Introduction to complex geometry
(Chapter 2)

Yalong Shi*

Abstract

Notes for 2020 BICMR Summer School for Differential Geometry.

Contents

2 A brief introduction to sheaf theory

2.1 Basic concepts in sheaf theory

Recall that a presheaf $\mathscr{F}$ of abelian groups over a topological space X is a rule assigning an abelian group $\mathscr{F}(U)$ for each open set $U \subset X$, and for each pair $V \subset U$ a homomorphism $r_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$ (called "restriction homomorphism"), satisfying $r_U^U = id$ and for any $W \subset V \subset U$, we have $r_W^U = r_W^V \circ r_V^U$. An element of $\mathscr{F}(U)$ is usually called a "section" of $\mathscr{F}$ over U. We also defined the stalk of $\mathscr{F}$ at a point $p \in X$ to be

$$\mathscr{F}_p := \varinjlim \mathscr{F}(U),$$

where the direct limit is taken with respect to open sets $p \in U$. This is $\coprod_{U \ni p} \mathscr{F}(U)/\sim$, with $s \in \mathscr{F}(U)$ equivalent to $t \in \mathscr{F}(V)$ iff we can find another open set $p \in W \subset U \cap V$ such that $r_W^U(s) = r_W^V(t)$.

By a morphism f between two presheaves $\mathscr{F}$ and $\mathscr{G}$ over X, we mean for each U open, we are given a homomorphism of abelian groups $f_U : \mathscr{F}(U) \to \mathscr{G}(U)$, such that whenever we have open sets $V \subset U$, we have a commutative diagram:

$$\begin{array}{ccc}
\mathscr{F}(U) & \xrightarrow{f_U} & \mathscr{G}(U) \\
{\scriptstyle r_V^U}\big\downarrow & & \big\downarrow{\scriptstyle \rho_V^U} \\
\mathscr{F}(V) & \xrightarrow[f_V]{} & \mathscr{G}(V).
\end{array}$$

Definition 2.1. *A presheaf of abelian groups $\mathscr{F}$ over X is called a sheaf, if it satisfies the following two properties:*

(S1) Assume we have a family of open sets $U_i \subset U, i \in \Lambda$ and $\cup_i U_i = U$. If $s \in \mathscr{F}(U)$ satisfies $r_{U_i}^U(s) = 0, \forall i \in \Lambda$, then $s = 0$.

(S2) Assume we have a family of open sets $U_i \subset U, i \in \Lambda$ and $\cup_i U_i = U$. If we also have a family of sections $s_i \in \mathscr{F}(U_i), \forall i \in \Lambda$, satisfying $r_{U_i \cap U_j}^{U_i}(s_i) = r_{U_i \cap U_j}^{U_j}(s_i)$ whenever $U_i \cap U_j \neq \emptyset$, then there is a section $s \in \mathscr{F}(U)$ such that $r_{U_i}^U(s) = s_i, \forall i \in \Lambda$.

A morphism between two sheaves is just a morphism between presheaves.

Note that by (S1), the section in (S2) is also unique.

Example 2.2. *Let X be a complex manifold, then $\mathscr{O}_X$ is a sheaf of commutative rings over X. We call it the "structure sheaf" of X.*

We can also define other sheaves on X. For example, define $\mathcal{E}(U) := C^\infty(U; \mathbb{C})$, then it is easy to see that $\mathcal{E}$ is a sheaf, called the "sheaf of smooth functions". Similarly, we can define the sheaf of continuous functions on X.

If $E \to X$ is a holomorphic vector bundle, then $\mathscr{O}(E)(U)$ defines a sheaf of abelian groups. It can also be viewed as a sheaf of $\mathscr{O}_X$-modules.

Example 2.3. *For $X = \mathbb{C}$, if we define $\mathscr{O}_b(U)$ to be the set of bounded holomorphic functions on $U \subset X$, then $\mathscr{O}_b$ is a presheaf over $\mathbb{C}$, but not a sheaf.*

Example 2.4. *Let G be a given abelian group, we define the constant presheaf over X to be $\underline{G}_{pre}(U) := G$ for any non-empty open set $U \subset X$, and $r^U_V = id$ for any non-empty pair $V \subset U$. Then it is in general not a sheaf.*

Example 2.5. *Let $\pi : Y \to X$ be a continuous surjective map between topological spaces. We define the sheaf of continuous sections of π as follows: for any open $U \subset X$, define $C_\pi(U) := \{\sigma : U \to Y | \pi \circ \sigma = id_U\}$. Then it is a sheaf over X. This example is in fact very general.*

Proposition 2.6. *For any presheaf $\mathscr{F}$ over X, there is a unique (up to isomorphism) sheaf $\mathscr{F}^+$ and a morphism $\theta : \mathscr{F} \to \mathscr{F}^+$ satisfying the following "universal property": for any sheaf $\mathscr{G}$ over X and any morphism of presheaves $f : \mathscr{F} \to \mathscr{G}$, there is a unique morphism of sheaves $f^+ : \mathscr{F}^+ \to \mathscr{G}$ such that $f = f^+ \circ \theta$.*

If $\mathscr{F}$ is already a sheaf, then θ is an isomorphism. $\mathscr{F}^+$ is called the "sheafification" of $\mathscr{F}$.

Outline of proof. I just outline one way of proof. From $\mathscr{F}$, we define a topological space, called the "étalé space" associated to $\mathscr{F}$:

$$\tilde{\mathscr{F}} := \coprod_{p \in X} \mathscr{F}_p.$$

We have a natural surjective projection map $\pi : \tilde{\mathscr{F}} \to X$. The topology on $\tilde{\mathscr{F}}$ is given as follows: If $s \in \mathscr{F}(U)$, then we have a natural map $\tilde{s} : U \to \tilde{\mathscr{F}}$, sending p to the germ of s at p, which is an element of $\mathscr{F}_p$. Then we require $\{\tilde{s}(U) |\ s \in \mathscr{F}(U), \forall U\}$ to be a topological basis for $\tilde{\mathscr{F}}$.

Now we can use the construction of Example2.5 to get a sheaf $\mathscr{F}^+$. The morphism θ is defined by $\theta_U : \mathscr{F}(U) \to \mathscr{F}^+(U)$, $\theta_U(s) := \tilde{s}$. □

Exercise: Check that we have the following concrete description of $\mathscr{F}^+$: a map $\tilde{s} : U \to \coprod_{p \in U} \mathscr{F}_p$ is an element of $\mathscr{F}^+(U)$ if and only if:

1. $\pi \circ \tilde{s} = id_U$;

2. For any $p \in U$, there is an open neighborhood $p \in V \subset U$ and a $s \in \mathscr{F}(V)$ such that for any $q \in V$, $\tilde{s}(q)$ equals the germ of s at q.

2.2 Sheaf cohomology (Čech's theory)

Sheaf is a useful tool to describe the obstructions to solve global problems when we can always solve a local one.

To illustrate this point, we come back to the Mittag-Leffler problem on a Riemann surface M. Suppose we are given finitely many points $p_1, \ldots, p_m \in M$, and for each p_i we are given a Laurant polynomial $\sum_{k=1}^{n_i} \frac{c_k^{(i)}}{z^k}$. We can view this as an element of $\mathscr{M}_p/\mathscr{O}_p$. We want to find a meromorphic function on M whose poles are precisely those p_i's with the given Laurant polynomial as its principal part at p_i.

This problem is always solvable locally: we can find a locally finite open covering $\mathcal{U} = \{U_i |\, i \in \Lambda\}$ of M such that each U_i contains at most one of the p_i's, and $f_i \in \mathscr{M}(U_i)$ such that the only poles of f_i are those of $\{p_i\}$ contained in U_i with principal part equals the given Laurent polynomial. The problem is that we can not patch them together: if $U_i \cap U_j$, there is no reason to have $f_i = f_j$. We have to define $f_{ij} := f_i - f_j$ and view the totality of these f_{ij}'s as the obstruction to solve the problem. Now by our choice of f_i, $f_{ij} \in \mathscr{O}(U_i \cap U_j)$. Note that we have $f_{ij} + f_{ji} = 0$ on $U_i \cap U_j$ and whenever $U_i \cap U_j \cap U_k \neq \emptyset$, we have on $U_i \cap U_j \cap U_k$: $f_{ij} + f_{jk} + f_{ki} = 0$. We call this the "cocycle" condition and $\{f_{ij}\}$ is a "Čech cocycle" for the sheaf $\mathscr{O}$ with respect to the cover $\mathcal{U}$.

When can we solve the Mittag-Leffler problem on M? We can solve it if we can modify the f_i by a holomorphic function $h_i \in \mathscr{O}(U_i)$ such that $\tilde{f}_i := f_i - h_i$ with patch together. This means that $f_{ij} = h_i - h_j$ on $U_i \cap U_j$. We call a cocycle of the form $\{h_i - h_j\}$ where each h_i is holomorphic a Čech coboundary. We get the conclusion that we can solve the Mittag-Leffler problem if the Čech cocycle $\{f_{ij}\}$ is a coboundary.

This motivates the introduction of the following Čech cohomology of a sheaf $\mathscr{F}$ with respect to a locally finite cover $\mathcal{U}$ of X: We first define the chain groups:

$$
\begin{aligned}
&C^0(\mathcal{U}, \mathscr{F}) := \Pi_{i\in\Lambda}\mathscr{F}(U_i)\\
&C^1(\mathcal{U}, \mathscr{F}) \subset \Pi_{i,j\in\Lambda}\mathscr{F}(U_i \cap U_j)\\
&\qquad\qquad \cdots\\
&C^p(\mathcal{U}, \mathscr{F}) \subset \Pi_{i_0,i_1,\ldots,i_p\in\Lambda}\mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_p})\\
&\qquad\qquad \cdots
\end{aligned}
$$

where $\{\sigma_{i_0,\ldots,i_p}\}$ is in $C^p(\mathcal{U}, \mathscr{F})$ if and only if:

(1) Whenever $i_k = i_l$ for some $k \neq l$, we have $\sigma_{i_0,\ldots,i_p} = 0$;

(2) For any permutation $\tau \in S_{p+1}$, we have $\sigma_{i_{\tau(0)},\ldots,i_{\tau(p)}} = (-1)^{\tau}\sigma_{i_0,\ldots,i_p}$.

Note that we always define $\mathscr{F}(U) = \{0\}$ if $U = \emptyset$.

We define the coboundary operator $\delta : C^p(\mathcal{U}, \mathscr{F}) \to C^{p+1}(\mathcal{U}, \mathscr{F})$ to be:

$$(\delta\sigma)_{i_0,\ldots,i_{p+1}} := \sum_{j=0}^{p+1}(-1)^j\sigma_{i_0,\ldots,\hat{i_j},\ldots,i_{p+1}}|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}}.$$

Here we use $\ldots|_{\ldots}$ to denote the restriction homomorphism of $\mathscr{F}$. It is direct to check that $\delta \circ \delta = 0$. So we have a chain complex

$$0 \to C^0(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} \ldots \xrightarrow{\delta} C^p(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} \ldots$$

We can define

$$Z^p(\mathcal{U}, \mathscr{F}) = Ker\delta \subset C^p(\mathcal{U}, \mathscr{F}),$$

whose elements are called Čech p-cocycles. Also define

$$B^p(\mathcal{U}, \mathscr{F}) = \delta C^{p-1}(\mathcal{U}, \mathscr{F}) \subset Z^p(\mathcal{U}, \mathscr{F}),$$

whose elements are called Čech p-coboundaries. Then we define the Čech cohomology of $\mathscr{F}$ with respect to $\mathcal{U}$:

$$H^p(\mathcal{U}, \mathscr{F}) := Z^p(\mathcal{U}, \mathscr{F})/B^p(\mathcal{U}, \mathscr{F}).$$

For example, an element of $H^0(\mathcal{U}, \mathscr{F})$ is given by a family of sections $f_i \in \mathscr{F}(U_i)$ such that $\delta\{f_i\} = 0$. This means precisely

$$r^{U_i}_{U_i \cap U_j}(f_i) = r^{U_j}_{U_i \cap U_j}(f_j)$$

whenever $U_i \cap U_j \neq \emptyset$. By sheaf axiom (S2), we get a global section of $\mathscr{F}$ over X. So $H^0(\mathcal{U}, \mathscr{F})$ is in fact independent of $\mathcal{U}$ and we have a canonical isomorphism

$$H^0(\mathcal{U}, \mathscr{F}) \cong \mathscr{F}(X).$$

When $p = 1$, $\{f_{ij}\} \in C^p(\mathcal{U}, \mathscr{F})$ is a cocycle if $f_{ij} + f_{ji} = 0$ and $f_{jk} - f_{ik} + f_{ij} = f_{ij} + f_{jk} + f_{ki} = 0$. This is precisely the "cocycle condition" we met before. However, this time the cohomology may depend on the cover.

Let $\mathcal{V} = \{V_\alpha\}_{\alpha\in\Gamma}$ be a locally finite refinement of $\mathcal{U}$. This means we have a map $\tau : \Gamma \to \Lambda$ (not unique) such that $V_\alpha \subset U_{\tau(\alpha)}$. Then we have a homomorphism $\Phi^{\mathcal{U}}_{\mathcal{V}} : H^p(\mathcal{U}, \mathscr{F}) \to H^p(\mathcal{V}, \mathscr{F})$ induced by

$$\{\sigma_{i_0,\ldots,i_p}\} \mapsto \{\sigma_{\tau(\alpha_0),\ldots,\tau(\alpha_p)}|_{V_{\alpha_0}\cap\cdots\cap V_{\alpha_p}}\}.$$

One can prove that $\Phi^{\mathcal{U}}_{\mathcal{V}}$ is in fact independent of the choice of the map τ. Then the cohomology of X with coefficients sheaf $\mathscr{F}$ is defined to be the direct limit:

$$H^p(X, \mathscr{F}) := \varinjlim H^p(\mathcal{U}, \mathscr{F}) = \coprod_{\mathcal{U}} H^p(\mathcal{U}, \mathscr{F})/\sim$$

where two cohomology classes $[\{\sigma_{i_0,\ldots,i_p}\}] \in H^p(\mathcal{U}, \mathscr{F})$ and $[\{\eta_{j_0,\ldots,j_p}\}] \in H^p(\mathcal{V}, \mathscr{F})$ are equivalent if we can find a common refinement $\mathcal{W}$ of $\mathcal{U}, \mathcal{V}$ such that

$$\Phi^{\mathcal{U}}_{\mathcal{W}}([\{\sigma_{i_0,\ldots,i_p}\}]) = \Phi^{\mathcal{V}}_{\mathcal{W}}([\{\eta_{j_0,\ldots,j_p}\}]).$$

Thus an element of $H^p(X, \mathscr{F})$ is an equivalent class of Čech cohomology classes, represented by an element of $H^p(\mathcal{U}, \mathscr{F})$, for some cover $\mathcal{U}$. But in many cases, in particular all the sheaves we use in this course, there exists sufficiently fine cover $\mathcal{U}$ such that $H^p(\mathcal{U}, \mathscr{F}) \cong H^p(X, \mathscr{F})$.

2.3 Useful results for sheaf cohomology

We present two useful results for sheaf cohomology. In many cases, it is safe to know only these results and forget the definition details.

Recall that a morphism $f : \mathscr{F} \to \mathscr{G}$ of sheaves over X induces for each point $p \in X$ a homomorphism of stalks: $f_p : \mathscr{F}_p \to \mathscr{G}_p$. We call a sequence of morphisms of sheaves an "exact sequence" if the induced sequence on stalks is so for each pint p.

The first result saying that a short exact sequence for morphisms of sheaves gives rise to a long exact sequence for sheaf cohomology:

Theorem 2.7. *If we have a short exact sequence for sheaves of abelian groups over X*

$$0 \to \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \to 0,$$

then we have a long exact sequence for cohomologies

$$0 \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{G}) \to H^0(X, \mathscr{H}) \to H^1(X, \mathscr{F}) \to \dots$$
$$\dots \to H^p(X, \mathscr{H}) \to H^{p+1}(X, \mathscr{F}) \to H^{p+1}(X, \mathscr{G}) \to \dots$$

We won't prove this, but will explain the meaning of this theorem.

For the given short exact sequence, we always get an exact sequence

$$0 \to \mathscr{F}(X) \to \mathscr{G}(X) \to \mathscr{H}(X),$$

(**Exercise:** Show that for any open set U, the sequence $0 \to \mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{H}(U)$ is always exact.)but the last homomorphism is in general not surjective. Let's explain why. Given an element $\sigma \in \mathscr{H}(X)$, we'd like to know whether we can find $\eta \in \mathscr{G}(X)$ such that $g_X(\eta) = \sigma$. But we already know that $0 \to \mathscr{F}_p \xrightarrow{f_p} \mathscr{G}_p \xrightarrow{g_p} \mathscr{H}_p \to 0$ is exact, so we can always find a germ $\eta_p \in \mathscr{G}_p$ such that $g_p(\eta_p) = \sigma_p$. This actually means that we can find a cover $\mathcal{U} = \{U_i\}$ of X and a sequence $\eta_i \in \mathscr{G}(U_i)$ such that $g_{U_i}(\eta_i) = \sigma|_{U_i}$. If all the $\eta_{ij} := \eta_j - \eta_i = 0$ on $U_i \cap U_j$, then we can patch these η_i's together, then we finish the problem. We'd like to modify η_i. Note that since $g_{U_i \cap U_j}(\eta_{ij}) = 0$, we can find $\mu_{ij} \in \mathscr{F}(U_i \cap U_j)$ such that $f_{U_i \cap U_j}(\mu_{ij}) = \eta_{ij}$. By the injectivity of f, we in fact get a cocycle $\{\mu_{ij}\} \in C^1(\mathcal{U}, \mathscr{F})$. So we get a homomorphism $\mathscr{H}(X) \to H^1(X, \mathscr{F})$. It is fairly easy to check that if σ goes to 0 in $H^1(X, \mathscr{F})$, then we can modify η_i properly (on a refinement of $\mathcal{U}$) such that they patch together to get an element of $\mathscr{G}(X)$.

A corollary of Theorem2.7 is the following "abstract de Rham theorem":

Theorem 2.8. *Suppose we have an exact sequence of the form:*

$$0 \to \mathscr{F} \to \mathscr{S}_0 \to \mathscr{S}_1 \to \dots \to \mathscr{S}_r \to \dots$$

where each $\mathscr{S}_r$ satisfies $H^p(X, \mathscr{S}_r) = 0, \forall p \geq 1$. (This is called an "acyclic resolution of $\mathscr{F}$".) Then $H^*(X, \mathscr{F})$ is isomorphic to the cohomology of the chain complex

$$0 \to \mathscr{S}_0(X) \to \mathscr{S}_1(X) \to \dots \to \mathscr{S}_r(X) \to \dots$$

i.e., $H^*(X, \mathscr{F}) \cong H^*(\Gamma(X, \mathscr{S}^*))$.

Proof. We break the sheaf sequence into a sequence of short exact sequences for $p \geq 1$: $0 \to \mathscr{K}_{p-1} \to \mathscr{S}_{p-1} \to \mathscr{K}_p \to 0$, where $\mathscr{K}_p = Ker(\mathscr{S}_p \to \mathscr{S}_{p+1}) = Im(\mathscr{S}_{p-1} \to \mathscr{S}_p)$. Note that $\mathscr{K}_0 \cong \mathscr{F}$. By the above theorem and the assumption for $\mathscr{S}_p$, we have an exact sequence

$$0 \to \mathscr{K}_{p-1}(X) \to \mathscr{S}_{p-1}(X) \to \mathscr{K}_p(X) \to H^1(X, \mathscr{K}_{p-1}) \to 0.$$

Also note that $\mathscr{K}_p(X) \cong Ker(\mathscr{S}_p(X) \to \mathscr{S}_{p+1}(X))$, so we get

$$H^1(X, \mathscr{K}_{p-1}) \cong Ker(\mathscr{S}_p(X) \to \mathscr{S}_{p+1}(X))/Im(\mathscr{S}_{p-1}(X) \to \mathscr{K}_p(X)) = H^p(\Gamma(X, \mathscr{S}^*)).$$

We need to prove $H^1(X, \mathscr{K}_{p-1}) \cong H^p(X, \mathscr{F}) = H^p(X, \mathscr{K}_0)$. For this, we only need to show for $2 \leq r \leq p$

$$H^{r-1}(X, \mathscr{K}_{p-r+1}) \cong H^r(X, \mathscr{K}_{p-r}).$$

But this again follows from the segment of long exact sequence:

$$\cdots \to H^{r-1}(X, \mathscr{S}_{p-r}) \to H^{r-1}(X, \mathscr{K}_{p-r+1}) \to H^r(X, \mathscr{K}_{p-r}) \to H^r(X, \mathscr{S}_{p-r}) \to \ldots.$$

□

When can we get an acyclic resolution? In particular, how can we find a lot of sheaves $\mathscr{S}_r$ such that $H^p(X, \mathscr{S}_r) = 0, \forall p \geq 1$?

Definition 2.9. *A sheaf $\mathscr{F}$ over X is called a "fine sheaf", if for any locally finite open cover $\mathcal{U} = \{U_i\}$, we can find a family of morphisms $\eta_i : \mathscr{F} \to \mathscr{F}$ such that:*

(1) For each i, $\eta_i(p) : \mathscr{F}_p \to \mathscr{F}_p$ equals 0 for p outside a compact set $W_i \subset U_i$;

(2) $\sum_i \eta_i = id_{\mathscr{F}}$.

It is obvious that in case we can use a smooth function to multiply the sections of $\mathscr{F}$, then a usual partition of unity will make $\mathscr{F}$ a fine sheaf.

Proposition 2.10. *If $\mathscr{F}$ is a fine sheaf, then $H^p(X, \mathscr{F}) = 0, \forall p \geq 1$.*

Proof. For any p-cocycle $\{\sigma_{i_0,\ldots,i_p}\} \in C^p(\mathcal{U}, \mathscr{F})$ for a locally finite cover $\mathcal{U} = \{U_i\}_{i\in\Lambda}$. Let η_i be the above morphisms in the definition. We define a $p-1$ cochain $\{\psi_{i_0,\ldots,i_{p-1}}\}$ as follows:

$$\psi_{i_0,\ldots,i_{p-1}} := \sum_i \eta_i(\sigma_{i,i_0,\ldots,i_{p-1}}).$$

Then (using the fact that $\delta\{\sigma_{\ldots}\} = 0$)

$$\begin{aligned}(\delta\psi)_{i_0,\ldots,i_p} &= \sum_{j=0}^{p}(-1)^j \psi_{i_0,\ldots,\hat{i_j},\ldots,i_p} \\ &= \sum_j \sum_i (-1)^j \eta_i(\sigma_{i,i_0,\ldots,\hat{i_j},\ldots,i_p}) \\ &= \sum_i \eta_i(\sigma_{i_0,\ldots,i_p}) = \sigma_{i_0,\ldots,i_p}.\end{aligned}$$

□

2.4 Applications of sheaf cohomology

Cohomology of constant sheaf

Let G be a given abelian group, we can define the constant sheaf $\underline{G}$ over X by $\underline{G}(U) =$ {locally constant maps $U \to G$}, then we usually denote $H^p(X, \underline{G})$ by $H^p(X, G)$. One can show that when X is a manifold, this is isomorphic to the singular cohomology or simplicial cohomology. But we won't prove this. For the isomorphism to simplicial cohomology when $G = \mathbb{Z}$, one can read Chapter 0 of Griffiths-Harris.

Picard group

Recall that when X is a complex manifold, then a holomorphic line bundle can be described by a family of "transition functions" $f_{ij} \in \mathscr{O}^*(U_i \cap U_j)$, satisfying the "cocycle" condition. So any holomorphic line bundle L determines an element of $H^1(X, \mathscr{O}^*)$. And on the other hand, given an element of $H^1(\mathcal{U}, \mathscr{O}^*)$, we can construct a holomorphic line bundle. In fact, one can show that $[\{f_{ij}\}] \in H^1(\mathcal{U}, \mathscr{O}^*)$ and $[\{h_{\alpha\beta}\}] \in H^1(\mathcal{V}, \mathscr{O}^*)$ determines isomorphic line bundles if and only if they define the same class in $H^1(X, \mathscr{O}^*)$. So we can in fact identify $H^1(X, \mathscr{O}^*)$ with the Picard group of X.

de Rham and Dolbeault theorem

We use the de Rham resolution of $\underline{\mathbb{C}}$:

$$0 \to \underline{\mathbb{C}} \to \mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \xrightarrow{d} \dots \xrightarrow{d} \mathscr{A}^{2n} \to 0$$

to get de Rham isomorphism:

$$H^p(X, \mathbb{C}) \cong H^p_{dR}(X, \mathbb{C}), \quad p = 0, \dots, 2n.$$

The reason for this to be a resolution is Poincaré's Lemma.

Similarly, we have a Dolbeault-Grothendieck Lemma, which says that a $\bar{\partial}$-closed form is locally $\bar{\partial}$-exact. So we get a fine resolution for any $0 \leq p \leq n$:

$$0 \to \Omega^p \to \mathscr{A}^{p,0} \xrightarrow{\bar{\partial}} \mathscr{A}^{p,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathscr{A}^{p,n} \to 0,$$

so we get

$$H^q(X, \Omega^p) \cong H^{p,q}_{\bar{\partial}}(X).$$

Also for a holomorphic vector bundle E, we have

$$H^q(X, \Omega^p(E)) \cong H^{p,q}_{\bar{\partial}}(X, E).$$

Divisor and line bundle

We define the sheaf of meromorphic functions $\mathscr{M}$ on X, where X is a compact complex manifold, to be the sheafification of the presheaf $U \mapsto$ quotient field of $\mathscr{O}(U)$. We define $\mathscr{M}^*$ to be the sheaf of meromorphic functions that are not identically 0, and let $\mathscr{O}^*$ be the subsheaf of $\mathscr{M}^*$, consisting of no-where vanishing holomorphic functions. The short exact sequence

$$1 \to \mathscr{O}^* \to \mathscr{M}^* \to \mathscr{M}^*/\mathscr{O}^* \to 1$$

gives us a long exact sequence, starting with

$$\{1\} \to \mathbb{C}^* \to \mathscr{M}^*(X) \to \mathscr{M}^*/\mathscr{O}^*(X) \to H^1(X, \mathscr{O}^*) \to \dots.$$

The global section of $\mathscr{M}^*/\mathscr{O}^*(X)$ can be equivalently described as a finite formal sum $\sum_i a_i D_i$, where $a_i \in \mathbb{Z}$ and D_i is codimension 1 irreducible analytic hypersurface of X. This is called a "divisor". We define the groups of divisor classes by

$$Div(X) := \left(\mathscr{M}^*/\mathscr{O}^*(X)\right)/\mathscr{M}^*(X).$$

Two divisors are called linearly equivalent, if their difference is a divisor of a global meromorphic function.

The map $\mathscr{M}^*/\mathscr{O}^*(X) \to H^1(X, \mathscr{O}^*)$ is given as follows: locally we can cover X by $\{U_i\}$ such that an element of $\mathscr{M}^*/\mathscr{O}^*(X)$ is given by $f_i \in \mathscr{M}^*(U_i)$. Then $g_{ij} := f_i/f_j$ defines a class in $H^1(X, \mathscr{O}^*)$.

<u>First Chern class</u>

A very useful exact sequence is the following

$$0 \to \underline{\mathbb{Z}} \to \mathscr{O} \xrightarrow{\exp(2\pi\sqrt{-1}\cdot)} \mathscr{O}^* \to 1.$$

We get the exact sequence

$$\cdots \to H^1(X, \mathscr{O}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to \dots.$$

We call $c_1 : H^1(X, \mathscr{O}^*) \to H^2(X, \mathbb{Z})$ the "first Chern class" map. We shall use differential forms to give another characterization of Chern classes in the next chapter.

References

[1] Phillip Griffiths and Joseph Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, Inc., 1978.

[2] R.O. Wells, *Differential analysis on complex manifolds*, 2nd edition, Springer, 1980.