorem" answers this question. This theorem is fundamental to the local theory of several complex variables.*

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