Introduction to complex geometry
(Chapter 1)

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Abstract

Notes for 2020 BICMR Summer School for Differential Geometry.

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1 Complex manifolds and complex vector bundles

1.1 Complex manifolds

Roughly speaking, a complex manifold is a topological space X on which we can talk about "holomorphic" functions. Since we know what does a holomorphic function means in Euclidean spaces, the first condition we impose on X is:

Condition 1:(existence of coordinate charts) X is locally homeomorphic to open sets of $\mathbb{C}^n$. To be precise, we require that there is an open covering $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ of X such that for each U_i we have a homeomorphism $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{C}^n$ onto an open set $\varphi_i(U_i)$ of $\mathbb{C}^n$.

Given these coordinates, we should define a function $f : \Omega \to \mathbb{C}$ to be holomorphic if all its coordinate-representations $f \circ \varphi_i^{-1} \in \mathscr{O}(\varphi_i(U_i \cap \Omega))$. But is this a well-defined notion? For example if $\Omega \subset U_i \cap U_j \neq \emptyset$, then on Ω we have two sets of coordinates. Is it possible that $f \circ \varphi_i^{-1} \in \mathscr{O}(\varphi_i(U_i \cap \Omega))$ but $f \circ \varphi_j^{-1} \notin \mathscr{O}(\varphi_i(U_j \cap \Omega))$? To avoid this, note that $f \circ \varphi_j^{-1} = (f \circ \varphi_i^{-1}) \circ (\varphi_i \circ \varphi_j^{-1})$, so we require:

Condition 2:(compatibility) Coordinate changes of Condition 1 should be holomorphic. To be precise, we require that whenever $U_i \cap U_j \neq \emptyset$, we have $\varphi_i \circ \varphi_j^{-1}$ is a biholomorphic map from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cap U_j)$.

Given these 2 conditions, one can check easily that the notion of "holomorphic function" makes perfect sense. However, to avoid pathology and use more analytic tools such as metrics and integration, we also require a complex manifold to be a nice topological space:

Condition 3: X satisfies T_2 and C_2 axioms, i.e. X is a Hausdorff space, and has a countable topological basis.

Definition 1.1. *A complex (analytic) manifold of dimension n is a topological space X satisfying Conditions 1,2,3 above. A 1-dimensional complex manifold is also known as a "Riemann surface". A map $f : X \to \mathbb{C}$ from a complex manifold X is called a "holomorphic function", if $f \circ \varphi_i^{-1} \in \mathscr{O}(\varphi_i(U_i))$ for all $i \in \Lambda$. In this case, we write $f \in \mathscr{O}(X)$.*

If X, Y are both complex manifolds of dimensions n and m respectively, a map $F : X \to Y$ is called "holomorphic", if for all coordinate charts (U, φ) of X and (V, ψ) of Y, the map $\psi \circ F \circ \varphi^{-1}$ is a holomorphic map on $\varphi(U \cap F^{-1}(V)) \subset \mathbb{C}^n$ whenever $U \cap F^{-1}(V) \neq \emptyset$. A holomorphic map with a holomorphic inverse is called "biholomorphic".

Remark 1.2. *In standard textbooks, the set of coordinate charts $\{(U_i, \varphi_i)\}_{i\in\Lambda}$ is assumed to be maximal, i.e., whenever a homeomorphism from an open set V, $\psi : V \to \psi(V) \subset \mathbb{C}^n$ is compatible with (U_i, φ_i) for all $U_i \cap V \neq \emptyset$, we have $(V, \psi) \in \{(U_i, \varphi_i)\}_{i\in\Lambda}$. It is easy* *to check that from the coordinate charts in our definition, one can always enlarge it to a unique maximal one satisfying the compatibility condition.*

Example 1.3. *1. Open subsets of $\mathbb{C}^n$ are complex manifolds.*

2. *Let $\{e_1, \dots, e_{2n}\}$ be any fixed $\mathbb{R}$-basis of $\mathbb{C}^n$, and let $\Lambda := \{m_1 e_1 + \cdots + m_{2n} e_{2n} | \, m_i \in \mathbb{Z}\}$ be a lattice of rank $2n$. Then we can define the quotient space $\mathbb{C}^n/\Lambda$, it is a compact Hausdorff space equipped with quotient topology. There is a natural complex manifold structure on $\mathbb{C}^n/\Lambda$, we call this complex manifold a "complex torus".*

3. *Let $P \in \mathbb{C}[z, w]$ be a polynomial of degree d. Define*

$$C := \{(z, w) | \, P(z, w) = 0\}.$$

We call it an "affine plane algebraic curve". Assume P is irreducible and $\frac{\partial P}{\partial z}, \frac{\partial P}{\partial w}$ have no common zeroes on C. Then C is a natural complex manifold. The coordinates can be chosen in the following way: if $\frac{\partial P}{\partial w}(z_0, w_0) \neq 0$, then we can apply the implicit function theorem **??** *to find a neighborhood $\Delta(z_0, \epsilon) \times \Delta(w_0, \delta)$ and a holomorphic function $g(z)$ such that $U := C \cap (\Delta(z_0, \epsilon) \times \Delta(w_0, \delta)) = \{(z, w) | \, z \in \Delta(z_0, \epsilon), w = g(z)\}$. We choose $\varphi : U \to \mathbb{C}$ to be $\varphi(z, w) = z$. If $\frac{\partial P}{\partial z}(z_0, w_0) \neq 0$, we use w as local coordinate. Exercise: what's the coordinates transformation function?*

The last example is a special case of "complex submanifold" we now define:

Definition 1.4. *A closed subset Y of a n-dimensional complex manifold X is called a "complex (analytic) submanifold" of dimension k, if for any $p \in Y$, we can find a compatible chart (U, φ) of X such that $p \in U$ and*

$$\varphi(U \cap Y) = \{(z_1, \dots, z_n) \in \varphi(U) | \, z_{k+1} = \cdots = z_n = 0\}.$$

One can check that the restriction of such charts (we call them "adapted charts") to Y makes Y a complex manifold and the inclusion $Y \subset X$ is a holomorphic map.

Example 1.5 (The complex projective space)**.** *We define an equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$: $(z_0, \dots, z_n) \sim (w_0, \dots, w_n)$ if and only if we can find a non-zero $\lambda \in \mathbb{C}$ (write $\lambda \in \mathbb{C}^*$ for short) such that $w_i = \lambda z_i$ for all $i = 0, \dots, n$. The equivalent class of $(z_0, \dots, z_n)$ is denoted by $[z_0, \dots, z_n]$. The n-dimensional complex projective space $\mathbb{C}P^n$ is defined to be the space of all equivalent classes, endowed with quotient topology. Then it is a compact Hausdorff space. We choose the holomorphic coordinate charts as follows: Define*

$$U_i := \{[z_0, \dots, z_n] \in \mathbb{C}P^n | \, z_i \neq 0\}, \quad i = 0, \dots, n.$$

These are open sets, and we define

$$\varphi_i : U_i \to \mathbb{C}^n, \quad \varphi_i([z_0, \dots, z_n]) := (\frac{z_0}{z_i}, \dots, \frac{\hat{z}_i}{z_i}, \dots, \frac{z_n}{z_i}).$$

The checking of compatibility is left to readers. Also it is easy to check that $\mathbb{C}P^1$ is our familiar S^2.

Let $F_1, \dots, F_k \in \mathbb{C}[z_0, \dots, z_n]$ be a set of irreducible homogeneous polynomials of degrees $d_1, \dots, d_k$ respectively. Then the set

$$V(F_1, \dots, F_k) := \{[z_0, \dots, z_n] | \ F_1(z_0, \dots, z_n) = \dots = F_k(z_0, \dots, z_n) = 0\}$$

is well-defined and is called a (complex) projective algebraic variety. If we assume that $V(F_1, \dots, F_k)$ is a complex submanifold of $\mathbb{C}P^n$, then it will be called a "projective algebraic manifold".

Example 1.6. *If $F \in \mathbb{C}[z_0, \dots, z_n]$ is irreducible and homogeneous of degree d. If we assume that the only common zero of $\frac{\partial F}{\partial z_0}, \dots, \frac{\partial F}{\partial z_n}$ in $\mathbb{C}^{n+1}$ is $(0, \dots, 0)$. Then $V(F)$ is a complex submanifold of dimension $n-1$. We check this on U_0. $V(F) \cap U_0$ is the zero locus of the holomorphic function $F(1, z_1, \dots, z_n) \in \mathscr{O}(U_0)$. We shall show that $\frac{\partial F}{\partial z_1}(1, z_1, \dots, z_n), \dots \frac{\partial F}{\partial z_n}(1, z_1, \dots, z_n)$ have no common zeroes on $V(F) \cap U_0$.*

Suppose

$$F(1, z_1^0, \dots z_n^0) = \frac{\partial F}{\partial z_1}(1, z_1^0, \dots, z_n^0) = \dots = \frac{\partial F}{\partial z_n}(1, z_1^0, \dots, z_n^0) = 0.$$

By Euler's theorem on homogeneous functions, we have

$$\frac{\partial F}{\partial z_0}(1, z_1^0, \dots, z_n^0) + z_1^0 \frac{\partial F}{\partial z_1}(1, z_1^0, \dots, z_n^0) + \dots + z_n^0 \frac{\partial F}{\partial z_n}(1, z_1^0, \dots, z_n^0) = dF(1, z_1^0, \dots, z_n^0) = 0.$$

This implies $\frac{\partial F}{\partial z_0}(1, z_1^0, \dots, z_n^0) = 0$, so $(1, z_1^0, \dots, z_n^0)$ is a common zero of $\frac{\partial F}{\partial z_0}, \dots, \frac{\partial F}{\partial z_n}$ in $\mathbb{C}^{n+1}$ different from $(0, \dots, 0)$.

For example, $V(z_0^d + \dots + z_n^d)$ is a smooth submanifold of $\mathbb{C}P^n$, called the "Fermat hypersurface" of degree d.

A generalization of submanifold is the following:

Definition 1.7. *A closed subset A of a complex manifold X is called an "analytic subvariety", if it is locally the common zeroes of finitely many holomorphic functions, i.e. $\forall p \in A$, there is an open set $U \subset X$ and $f_1, \dots, f_k \in \mathscr{O}(U)$ such that $A \cap U = \{z \in U | \ f_1(z) = \dots = f_k(z) = 0\}$.*

An analytic subvariety A is called a "hypersurface" if it is locally the zero locus of a holomorphic function.

Note that a complex submanifold is an analytic subvariety, we just choose U to be the domain of the adapted chart and f_i to be $z_{k+1}, \dots, z_n$.

Let $A \subset X$ be an analytic subvariety. $p \in A$ is called a *"regular point"*, if we can find open $U \subset X$ and $f_1, \dots, f_k \in \mathscr{O}(U)$ such that $A \cap U = \{z \in U | \ f_1(z) = \dots = f_k(z) = 0\}$ and

$$rank \frac{\partial(f_1, \dots, f_k)}{\partial(z_1, \dots, z_n)}(p) = k.$$

In this case, A is locally near p a complex submanifold of dimension $n-k$: without loss of generality, assume

$$\det \frac{\partial(f_1,\ldots,f_k)}{\partial(z_1,\ldots,z_k)}(p) \neq 0,$$

then we can choose a new compatible coordinate system $(f_1,\ldots,f_k,z_{k+1},\ldots,z_n)$. This is an adapted chart for A near p.

The locus of regular points of A is denoted by A_{reg}. Its complement in A is called the *"singular locus"*, and its elements are called *"singular points of A"*.

Chow's theorem says that any complex analytic subvariety of $\mathbb{C}P^n$ is projective algebraic, i.e., the common zeroes of finitely many homogeneous polynomials.

To end this section, we say something about the existence of complex structures on a given differential manifold. A complex manifold is an even dimensional orientable differential manifold. However, for a given even dimensional oriented manifold, it is not always clear whether or not we can make it a complex manifold. There are topological obstructions to "almost complex structure", this can rule out all even dimensional spheres except S^2 and S^6. We already knew S^2 is a complex manifold. But the S^6 case is still open. In this view, we give an example of complex structures on product of odd dimensional spheres:

Example 1.8 (Calabi-Eckman)**.** *We can make $S^{2p+1}\times S^{2q+1}$ into a complex manifold. The idea is that we can write*

$$S^{2p+1} = \{z \in \mathbb{C}^{p+1} | \sum_{i=0}^{p} |z_i|^2 = 1\}, \quad S^{2q+1} = \{z \in \mathbb{C}^{q+1} | \sum_{j=0}^{q} |z_j|^2 = 1\},$$

and we have the Hopf fibration maps:

$$\pi_p : S^{2p+1} \to \mathbb{C}P^p, \quad \pi_q : S^{2q+1} \to \mathbb{C}P^q,$$

each with fiber S^1. So if we consider the map $\pi = (\pi_p,\pi_q) : S^{2p+1}\times S^{2q+1} \to \mathbb{C}P^p \times \mathbb{C}P^q$, then we can view $S^{2p+1}\times S^{2q+1}$ as a fiber bundle on $\mathbb{C}P^p\times\mathbb{C}P^q$, which is a complex manifold, with fiber $S^1\times S^1 = T^2$, which can also be made a complex manifold.

To be precise, fix a $\tau\in\mathbb{C}$ with $Im\tau > 0$. We donote by T_τ the complex torus $\mathbb{C}/ < 1,\tau >$. Consider the open sets:

$$U_{kj} := \{(z,z') \in S^{2p+1}\times S^{2q+1} | z_k z'_j \neq 0\},$$

and the map $h_{kj} : U_{kj} \to \mathbb{C}^{p+q}\times T_\tau$ given by

$$h_{kj}(z,z') = (\frac{z_0}{z_k},\ldots,\frac{\hat{z_k}}{z_k},\ldots,\frac{z_p}{z_k},\frac{z'_0}{z'_j},\ldots,\frac{\hat{z'_j}}{z'_j},\ldots,\frac{z'_q}{z'_j},t_{kj}),$$

where $t_{kj} := \frac{1}{2\pi\sqrt{-1}}(\log z_k + \tau\log z'_j) \mod < 1,\tau >$. Exercise: check that these charts makes $S^{2p+1}\times S^{2q+1}$ a complex manifold.

A direct application of the maximum principle gives:

Theorem 1.9. *Any holomorphic function on a compact connected complex manifold should be a constant.*

Let M be a complex submanifolds of $\mathbb{C}^n$. Since the restriction of complex coordinate functions of $\mathbb{C}^n$ to M are holomorphic functions on M, we get:

Corollary 1.10. *There are no compact complex submanifolds of $\mathbb{C}^n$ of positive dimension.*

Remark 1.11. *Those non-compact complex manifolds which admit proper holomorphic embeddings into $\mathbb{C}^N$ for some large N are precisely "Stein manifolds" in complex analysis (Remmert's theorem).*

The triumph of this short course is Kodaira's "projective embedding theorem", characterizing those compact complex manfolds which admit holomorphic embeddings into $\mathbb{C}P^N$ for some large N, i.e., projective algebraic manifolds.

1.2 Vector bundles

Roughly speaking, a holomorphic vector bundle over a complex manifold is a family of vector spaces, varying holomorphically.

Definition 1.12. *A holomorphic vector bundle of rank r over a n-dimensional complex manifold X is a complex manifold E of dimension $n+r$, together with a holomorphic surjective map $\pi : E \to X$ satisfying:*

1. *(**Fiberwise linear**) Each fiber $E_p := \pi^{-1}(p)$ has the structure of r-dimensional vector space over $\mathbb{C}$;*
2. *(**Locally trivial**) There is an open cover of X, $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ such that each $\pi^{-1}(U_i)$ is biholomorphic to $U_i \times \mathbb{C}^r$ via $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$, and $E_p \hookrightarrow \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ is a linear isomorphism onto $\{p\} \times \mathbb{C}^r$ for any $p \in U_i$. φ_i is called a "local trivialization".*

In this case, whenever $U_i \cap U_j \neq \emptyset$, we have a holomorphic map, called the "transition map", $\psi_{ij} : U_i \cap U_j \to GL(r, \mathbb{C})$ (viewed as an open subset of $\mathbb{C}^{r^2}$) such that $\varphi_i \circ \varphi_j^{-1}(z, v) = (z, \psi_{ij}(z)v)$. These families of transition maps satisfies the "cocycle condition":

(1) $\psi_{ij}\psi_{ji} = I_r$ on $U_i \cap U_j$;

(2) Whenever $U_i \cap U_j \cap U_k \neq \emptyset$, we have $\psi_{ij}\psi_{jk}\psi_{ki} = I_r$ on $U_i \cap U_j \cap U_k$.

The name "cocycle" is no coincidence. In fact we will see later that $\{\psi_{ij}\}$ above is indeed a cocycle in Čech's approach to sheaf cohomology theory.

Remark 1.13. *On the other hand, if we are given a set of holomorphic transition maps $\psi_{ij} : U_i \cap U_j \to GL(r, \mathbb{C})$ satisfying the cocycle condition, we can construct a holomorphic vector bundle by setting $E = \coprod_{i\in\Lambda}(U_i \times \mathbb{C}^r)/\sim$, where $(z, v) \sim (z', w)$ for $(z, v) \in U_i \times \mathbb{C}^r$ and $(z', w) \in U_j \times \mathbb{C}^r$ if and only if $z = z'$ and $v = \psi_{ij}(z)w$. We leave the detail as an exercise.*

A holomorphic vector bundle of rank 1 is usually called a *"holomorphic line bundle"*.

Definition 1.14 (holomorphic section)**.** *Let $\pi : E \to X$ be a holomorphic vector bundle over X. Let $U \subset X$ be an open set. A holomorphic section of E over U is a holomorphic map $s : U \to E$ such that $\pi \circ s = id_U$, i.e., $s(p) \in E_p$ for any $p \in U$. The set of holomorphic sections over U is usually denoted by $\Gamma(U, \mathscr{O}(E))$ or $\mathscr{O}(E)(U)$.*

One of the fundamental problem for the theory of vector bundles is the construction of global holomorphic sections of a given bundle. An important tool is the L^2-method for the $\bar{\partial}$-equation. One can find the basics from Hörmander's book. It is interesting that whether or not we can solve the equation depends on the geometry, in particular, the curvature of the bundle.

Definition 1.15 (bundle map)**.** *Let $\pi^E : E \to X$ and $\pi^F : F \to X$ are holomorphic vector bundles of ranks r and s respectively. A bundle map from E to F is a holomorphic map $f : E \to F$ such that f maps E_p to F_p for any $p \in X$ and $f|_{E_p} : E_p \to F_p$ is linear. When a bundle map has an inverse bundle map, we will say that these two bundles are isomorphic.*

Another fundamental problem is the classification problem. One important tool is the theory of characteristic classes that we shall discuss later. Also the set of isomorphic classes of holomorphic vector bundles over a given complex manifold has rich structures and is an important invariant for the complex manifold.

Example 1.16 (trivial bundle)**.** *$X \times \mathbb{C}^r$ with $\pi_1 : X \times \mathbb{C}^r \to X$ is a holomorphic vector bundle over X, called the "trivial bundle" over X, denoted by $\underline{\mathbb{C}^r}$.*

Example 1.17 (holomorphic tangent bundle)**.** *Let X be a complex manifold of dimension n. We shall now construct its "holomorphic tangent bundle" TX as follows:*

Let $p \in X$, we first define the ring

$$\mathscr{O}_{X,p} := \varinjlim \mathscr{O}_X(U),$$

where the direct limit is taken with respect to open sets $p \in U$. For persons not familiar with direct limit, this is $\coprod_{U \ni p} \mathscr{O}_X(U)/\sim$, with $f \in \mathscr{O}_X(U)$ equivalent to $g \in \mathscr{O}_X(V)$ iff we can find another open set $p \in W \subset U \cap V$ such that $f|_W = g|_W$. As an exercise, we can see that $\mathscr{O}_{X,p}$ is isomorphic to the ring of convergent power series $\mathbb{C}\{z_1, \dots, z_n\}$. An element of $\mathscr{O}_{X,p}$ is called a "germ of holomorphic function" at p.

A tangent vector at p is a derivation $v : \mathscr{O}_{X,p} \to \mathbb{C}$, i.e., a $\mathbb{C}$-linear map satisfying the Leibniz rule

$$v(fg) = v(f)g(p) + f(p)v(g).$$

The set of tangent vectors at p is easily seen to be a $\mathbb{C}$-vector space. We call it the (holomorphic) tangent space of X at p, denoted by T_pX.

If $\varphi : U_i \to \mathbb{C}^n$ is a holomorphic coordinate chart with $\varphi_i = (z_1, \dots, z_n)$. Then we can define $\frac{\partial}{\partial z_i}|_p \in T_pX$ to be

$$\frac{\partial}{\partial z_i}|_p(f) := \frac{\partial(f \circ \varphi_i^{-1})}{\partial z_i}(\varphi_i(p)).$$

Then one can show that $\{\frac{\partial}{\partial z_i}|_p\}_{i=1}^n$ is a basis of T_pX.

Let $TX := \coprod_{p \in X} T_pX$, and define $\pi : TX \to X$ in the obvious way. We can make it a holomorphic vector bundle of rank n over X as follows: Let (U_i, φ_i) be a holomorphic chart. Then we can define the local trivialization $\tilde{\varphi}_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$ to be

$$\tilde{\varphi}_i(q, \sum_i a_i \frac{\partial}{\partial z_i}|_q) := (q, a_1, \dots, a_n).$$

This gives a complex structure on TX and at the same time gives a local trivialization of TX over U_i.

A holomorphic section of TX over U is called a "holomorphic vector field" *on U.*

Example 1.18 (holomorphic cotangent bundle)**.** *Any $f \in \mathcal{O}_{X,p}$ defines a linear functional on T_pX by $v \mapsto v(f)$. We call this $df|_p \in (T_pX)^* =: T_p^*X$. T_p^*X is called the (holomorphic) cotangent space of X at p. It is easy to see that if (U_i, φ_i) is a holomorphic chart, then $\{dz_i|_p\}_{i=1}^n$ is the basis of T_p^*X dual to $\{\frac{\partial}{\partial z_i}|_p\}_{i=1}^n$.*

We can similarly give $T^*X := \coprod_{p\in X} T_p^*X$ a holomorphic bundle structure, called the "(holomorphic) cotangent bundle" of X. We leave this as an exercise.

A holomorphic section of T^*X over U is called a "holomorphic 1-form" on U.

In this course, holomorphic line bundles play very important roles. Let $\pi : L \to X$ be a holomorphic line bundle and $\{U_i\}_{i\in\Lambda}$ an open cover by trivialization neighborhoods, and $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ the trivialization map. Since $GL(1, \mathbb{C}) = \mathbb{C}^*$, now the transition maps ψ_{ij} become non-vanishing holomorphic functions on $U_i \cap U_j$. Let $s \in \Gamma(X, \mathcal{O}(L))$, then $\varphi_i \circ s|_{U_i} : U_i \to U_i \times \mathbb{C}$ could be represented by a holomorphic function $f_i \in \mathcal{O}(U_i)$, such that $\varphi_i \circ s|_{U_i}(p) = (p, f_i(p))$. When $U_i \cap U_j \neq \emptyset$, since $s|_{U_i} = s|_{U_j}$ on $U_i \cap U_j$, we have for any $p \in U_i \cap U_j$:

$$\begin{aligned}(p, f_i(p)) &= \varphi_i(s(p)) \\ &= (\varphi_i \circ \varphi_j^{-1}) \circ \varphi_j(s(p)) \\ &= (\varphi_i \circ \varphi_j^{-1})(p, f_j(p)) \\ &= (p, \psi_{ij}(p) f_j(p)).\end{aligned}$$

So we have $f_i = \psi_{ij} f_j$ on $U_i \cap U_j$. On the other hand, it is direct to check that given a family of holomorphic functions $f_i \in \mathcal{O}(U_i)$, satisfying $f_i = \psi_{ij} f_j$ on $U_i \cap U_j$, then there corresponds a unique $s \in \Gamma(X, \mathcal{O}(L))$.

Example 1.19 (Universal line bundle over $\mathbb{C}P^n$)**.** [1] *We define a holomorphic line bundle $U \to \mathbb{C}P^n$ as follows: As a set,*

$$U = \{([z], v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} | \ v \in [z]\},$$

where we view $[z]$ as the 1-dimensional subspace of $\mathbb{C}^{n+1}$ determined by z. As one can check easily, we can write

$$U = \{([z], v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} | \ v_i z_j - z_j z_i = 0, \forall i, j = 0, \ldots, n\}.$$

From this, it is easy to see that U is a complex submanifold of $\mathbb{C}P^n \times \mathbb{C}^{n+1}$, and hence a complex manifold. The projection onto its first component $\mathbb{C}P^n$ is clearly a holomorphic map, with fiber the 1-dimensional linear subspace of $\mathbb{C}^{n+1}$ generated by $(z_0, \ldots, z_n)$.

For local triviality, we use the holomorphic charts $\{(U_i, \varphi_i)\}_{i=0}^n$ defined before. On $\pi^{-1}(U_i)$, each $v \in U_{[z]}$ can be uniquely write as $t \cdot (\frac{z_0}{z_i}, \ldots, 1, \ldots, \frac{z_n}{z_i})$, so we define

$$\tilde{\varphi}_i([z_0, \ldots, z_n], t \cdot (\frac{z_0}{z_i}, \ldots, 1, \ldots, \frac{z_n}{z_i})) = ([z_0, \ldots, z_n], t) \in U_i \times \mathbb{C}.$$

This is easily seen to be a biholomorphic map.

It is easy to write down the transition functions: $\psi_{ij}([z]) = \frac{z_i}{z_j}$.

[1] Also called the "tautological bundle"

Construct new bundles from old ones: The usual constructions in linear algebra all have counterparts in the category of vector bundles over X.

Direct sum

Let E, F be vector bundles over X of rank r and s respectively. Then their direct sum is a vector bundle of rank $r+s$ with fiber $E_p \oplus F_p$. To describe it, it suffices to write down the transition maps: if $\{U_i\}_{i\in\Lambda}$ is a common trivializing covering of X for E and F. The transition maps are ψ_{ij} and η_{ij} respectively, then the transition maps for $E\oplus F$ are precisely $diag(\psi_{ij}, \eta_{ij})$ with values in $GL(r+s, \mathbb{C})$.

Tensor product

Let E, F be vector bundles over X of rank r and s respectively. Then their tensor product is a vector bundle of rank rs with fiber $E_p \otimes F_p$. In applications, we only use the tensor product of a line bundle L with a general vector bundle E. In this case, if the transition maps for E and L with respect to a common trivializing covering are ψ_{ij} and η_{ij}, then the transition maps of $E \otimes L$ are $\eta_{ij}\psi_{ij}$.

$\underline{Hom}(E, F)$

Let E, F be vector bundles over X of rank r and s respectively. Then $\underline{Hom}(E, F)$ is a vector bundle of rank rs with fiber $Hom(E_p, F_p)$, the space of linear maps from E_p to F_p. In particular, we define the dual of E to be $E^* := \underline{Hom}(E, \underline{\mathbb{C}})$, whose fiber over p is exactly the dual space of E_p, $(E_p)^*$.

When $L \to X$ is a holomorphic line bundle, we can easily describe L^* in terms of transition functions: if the transition functions of L are ψ_{ij}, then the transition functions of L^* are ψ_{ij}^{-1}. For this reason, we usually also write L^{-1} for L^*.

Exercise: Prove that the bundle $\underline{Hom}(E, F)$ is isomorphic to $E^* \otimes F$.

Example 1.20. *Let $U \to \mathbb{C}P^n$ be the universal bundle, its dual is usually denoted by H, we call it the "hyperplane line bundle".* [2] *Another common notation for H is $\mathcal{O}(1)$. We also write the H^k, or $\mathcal{O}(k)$, short for the k-times tensor product of H, $H^k := H^{\otimes k} = H \otimes \cdots \otimes H$, and $\mathcal{O}(-k) := H^{-k} := U^{\otimes k}$.*

We now study the holomorphic sections of H^k for $k > 0$. Let $s \in \Gamma(\mathbb{C}P^n, \mathcal{O}(H^k))$, we know that s can be represented by a family of holomorphic functions $f_\alpha \in \mathcal{O}(U_\alpha)$, where $U_\alpha = \{[z] \in \mathbb{C}P^n | \ z_\alpha \neq 0\}$. These f_α's satisfy the condition

$$f_\alpha([z]) = \left(\frac{z_\beta}{z_\alpha}\right)^k f_\beta([z])$$

on $U_\alpha \cap U_\beta$.

[2] The reason for this name should be clear after we find out what are the zero locus of its holomorphic sections.

Pulling back to $\mathbb{C}^{n+1} \setminus \{0\}$, we can view $z_\alpha^k f_\alpha([z])$ as a homogeneous function of degree k on $\mathbb{C}^{n+1} \setminus \{z_\alpha = 0\}$, which is also holomorphic. Now the above compatibility condition means that these $z_\alpha f_\alpha([z])$'s could be "glued" together to form a holomorphic function on $\mathbb{C}^{n+1} \setminus \{0\}$, homogeneous of degree k. By Hartogs extension theorem **??**, *this function extends to a holomorphic function $F(z_0, \ldots, z_n) \in \mathscr{O}(\mathbb{C}^{n+1})$. We necessarily have $F(0) = 0$ by homogeneity and continuity. From this we easily conclude that F is a homogeneous polynomial of degree k.*

On the other hand, it is easy to see that any homogeneous polynomial of degree k in $\mathbb{C}[z_0, \ldots, z_n]$ determines uniquely a holomorphic section of H^k. So we have

$$\dim_{\mathbb{C}} \Gamma(\mathbb{C}P^n, \mathscr{O}(H^k)) = \binom{n+k}{n}.$$

Exercise: Prove that when $k < 0$, $\Gamma(\mathbb{C}P^n, \mathscr{O}(H^k)) = \{0\}$.

Definition 1.21. *The isomorphic classes of holomorphic line bundles over X is called the "Picard group" of X, denoted by $Pic(X)$.*

$Pic(X)$ is indeed a group: we define $[L_i] \cdot [L_2] := [L_1 \otimes L_2]$, then $\underline{\mathbb{C}}$ is the identity element and $[L]^{-1}$ is just $[L^*]$.

For $\mathbb{C}P^n$, we have $Pic(\mathbb{C}P^n) \cong \mathbb{Z}$, and any holomorphic line bundle is isomorphic to $\mathscr{O}(k)$ for some $k \in \mathbb{Z}$. However, this is rather deep, and we can not prove it here. One can find a proof in Chapter 1 of [1].

Wedge product

Let E be vector bundles over X of rank r, for $k \in \mathbb{N}$ and $k \le r$, the degree k wedge product of E is a vector bundle $\Lambda^k E$ with fiber $\Lambda^k E_p$ at p. The highest degree wedge product $\Lambda^r E$ is also called the "determinant line bundle" of E, since its transition functions are precisely $\det \psi_{ij}$.

$\Omega^p(X) := \Lambda^p T^* X$ is the bundle of holomorphic p-forms.

Pull back via holomorphic map

Let $E \to X$ be a holomorphic vector bundle of rank r, $f : Y \to X$ be a holomorphic map between complex manifolds, then we can define a "pull back" holomorphic vector f^*E over Y. In fact, we can simply define the total space of f^*E to be

$$f^*E := \{(y, (x, v)) \in Y \times E |\ x = f(y)\},$$

and $p : f^*E \to Y$ is just the projection to its first component.

We can also describe f^*E via transition maps: if $\{U_i\}_{i \in \Lambda}$ is a trivializing covering of X for E with transition maps $\psi_{ij} : U_i \cap U_j \to GL(r, \mathbb{C})$, and we choose an open covering $\{V_\alpha\}_{\alpha \in I}$ such that $f(V_\alpha) \subset U_i$ for some $i \in \Lambda$. We fix a map $\tau : I \to \Lambda$ such that $f(V_\alpha) \subset U_{\tau(\alpha)}$. Then the transition maps for f^*E with respect to $\{V_\alpha\}_{\alpha \in I}$ are just $f^*\psi_{\tau(\alpha)\tau(\beta)} = \psi_{\tau(\alpha)\tau(\beta)} \circ f : V_\alpha \cap V_\beta \to GL(r, \mathbb{C})$.

1.3 Almost complex manifolds

The definition of a n-dimensional differential manifold is similar to that of complex manifolds. Just replace every $\mathbb{C}^n$ by $\mathbb{R}^n$ and every "holomorphic" by "smooth" or C^∞. Similar for differential vector bundles over a differential manifold. A differential manifold is called orientable, if we can find a coordinate covering such that whenever two coordinate charts intersect, the Jacobian determinant of the coordinate transform is positive.

Lemma 1.22. *A n-dimensional complex manifold X is also a $2n$-dimensional orientable differential manifold.*

This follows from the computation we did before in the proof of Theorem **??**. Here if we have a holomorphic coordinate chart (U, φ) with $\varphi = (z_i, \dots, z_n)$, then the corresponding chart to define the oriented differential structure is $(x_1, \dots, x_n, y_1, \dots, y_n)$.

For $p \in X$, we can define a real tangent vector at p and the corresponding real tangent space at p, $T_p^{\mathbb{R}}X$. In terms of coordinate chart $\varphi = (z_1, \dots, z_n)$, we have

$$T_p^{\mathbb{R}}X = \mathbb{R} < \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} >_{i=1}^{n} .$$

We can give $\coprod_{p\in X} T_p^{\mathbb{R}}X$ a structure of $\mathbb{R}$-vector bundle of rank $2n$, called the "real tangent bundle" of X, and denoted by $T^{\mathbb{R}}X$. Similarly, we can define the real cotangent bundle $T^{*\mathbb{R}}X$.

There are two ways to get from this our previous holomorphic tangent and cotangent bundles.

Recall that any real vector space V of dimension $2n$ can be regarded as $\mathbb{C}$-vector space of dimension n once we know what does it mean to multiply $\sqrt{-1}$ to an element of V. This is equivalent to giving a $\mathbb{R}$-linear map $J : V \to V$ such that $J^2 := J \circ J = -id$. We call such a J a "complex structure" on V. In this case, V can be regarded as a $\mathbb{C}$-vector space by defining

$$(\alpha + \sqrt{-1}\beta)v := \alpha v + \beta J v, \quad \forall \alpha, \beta \in \mathbb{R}, \forall v \in V.$$

Definition 1.23. *Let M be a real orientable differential manifold of dimension $2n$. An almost complex structure on M is a bundle map $J : TM \to TM$ satisfying $J^2 = -id$.*

Note that a complex manifold X has a natural almost complex structure: just define

$$J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}, \quad J\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}.$$

If an almost complex structure is induced from a complex structure as above, we will call it "integrable".

Example 1.24. *For S^2, we can define $J : TS^2 \to TS^2$ as follows: we identify T_xS^2 with the subspace of $\mathbb{R}^3$:*

$$T_xS^2 \cong \{y \in R^3 | \, x \cdot y = 0\}.$$

Then we define $J_x : T_xS^2 \to T_xS^2$ *by*

$$J_x(y) := x \times y.$$

On can check that this is an integrable almost complex structure, induced by the complex structure of $S^2 \cong \mathbb{C}P^1$.

Example 1.25. *For* S^6*, we have a similar almost complex structure given by "wedge product" in* $\mathbb{R}^7$*. Note that the wedge product in* $\mathbb{R}^3$ *can be defined as the product of purely imaginary quaternions. To define this wedge product in* $\mathbb{R}^7$*, we shall use Cayley's theory of octonions.*

We write $\mathbb{H} \cong \mathbb{R}^4$ *the space of quaternions* $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ *with* $a, b, c, d \in \mathbb{R}$*, satisfying* $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ *and* $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$*,* $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$*, and* $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$*. Then this multiplication is still associative but not commutative. For* $q \in \mathbb{H}$*, we define* $\bar{q} := a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$*, then* $|q|^2 = q\bar{q}$*.*

Now we define the space of octonians, $\mathbb{O} \cong \mathbb{R}^8$*, as* $\mathbb{O} := \{x = (q_1, q_2) | \, q_1, q_2 \in \mathbb{H}\}$*. The multiplication is defined by*

$$(q_1, q_2)(q_1', q_2') := (q_1q_1' - \bar{q}_2'q_2, q_2'q_1 + q_2\bar{q}_1').$$

And we also define $\bar{x} := (\bar{q}_1, -q_2)$*. Then we still have* $x\bar{x} = x \cdot x = |x|^2$*, here the* $\cdot$ *means the usual inner product in* $\mathbb{R}^8$*. Note that this multiplication is even not associative.*

We identify $\mathbb{R}^7$ *as the space of purely imaginary octonians. If* $x, x' \in \mathbb{R}^7$*, we define* $x \times x'$ *as the imaginary part of* xx'*. Then one can check that* $xx = -|x|^2$*,* $x \times x' = -x' \times x$*, and* $(x \times x') \cdot x'' = x \cdot (x' \times x'')$*.*

From this, one can define an almost complex structure on $S^6 \subset \mathbb{R}^7$ *in a similar way as* S^2*: identify* T_xS^6 *with* $\{y \in \mathbb{R}^7 | \, x \cdot y = 0\}$*, then define*

$$J_x(y) := x \times y.$$

Remark 1.26. *For spheres of even dimension* $2n$*, it is known (Borel-Serre) that there are no almost complex structures unless* $n = 1, 3$*. A modern proof of this fact using characteristic classes can be found in P. May's book on algebraic topology. It is generally believed that there are no integrable almost complex structures on* S^6*, however S.T. Yau has a different conjecture saying that one can make* S^6 *into a complex manifold. This is still open.*

Now given $J : T^{\mathbb{R}}X \to T^{\mathbb{R}}X$, we can view $T^{\mathbb{R}}X$ as a $\mathbb{C}$-vector bundle. One can check that, when X is a complex manifold, $(T^{\mathbb{R}}X, J)$ is isomorphic to the holomorphic tangent bundle TX as $\mathbb{C}$-vector bundles. This is the first approach.

The second approach also uses J. Let again V be a real vector space with complex structure J. But now we simply complexify V to get

$$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}.$$

We also extend J $\mathbb{C}$-linearly to $V_{\mathbb{C}}$, again $J^2 = -id$.

There is a direct sum decomposition of $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, which are $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of J respectively. In fact we have a very precise description of $V^{1,0}$ and $V^{0,1}$:

$$V^{1,0} = \{v - \sqrt{-1}Jv | \, v \in V\}, \quad V^{0,1} = \{v + \sqrt{-1}Jv | \, v \in V\}.$$

It is direct to check that they are both $\mathbb{C}$-linear subspaces of $V_{\mathbb{C}}$ and $V^{0,1} = \overline{V^{1,0}}$.

Now apply this to $(T^{\mathbb{R}}X, J)$ for a manifold with an almost complex structure: define the complexified tangent bundle to be

$$T^{\mathbb{C}}X := T^{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C}$$

and we have the decomposition

$$T^{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X,$$

which are the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of J, respectively. When J is integrable, $T^{1,0}X$ is locally generated by $\{\frac{\partial}{\partial z_i}\}_{i=1}^n$, so we can again identify it with our previous holomorphic vector bundle TX.

We define $T^{*1,0}X$ to be the subspace of $T^{*\mathbb{C}}X := T^{*\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C}$ that annihilates $T^{0,1}X$. And similarly define $T^{*0,1}X$. Then

$$T^{*\mathbb{C}}X = T^{*1,0}X \oplus T^{*0,1}X.$$

When J is integrable, $T^{*1,0}X$ is locally generated by $\{dz_i\}_{1\leq i\leq n}$ and $T^{*0,1}X$ is generated by $\{d\bar{z}_i\}_{1\leq i\leq n}$. We define the vector bundle $\Lambda^{p,q}T^*X$, the bundle of (p,q)-forms to be the sub-bundle of $\Lambda^{p+q}T^{*\mathbb{C}}X$, generated by $\Lambda^p T^{*1,0}X$ and $\Lambda^q T^{*0,1}X$. Then we have

$$\Lambda^k T^{*\mathbb{C}}X = \bigoplus_{p=0}^{k} \Lambda^{p,k-p}T^*X,$$

and we denote the projection map of $\Lambda^{p+q}T^{*\mathbb{C}}X$ onto $\Lambda^{p,q}T^*X$ by $\Pi_{p,q}$. The set of smooth sections of $\Lambda^{p,q}T^*X$ over an open set U is denoted by $A^{p,q}(U)$, while the set of smooth sections of $\Lambda^k T^{*\mathbb{C}}X$ is denoted by $A^k(U)$.

When J is integrable, a smooth section of $\Lambda^{p,q}T^*X$ over a coordinate open set U is of the forms

$$\sum_{1\leq i_1<\cdots<i_p\leq n, 1\leq j_1<\cdots<j_q\leq n} a_{i_1\ldots i_p,\bar{j}_1\ldots\bar{j}_q} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q},$$

where $a_{i_1\ldots i_p,\bar{j}_1\ldots\bar{j}_q} \in C^\infty(U;\mathbb{C})$.

The exterior differential operator d extends $\mathbb{C}$-linearly to $d : A^k(U) \to A^{k+1}(U)$. We define the operators

$$\partial := \Pi_{p+1,q} \circ d : A^{p,q}(U) \to A^{p+1,q}(U),$$

and

$$\bar{\partial} := \Pi_{p,q+1} \circ d : A^{p,q}(U) \to A^{p,q+1}(U).$$

When J is integrable, then for $\eta = \sum_{|I|=p,|J|=q} a_{I\bar{J}} dz_I \wedge d\bar{z}_J \in A^{p,q}(U)$, we have

$$\begin{aligned} d\eta &= \sum_{I,J} da_{I\bar{J}} \wedge dz_I \wedge d\bar{z}_J \\ &= \sum_{I,J} \partial a_{I\bar{J}} \wedge dz_I \wedge d\bar{z}_J + \sum_{I,J} \bar{\partial} a_{I\bar{J}} \wedge dz_I \wedge d\bar{z}_J \in A^{p+1,q}(U) \oplus A^{p,q+1}(U). \end{aligned}$$

So we always have $d = \partial + \bar{\partial}$. Conversely, we have:

Theorem 1.27 (Newlander-Nirenberg)**.** *An almost complex structure is integrable if and only if* $d = \partial + \bar{\partial}$ *(equivalently,* $[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$*) for any* $A^{p,q}(U)$.

Besides the original proof of Newlander-Nirenberg, there is another proof by J.J. Kohn based on techniques for solving the "$\bar{\partial}$-equation", which can be found in Hörmander's book.

1.4 De Rham cohomology and Dolbeault cohomology

In the following, we always assume the almost complex structure J is integrable, i.e., X is a complex manifold.

Now $d = \partial + \bar{\partial}$. Since we always have $d^2 = 0$, a fact first noticed by Poincaré, we have

$$0 = \partial^2 + \bar{\partial}^2 + (\partial\bar{\partial} + \bar{\partial}\partial),$$

acting on $A^{p,q}(X)$. Comparing types, we get

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

We can define from these identities several differential cochain complexes:
The de Rham complex

$$0 \to A^0(X) \xrightarrow{d} A^1(X) \xrightarrow{d} \dots \xrightarrow{d} A^{2n}(X) \to 0$$

We define the *de Rham cohomology* (with coefficient $\mathbb{C}$)

$$H^k_{dR}(X, \mathbb{C}) := Ker\Big(A^k(X) \xrightarrow{d} A^{k+1}(X)\Big)/dA^{k-1}(X).$$

The Dolbeault complex

$$0 \to A^{p,0}(X) \xrightarrow{\bar{\partial}} A^{p,1}(X) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} A^{p,n}(X) \to 0.$$

We define the *Dolbeault cohomology*

$$H^{p,q}_{\bar{\partial}}(X) := Ker\Big(A^{p,q}(X) \xrightarrow{\bar{\partial}} A^{p,q+1}(X)\Big)/\bar{\partial}A^{p,q-1}(X).$$

The holomorphic de Rham complex

$$0 \to \Omega^0(X) \xrightarrow{d=\partial} \Omega^1(X) \xrightarrow{d=\partial} \dots \xrightarrow{d=\partial} \Omega^n(X) \to 0$$

We define the *holomorphic de Rham cohomology*

$$H^k_{dR}(X, hol) := Ker\Big(\Omega^k(X) \xrightarrow{d} \Omega^{k+1}(X)\Big)/d\Omega^{k-1}(X).$$

The relation between these cohomology theories, as well as computational tools will be discussed when we finish sheaf cohomology theory and Hodge theorem.

References

[1] Phillip Griffiths and Joseph Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, Inc., 1978.

[2] R.O. Wells, *Differential analysis on complex manifolds*, 2nd edition, Springer, 1980.