Introduction to complex geometry
(Chapter 3)

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Abstract

Notes for 2020 BICMR Summer School for Differential Geometry.

Contents

3 Differential geometry of vector bundles

3.1 Metrics, connections and curvatures

Definition 3.1. *Let $E \to X$ be a complex vector bundle of rank r over a smooth manifold X. A smooth Hermitian metric on E is an assignment of Hermitian inner products $h_p(\cdot, \cdot) =< \cdot, \cdot >_p$ on each fiber E_p, such that if ξ, η are smooth sections of E over an open set U, then $h(\xi, \eta) \in C^\infty(U; \mathbb{C})$.*

If U is a local triviliazation neighborhood of E via $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$, then we can define r smooth sections of E over U:

$$e_\alpha(p) := \varphi_U^{-1}(p, 0, \ldots, 0, 1, 0 \ldots, 0).$$

Then at any point $p \in U$, $\{e_\alpha(p)\}_{\alpha=1}^r$ is a basis of E_p. We call $\{e_\alpha\}_{\alpha=1}^r$ a local frame of E over U. Note that when E is a holomorphic bundle and (U, φ_U) a holomorphic trivialization, then these e_α's are also holomorphic sections, and we call it a holomorphic frame.

If ξ is a smooth section over U, then we can write in a unique way $\xi = \xi^\alpha e_\alpha$, with $\xi_\alpha \in C^\infty(U; \mathbb{C}), \alpha = 1, \ldots, r$. If we define the (positive definite) Hermitian matrix-valued smooth functions: $h_{\alpha\bar{\beta}} := h(e_\alpha, e_\beta)$, then we have

$$h(\xi, \eta) = h(\xi^\alpha e_\alpha, \eta^\beta e_\beta) = h_{\alpha\bar{\beta}} \xi^\alpha \bar{\eta}^\beta.$$

Sometimes, we also denote the matrix-valued smooth function $(h_{\alpha\bar{\beta}})$ by h. Hopefully this will cause no confusion.

Notation: We shall denote the space of smooth sections of E over U by $C^\infty(U; E)$. When E is a holomorphic bundle, the set of holomorphic sections over U is denoted by $\Gamma(U; E)$ or $\mathscr{O}(E)(U)$.

Definition 3.2. *A connection on a smooth rank r complex vector bundle over a manifold X is a map $D : C^\infty(X; E) \to C^\infty(X, T^{*\mathbb{C}}X \otimes E)$ satisfying :*

1. *D is $\mathbb{C}$-linear;*
2. *(Leibniz rule) $D(f\xi) = df \otimes \xi + fD\xi$, $\forall f \in C^\infty(X; \mathbb{C}), \xi \in C^\infty(X; E)$.*

If $\{e_\alpha\}$ is a local frame, then we can define a family of local smooth 1-forms $\theta_\alpha^\beta \in A^1(U)$ satisfying:

$$De_\alpha = \theta_\alpha^\beta \otimes e_\beta.$$

Sometimes we just write $De_\alpha = \theta_\alpha^\beta e_\beta$ for short. We call these $\{\theta_\alpha^\beta\}$ "connection one-forms". For $\xi = \xi^\alpha e_\alpha \in C^\infty(U; E)$, we then have

$$D\xi = D(\xi^\alpha e_\alpha) = (d\xi^\alpha + \xi^\beta \theta_\beta^\alpha) e_\alpha.$$

Convention: We always regard ξ^α as a column vector, and for θ^α_β we always regard the upper index as line index and the lower index the column index.

So if we identify ξ with its coordinate representation with respect to the frame $\{e_\alpha\}$, then we can write $D\xi = d\xi + \theta\xi$, or $D = d + \theta$. Physicists always use this way to represent a connection.

We can extend the action of D to bundle-valued differential forms. We write $A^k(X, E) := C^\infty(X; \Lambda^k T^{*\mathbb{C}} X \otimes E)$. Then we define $D : A^k(X, E) \to A^{k+1}(X, E)$ by

$$D(\varphi\xi) := (d\varphi)\xi + (-1)^k \varphi \wedge D\xi,$$

where φ is a $\mathbb{C}$-valued k-form and ξ is a smooth section of E.

Definition 3.3. *We define the curvature of D to be* $\Theta := D^2 : A^0(X; E) \to A^2(X, E)$.

If f is a smooth function and $\xi \in A^0(X, E)$, we have

$$\begin{aligned}\Theta(f\xi) &= D(df\xi + fD\xi)\\ &= d(df)\xi - df \wedge D\xi + df \wedge D\xi + fD^2\xi\\ &= f\Theta(\xi).\end{aligned}$$

Locally if we define the 2-forms Θ^β_α by

$$\Theta(e_\alpha) = \Theta^\beta_\alpha e_\beta.$$

Then we have

$$\begin{aligned}\Theta(\xi) &= \Theta(\xi^\alpha e_\alpha)\\ &= \xi^\alpha \Theta(e_\alpha)\\ &= \Theta^\alpha_\beta \xi^\beta e_\alpha.\end{aligned}$$

From this, we conclude that $\Theta \in A^2(X, End(E))$.

We can also represent Θ^α_β in terms of θ^α_β:

$$\begin{aligned}\Theta^\beta_\alpha e_\beta &= D(De_\alpha) = D(\theta^\gamma_\alpha e_\gamma)\\ &= d\theta^\gamma_\alpha e_\gamma - \theta^\gamma_\alpha \wedge De_\gamma\\ &= d\theta^\beta_\alpha e_\beta - \theta^\gamma_\alpha \wedge \theta^\beta_\gamma e_\beta\\ &= (d\theta^\beta_\alpha + \theta^\beta_\gamma \wedge \theta^\gamma_\alpha) e_\beta.\end{aligned}$$

So we get

$$\Theta^\alpha_\beta = d\theta^\alpha_\beta + \theta^\alpha_\gamma \wedge \theta^\gamma_\beta,$$

or $\Theta = d\theta + \theta \wedge \theta$ for short. Note that our sign convention is different from Griffiths-Harris, since they regard the upper index as the column index.

We now study the change of connection forms and curvature forms under the change of frames.

Suppose $\{\tilde{e}_\alpha\}$ is another local frame on U, then we can write $\tilde{e}_\alpha = a_\alpha^\beta e_\beta$, where (a_α^β) is a $GL(r, \mathbb{C})$-valued smooth function on U. (When both frames are local holomorphic frames of a holomorphic bundle, then (a_α^β) is a $GL(r, \mathbb{C})$-valued holomorphic function on U.) The new connection forms and curvature forms are denoted by $\tilde{\theta}$ and $\tilde{\Theta}$. We have

$$\begin{aligned}\tilde{\theta}_\alpha^\gamma \tilde{e}_\gamma = D\tilde{e}_\alpha &= D(a_\alpha^\beta e_\beta)\\ &= da_\alpha^\beta e_\beta + a_\alpha^\beta \theta_\beta^\gamma e_\gamma\\ &= (da_\alpha^\beta + \theta_\gamma^\beta a_\alpha^\gamma) e_\beta.\end{aligned}$$

On the other hand, the left equals

$$\tilde{\theta}_\alpha^\gamma a_\gamma^\beta e_\beta.$$

So we get

$$a\tilde{\theta} = da + \theta a,$$

or

$$\tilde{\theta} = a^{-1} da + a^{-1} \theta a. \tag{3.1}$$

From this, we get

$$\begin{aligned}\tilde{\Theta} &= d\tilde{\theta} + \tilde{\theta} \wedge \tilde{\theta}\\ &= d(a^{-1}da + a^{-1}\theta a) + (a^{-1}da + a^{-1}\theta a) \wedge (a^{-1}da + a^{-1}\theta a)\\ &= -a^{-1}da \wedge a^{-1}da - a^{-1}da \wedge a^{-1}\theta a + a^{-1}d\theta a - a^{-1}\theta \wedge da\\ &\quad + a^{-1}da \wedge a^{-1}da + a^{-1}da \wedge a^{-1}\theta a + a^{-1}\theta \wedge da + a^{-1}\theta \wedge \theta a\\ &= a^{-1}(d\theta + \theta \wedge \theta)a.\end{aligned}$$

So we conclude

$$\tilde{\Theta} = a^{-1} \Theta a. \tag{3.2}$$

From this, we can construct a family of globally defined differential forms:

$$\det\Big(I_r + \frac{\sqrt{-1}}{2\pi}\Theta\Big) := 1 + c_1(E, D) + \cdots + c_r(E, D),$$

where $c_k(E, D) \in A^{2k}(X)$ is called the "k-th" Chern form of E associated to the connection D.

In physicists' language, a connection is a "field", the curvature is the "strength" of the field, and choosing a local frame is called "fixing the gauge". The reason for these names comes from H. Weyl's work, rewriting Maxwell's equations. The "vector potential" and "scalar potential" together form the connection 1-form, and the curvature 2-form has 6 components, consisting the components of the electric field and the magnetic field.

3.2 Chern connection on holomorphic vector bundles

In general, there is no "canonical connections" on a given vector bundle with a smooth Hermitian metric. However, if the bundle is a holomorphic vector bundle, there is indeed a canonical connection, called the "Chern connection":

Theorem 3.4. *On a given holomorphic vector bundle E with a smooth Hermitian metric h, there is a unique connection D, called the "Chern connection" satisfying the following two additional conditions:*

1. **(Compatibility with the metric)** *If ξ, η are two smooth sections, then we have*
$$dh(\xi, \eta) = h(D\xi, \eta) + h(\xi, D\eta).$$
2. **(Compatibility with the complex structure)** *If ξ is a holomorphic section of E, then $D\xi$ is a E-valued $(1,0)$-form.*

Proof. We first prove the uniqueness part. Let $\{e_\alpha\}_{\alpha=1}^r$ be a local holomorphic frame, and the connection 1-form with respect to this frame is $(\theta_\alpha^\beta)_{1\le\alpha,\beta\le r}$, satisfying $De_\alpha = \theta_\alpha^\beta e_\beta$. By the compatibility with complex structure, each θ_α^β is a smooth $(1,0)$-form. Now we use the compatibility with metric to get

$$\begin{aligned} dh_{\alpha\bar\beta} &= h(De_\alpha, e_\beta) + h(e_\alpha, De_\beta) \\ &= \theta_\alpha^\gamma h_{\gamma\bar\beta} + \bar\theta_\beta^\gamma h_{\alpha\bar\gamma}. \end{aligned}$$

On the other hand, we have $dh_{\alpha\bar\beta} = \partial h_{\alpha\bar\beta} + \bar\partial h_{\alpha\bar\beta}$. Comparing types, we get $\partial h = \theta^t h$, so $\theta^t = \partial h \cdot h^{-1}$. Denote $h^{-1} = (h^{\bar\beta\alpha})$, then we can rewrite this as

$$\theta_\alpha^\beta = h^{\bar\nu\beta}\partial h_{\alpha\bar\nu}.$$

Also, since $\bar h^t = h$, the $(0,1)$-part gives the same equation. This proves the uniqueness.

For existence, we simply set locally $\theta_\alpha^\beta := h^{\bar\nu\beta}\partial h_{\alpha\bar\nu}$, and define for $s = f^\alpha e_\alpha$:

$$Ds := (df^\alpha + f^\beta\theta_\beta^\alpha)e_\alpha.$$

We need to check that this is globally well-defined. For this, if $\tilde e_\alpha = a_\alpha^\beta e_\beta$ is another holomorphic frame on V with $U \cap V \ne \emptyset$. Then a is a holomorphic matrix. We have $\tilde h = a^t h \bar a$, so we have $\tilde\theta := (\tilde h^t)^{-1}\partial\tilde h^t = a^{-1}\partial a + a^{-1}\theta a$. Since $s = \tilde f^\alpha\tilde e_\alpha = f^\alpha e_\alpha$, we have $\tilde f = a^{-1}f$, so

$$\begin{aligned} \tilde e(d\tilde f + \tilde\theta\tilde f) &= ea(-a^{-1}daa^{-1}f + a^{-1}df + a^{-1}\partial aa^{-1}f + a^{-1}\theta aa^{-1}f) \\ &= e(df + \theta f). \end{aligned}$$

So D is globally defined. It is direct to check that D is compatible with both the metric and the complex structure of the bundle. □

It is worth pointing out that the line bundle case is particularly simple: if e is a local holomorphic frame and we set $h = h(e,e) > 0$. Then the connection 1-form is $\theta = h^{-1}\partial h = \partial\log h$. Then the curvature is $\Theta = d\theta + \theta\wedge\theta = d\theta = d\partial\log h = \bar\partial\partial\log h$. It is already a globally defined closed $(1,1)$-form.

3.3 Chern classes

We give a very elementary introduction to Chern-Weil theory in this section, following Professor Weiping Zhang's book [3].

We first define a trace map $tr : A^k(X, EndE) \to A^k(X)$. For a $EndE$-valued form $\eta \in A^k(X, EndE)$, the trace of η is the k-form $tr(\eta)$ obtained by tracing out the $EndE$ factor. Locally, we can write η as a matrix of k-forms, and $tr(\eta)$ is just the trace of this matrix. Or equivalently, we can write η as $\sum_i \omega_i \otimes A_i$ with ω_i a family of k-forms and A_i a family of local sections of $EndE$, and then $tr(\eta) = \sum_i tr(A_i)\omega_i$.

Another tool we shall use is the (super)-commutator, defined by $[\omega \otimes A, \eta \otimes B] := (\omega \wedge \eta) \otimes [A, B]$, where ω, η are locally defined forms and A, B are local sections of $EndE$. It is easy to see that

$$[\omega \otimes A, \eta \otimes B] = \omega A \wedge \eta B - (-1)^{deg(\omega)deg(\eta)} \eta B \wedge \omega A.$$

The appearance of the extra factor $(-1)^{deg(\omega)deg(\eta)}$ is the reason why sometimes it is called a "super"-commutator. We sometimes extend the definition: we define for the connection D and $\omega \otimes A$: $[D, \omega \otimes A]s := D(\omega \otimes As) - (-1)^{deg(\omega)} \omega \otimes A \wedge Ds$.

We state two useful lemmas, whose proofs are left as exercises.

Lemma 3.5. *If $\tilde{D}$ is another connection on E, then $\tilde{D} - D \in A^1(X, EndE)$.*

Lemma 3.6. *If P, Q are both $EndE$-valued differential forms, then $tr[P, Q] = 0$.*

The first nontrivial lemma is:

Lemma 3.7 (Bianchi identity)**.** *We have $[D, \Theta^k] = 0$, for any $k \in \mathbb{N}$.*

Proof. Simply note that $\Theta = D^2$, so $[D, \Theta^k] = [D, D^{2k}] = 0$. □

Exercise: Check that under local frames $[D, \Theta] = 0$ means $d\Theta = [\Theta, \theta]$.

The next lemma is one of our key tool:

Lemma 3.8. *For $A \in A^k(X, EndE)$, we have*

$$d\ tr(A) = tr[D, A].$$

Proof. First note that the left hand side is obviously independent of the connection. For the right hand side, if we use another connection $\tilde{D}$, by Lemma3.5 and Lemma3.6, we have $tr[\tilde{D}, A] = tr[\tilde{D} - D, A] + tr[D, A] = tr[D, A]$. So the right hand side is also independent of the connection.

So we can in fact choose a trivial connection locally to carry out the computation: Let $D_0 = d$ be a trivial connection on $E|_U \to U$, then

$$\begin{aligned}
[D_0, A]s &= D_0(As) - (-1)^{deg(A)} A \wedge D_0 s \\
&= d(A_\alpha^\beta f^\alpha) e_\beta - (-1)^{deg(A)} A_\alpha^\beta \wedge df^\alpha e_\beta \\
&= dA_\alpha^\beta f^\alpha e_\beta.
\end{aligned}$$

Hence $tr[D_0, A] = d\ tr(A)$. □

For any formal power series in one variable $f(x) = a_0 + a_1 x + \dots$, we define $f(\Theta) := a_0 + a_1\Theta + \cdots + a_n\Theta^n \in A^*(X)$.

Theorem 3.9 (Chern-Weil)**.** *For f as above, we have:*

1. $d\, tr f(\Theta) = 0$*;*
2. *If $\tilde{D}$ is another connection with curvature $\tilde{\Theta}$, there is a differential form $\eta \in A^*(X)$ such that $tr\ f(\tilde{\Theta}) - tr\ f(\Theta) = d\eta$.*

So the cohomology class of $tr\ f(\Theta)$ is independent of the connection. We call it the "characteristic class" of E associated to f, and $tr\ f(\Theta)$ the corresponding "characteristic form" of E associated to f and D.

Example 3.10. *Since $det(I_r + \frac{\sqrt{-1}}{2\pi}\Theta) = \exp\Big(tr \log(I_r + \frac{\sqrt{-1}}{2\pi}\Theta)\Big)$. So $c_i(E, D) \in A^{2i}(X)$ are all closed forms, whose cohomology classes are all independent of D. These are called "Chern classes". For example we have*

$$c_1(E, D) = \frac{\sqrt{-1}}{2\pi} tr\Theta, \quad c_2(E, D) = \frac{1}{8\pi^2}\Big(tr(\Theta^2) - (tr\Theta)^2\Big).$$

Proof of Theorem 3.9: For the first conclusion, by Lemma3.8, we have

$$\begin{aligned} d\, tr f(\Theta) &= tr[D, f(\Theta)] \\ &= \sum_k a_k tr[D, \Theta^k] = 0, \end{aligned}$$

where we used Lemma3.7 in the last step.

For the second one, we choose a family of connections $D_t := t\tilde{D} + (1-t)D$. Then

$$\dot{D}_t := \frac{dD_t}{dt} = \tilde{D} - D \in A^1(X, EndE),$$

and

$$\dot{\Theta}_t := \frac{d\Theta_t}{dt} = \frac{dD_t}{dt} D_t + D_t \frac{dD_t}{dt} = [D_t, \frac{dD_t}{dt}] = [D_t, \dot{D}_t].$$

So we have (by Lemma3.6, we can change the positions of Θ and $\dot{\Theta}$)

$$\begin{aligned} \frac{d}{dt} tr\ f(\Theta_t) &= tr(\dot{\Theta}_t f'(\Theta_t)) \\ &= tr([D_t, \dot{D}_t] f'(\Theta_t)) \\ &\overset{\text{Bianchi}}{=} tr[D_t, \dot{D}_t f'(\Theta_t)] \\ &= d\ tr\Big(\dot{D}_t f'(\Theta_t)\Big). \end{aligned}$$

So we conclude that $tr\ f(\tilde{\Theta}) - tr\ f(\Theta) = d \int_0^1 tr\Big(\dot{D}_t f'(\Theta_t)\Big) dt$. □

3.4 Hermitian metrics and Kähler metrics

Let X be a complex manifold of dimension n. We denote the canonical almost complex structure by J. A Riemannian metric g on X is called "Hermitian", if g is J-invariant, i.e.

$$g(Ju, Jv) = g(u, v), \quad \forall u, v \in T_x^{\mathbb{R}}X, \forall x \in X.$$

As before, we extend g to $T^{\mathbb{C}}X$ as a complex bilinear form. For simplicity, we also denote this bilinear form by g. Then we have

$$g(T^{1,0}, T^{1,0}) = 0 = g(T^{0,1}, T^{0,1})$$

and $\langle Z, W\rangle := g(Z, \bar{W})$ defines an Hermitian metric on the rank n holomorphic vector bundle $T^{1,0}X$. Conversely, any Hermitian metric on $T^{1,0}X$ determines uniquely a J-invariant Riemannian metric on X.

For an Hermitian metric g on (X, J), we define the associated Kähler form ω_g by

$$\omega_g(u, v) := g(Ju, v).$$

It is direct to check that ω_g is a real 2-form on X.

Definition 3.11. *An Hermitian metric g on X is called a Kähler metric, if $d\omega_g = 0$. Its cohomology class in $H^2_{dR}(X)$ is call the "Kähler class" of g. If a (compact) complex manifold admits a Kähler metric, we call it a "Kähler manifold".*

Locally, if $(z_1, \ldots, z_n)$ is a holomorphic coordinate system, then g is determined by

$$g_{i\bar{j}} := g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}),$$

since $g_{ij} = g_{\bar{i}\bar{j}} = 0$. Then we have

$$\omega_g = \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

where Einstein's summation convention is always used. Now we have

$$\begin{aligned}
0 = d\omega_g &= \sqrt{-1} dg_{i\bar{j}} dz_i \wedge d\bar{z}_j \\
&= \sqrt{-1} \frac{\partial g_{i\bar{j}}}{\partial z_k} dz_k \wedge dz_i \wedge d\bar{z}_j - \sqrt{-1} \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_l} dz_i \wedge d\bar{z}_l \wedge d\bar{z}_j \\
&= \sqrt{-1} \sum_j \sum_{k<i} \Big(\frac{\partial g_{i\bar{j}}}{\partial z_k} - \frac{\partial g_{k\bar{j}}}{\partial z_i}\Big) dz_k \wedge dz_i \wedge d\bar{z}_j \\
&\quad + \sqrt{-1} \sum_i \sum_{j<l} \Big(\frac{\partial g_{i\bar{j}}}{\partial \bar{z}_l} - \frac{\partial g_{i\bar{l}}}{\partial \bar{z}_j}\Big) dz_i \wedge d\bar{z}_j \wedge d\bar{z}_l.
\end{aligned}$$

So being Kähler mean that $g_{i\bar{j}}$ have the additional symmetries:

$$\frac{\partial g_{i\bar{j}}}{\partial z_k} = \frac{\partial g_{k\bar{j}}}{\partial z_i}, \quad \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_l} = \frac{\partial g_{i\bar{l}}}{\partial \bar{z}_j}, \quad \forall i, j, k, l.$$

Example 3.12. *The Euclidean metric $g = \sum_{i=1}^n (dx_i \otimes dx_i + dy_i \otimes dy_i)$ is a Kähler metric, since we have*

$$\omega_g = \frac{\sqrt{-1}}{2} \sum_{i=1}^n dz_i \wedge d\bar{z}_i.$$

To give more examples, note that to define a Kähler metrics, it suffices to define its associated Kähler form, since we have $g(u, v) = g(Ju, Jv) = \omega_g(u, Jv)$. So sometimes we will also say "Let ω_g be a Kähler metric..."

Example 3.13. *Let $X = B(1) \subset \mathbb{C}^n$ be the unit ball in $\mathbb{C}^n$. We define a Kähler metric:*

$$\omega_g := \sqrt{-1}\partial\bar{\partial} \log \frac{1}{1 - |z|^2}.$$

This is called the "complex hyperbolic metric".

Example 3.14. *Let $X = \mathbb{C}P^n$ with homogeneous coordinates $[Z_0, \ldots, Z_n]$, we define a Kähler metric:*

$$\omega_g := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} \log(|Z_0|^2 + \cdots + |Z_n|^2).$$

It is easy to check that this is well-defined. It is called the "Fubini-Study metric".

Not every compact complex manifold is Kähler, since, for example, $H^2_{dR}(X)$ must be non-trivial[1]. So Calabi-Eckman manifolds are never Kähler. However, we have the following:

Lemma 3.15. *If X is Kähler and Y is a complex analytic submanifold of X, then Y is also Kähler.*

Proof. (Outline) Let g be a Kähler metric on X and $\iota : Y \to X$ be the embedding map, then $\iota^* g$ is a Kähler metric on Y and the associated Kähler form is just $\iota^* \omega_g$. □

By this lemma, all projective algebraic manifolds are Kähler.

In Riemannian geometry, normal coordinates are very useful in tensor calculations. The next lemma shows that being Kähler is both necessary and sufficient for the existence of complex analogue of normal coordinates.

Lemma 3.16. *For an Hermitian metric g on X, the follows two properties are equivalent:*

(1) g is Kähler;

(2) For any point $p \in X$, there are local holomorphic coordinates $(z_1, \ldots, z_n)$ such that $z_i(p) = 0$, $g_{i\bar{j}}(p) = \delta_{ij}$ and $dg_{i\bar{j}}(p) = 0$.

[1] If not, ω_g will be exact, so $\int_X \omega_g^n = 0$ by Stokes theorem. But this is impossible since $\int_X \omega_g^n > 0$.

Proof. (2) $\Longrightarrow$ (1): For any given point p, we choose the coordinate in (2), then since first order derivatives of $g_{i\bar{j}}$ at p vanish, we will have $d\omega_g(p) = 0$. This implies $d\omega_g = 0$, i.e., g is Kähler.

(1) $\Longrightarrow$ (2): Suppose g is Kähler. Given any point $p \in X$, we can first choose local holomorphic coordinates $(w_1, \ldots, w_n)$ such that $w_i(p) = 0$ and $g(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial \bar{w}_j})(p) = \delta_{ij}$. We want to find holomorphic coordinate transformation of the form $w_i = z_i + \frac{1}{2}a_{ijk}z_j z_k$ with $a_{ijk} = a_{ikj}$ such that

$$\omega_g = \sqrt{-1}(\delta_{ij} + O(|z|^2))dz_i \wedge d\bar{z}_j.$$

Direct computation shows that

$$\begin{aligned}
\omega_g &= \sqrt{-1}\Big(\delta_{ij} + g_{i\bar{j},k}(0)w_k + g_{i\bar{j},\bar{l}}(0)\bar{w}_l + O(|w|^2)\Big)dw_i \wedge d\bar{w}_j \\
&= \sqrt{-1}\Big(\delta_{ij} + g_{i\bar{j},k}(0)z_k + g_{i\bar{j},\bar{l}}(0)\bar{z}_l + O(|z|^2)\Big)(dz_i + a_{ipq}z_p dz_q) \wedge (d\bar{z}_j + \bar{a}_{jst}\bar{z}_s d\bar{z}_t) \\
&= \sqrt{-1}\Big(\delta_{ij}dz_i \wedge d\bar{z}_j + \bar{a}_{ilj}\bar{z}_l dz_i \wedge d\bar{z}_j + a_{jki}z_k dz_i \wedge d\bar{z}_j \\
&\quad + (g_{i\bar{j},k}(0)z_k + g_{i\bar{j},\bar{l}}(0)\bar{z}_l)dz_i \wedge d\bar{z}_j + O(|z|^2)\Big).
\end{aligned}$$

So the condition we need is $a_{jki} + g_{i\bar{j},k}(0) = 0$ and $\bar{a}_{ilj} + g_{i\bar{j},\bar{l}}(0) = 0$. So we simply take

$$a_{jki} := -\frac{\partial g_{i\bar{j}}}{\partial w_k}(0).$$

The Kähler condition makes sure that this is well-defined. □

Remark 3.17. *We shall call such a holomorphic coordinate system a "Kähler normal coordinate system".*

Recall that for a connection ∇ on a vector bundle E, we can define the covariant derivative of a section s with respect to a tangent vector $v \in T_pX$ by setting $\nabla_v s := \nabla s(v)$. If e_α is a local frame of E, then we have $\nabla e_\alpha = \omega_\alpha^\beta e_\beta$, and $\nabla_v e_\alpha = \omega_\alpha^\beta(v)e_\beta$. Another good feature of the Kähler condition is that if we complexify the usual Levi-Civita connection, we will automatically get the Chern connection on $T^{1,0}X$.

Proposition 3.18. *Let (X, J, g) be a Kähler manifold. Then the complexification of the Levi-Civita connection restricts to the Chern connection on $T^{1,0}X$.*

Proof. We also denote the complexified Levi-Civita connection by ∇. Recall that ∇ is characterized as the only connection on $T^{\mathbb{R}}X$ that is both torsion free and compatible with g. For short, we write $\partial_i := \frac{\partial}{\partial z_i}$ and $\partial_{\bar{j}} := \frac{\partial}{\partial \bar{z}_j}$. By definition, we can assume $\nabla_{\partial_i}\partial_j := \Gamma_{ij}^k\partial_k + \Gamma_{ij}^{\bar{k}}\partial_{\bar{k}}$, $\nabla_{\partial_{\bar{i}}}\partial_j := \Gamma_{\bar{i}j}^k\partial_k + \Gamma_{\bar{i}j}^{\bar{k}}\partial_{\bar{k}}$. Since ∇ is a real operator, we also have $\nabla_{\partial_{\bar{i}}}\partial_{\bar{j}} := \overline{\Gamma_{ij}^k}\partial_{\bar{k}} + \overline{\Gamma_{ij}^{\bar{k}}}\partial_k$, $\nabla_{\partial_i}\partial_{\bar{j}} := \overline{\Gamma_{\bar{i}j}^k}\partial_{\bar{k}} + \overline{\Gamma_{\bar{i}j}^{\bar{k}}}\partial_k$. Since ∇ is torsion free, we have $\Gamma_{ij}^k = \Gamma_{ji}^k$, $\Gamma_{ij}^{\bar{k}} = \Gamma_{ji}^{\bar{k}}$, and $\Gamma_{\bar{i}j}^k = \overline{\Gamma_{\bar{j}i}^{\bar{k}}}$, $\Gamma_{\bar{i}j}^{\bar{k}} = \overline{\Gamma_{\bar{j}i}^k}$. Now we use the metric compatibility:

$$\begin{aligned}
0 &= \partial_i g(\partial_k, \partial_l) = g(\nabla_{\partial_i}\partial_k, \partial_l) + g(\partial_k, \nabla_{\partial_i}\partial_l) \\
&= \Gamma_{ik}^{\bar{q}} g_{l\bar{q}} + \Gamma_{il}^{\bar{q}} g_{k\bar{q}},
\end{aligned}$$

Exchange i and k, we get $0 = \Gamma_{ki}^{\bar{q}} g_{l\bar{q}} + \Gamma_{kl}^{\bar{q}} g_{i\bar{q}}$, and hence $\Gamma_{kl}^{\bar{q}} g_{i\bar{q}} = \Gamma_{il}^{\bar{q}} g_{k\bar{q}}$. So

$$\Gamma_{ik}^{\bar{q}} g_{l\bar{q}} = \Gamma_{lk}^{\bar{q}} g_{i\bar{q}} = \Gamma_{kl}^{\bar{q}} g_{i\bar{q}} = \Gamma_{il}^{\bar{q}} g_{k\bar{q}}.$$

This implies $\Gamma_{ik}^{\bar{q}} g_{l\bar{q}} = 0$ and hence $\Gamma_{ik}^{\bar{q}} = 0$. This means

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k, \quad \nabla_{\partial_{\bar{i}}} \partial_{\bar{j}} = \overline{\Gamma_{ij}^k} \partial_{\bar{k}}. \tag{3.3}$$

On the other hand,

$$\begin{aligned} \partial_i g(\partial_k, \partial_{\bar{l}}) &= g(\nabla_{\partial_i} \partial_k, \partial_{\bar{l}}) + g(\partial_k, \nabla_{\partial_i} \partial_{\bar{l}}) \\ &= \Gamma_{ik}^p g_{p\bar{l}} + \Gamma_{\bar{l}i}^{\bar{q}} g_{k\bar{q}}. \end{aligned}$$

By Kähler condition, the last quantity also equals

$$\partial_k g(\partial_i, \partial_{\bar{l}}) = \Gamma_{ki}^p g_{p\bar{l}} + \Gamma_{\bar{l}k}^{\bar{q}} g_{i\bar{q}},$$

so we get $\Gamma_{\bar{l}k}^{\bar{q}} g_{i\bar{q}} = \Gamma_{\bar{l}i}^{\bar{q}} g_{k\bar{q}}$. But the sum of these two quantity equals $\partial_{\bar{l}} g(\partial_i, \partial_k) = 0$, we get $\Gamma_{\bar{l}i}^{\bar{q}} g_{k\bar{q}} = 0$ and hence $\Gamma_{\bar{l}i}^{\bar{q}} = 0$. This also implies $\Gamma_{\bar{i}l}^{q} = 0$. So we get

$$\nabla_{\partial_i} \partial_{\bar{j}} = 0 = \nabla_{\partial_{\bar{j}}} \partial_i, \tag{3.4}$$

and also

$$\partial_i g_{k\bar{l}} = \Gamma_{ik}^p g_{p\bar{l}},$$

equivalently,

$$\Gamma_{ij}^k = g^{\bar{l}k} \frac{\partial g_{i\bar{l}}}{\partial z_j}. \tag{3.5}$$

This is precisely the formula for the Chern connection. $\square$

For curvature, we also extend the curvature tensor $\mathbb{C}$-linearly to the complexified tangent bundle. Then this curvature tensor automatically satisfies the Bianchi identities. The Kähler condition also implies that the curvature tensor has more symmetries, and hence has much simpler formula. We leave this to later sections. Here we only add one warming exercise:

Exercise: Let (E, ∇) be a vector bundle with connection. We define for $u, v \in \Gamma(TX)$ and $s \in \Gamma(E)$, $R(u, v)s := (\nabla_u \nabla_v - \nabla_u \nabla_v - \nabla_{[u,v]})s$. Show that $R(u, v)s = \Omega(u, v)s$, where $\Omega \in \Gamma(\Lambda^2 T^* X \otimes End(E))$ is the curvature form of ∇.

Let (X, J, g) be a Kähler manifold. We know from Proposition 3.18 that the connection of g has very special properties. We now explore its implication for the curvature.

Lemma 3.19. *For a Kähler manifold (X, J, g), we always have $\nabla J = 0$.*

Proof. For any given point $p \in X$, we compute using Kähler normal coordinates in Lemma 3.16. Now in complex coordinates, J has constant coefficients, this implies ∇J vanishes at p. Since p is arbitrary, we have $\nabla J = 0$. □

By definition, this implies that $\nabla(JX) = J\nabla X$, so for the curvature $R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, we have $R(X, Y)JZ = JR(X, Y)Z$. Also, by symmetry of curvature tensor, we have

$$\langle R(JX, JY)Z, W\rangle = \langle R(Z, W)JX, JY\rangle = \langle R(Z, W)X, Y\rangle = \langle R(X, Y)Z, W\rangle.$$

Since W is arbitrary, we also have $R(JX, JY)Z = R(X, Y)Z$. Moreover, we have:

Proposition 3.20. *We $\mathbb{C}$-linearly extend the curvature tensor of the Kähler metric g, then $\langle R(\partial_i, \partial_j)\cdot, \cdot\rangle = 0 = \langle R(\partial_{\bar{i}}, \partial_{\bar{j}})\cdot, \cdot\rangle$, and the only essentially non-trivial term is*

$$R_{i\bar{j}k\bar{l}} := \langle R(\partial_i, \partial_{\bar{j}})\partial_k, \partial_{\bar{l}}\rangle = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + g^{\bar{q}p}\frac{\partial g_{i\bar{q}}}{\partial z_k}\frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}.$$

In particular, besides Bianchi identities, we have an extra symmetry: $R_{i\bar{j}k\bar{l}} = R_{i\bar{l}k\bar{j}} = R_{k\bar{j}i\bar{l}}$. The Ricci curvature Rc is also J-invariant, and the 2-form $Ric(\omega_g) := Rc(J\cdot, \cdot)$ is called the Ricci form, and we have $Ric(\omega_g) = \sqrt{-1}R_{i\bar{j}}dz_i \wedge d\bar{z}_j$, with

$$R_{i\bar{j}} = Rc(\partial_i, \partial_{\bar{j}}) = g^{\bar{l}k}R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\log\det(g_{p\bar{q}}).$$

Proof. We compute by definition:

$$\begin{aligned}
R_{i\bar{j}k\bar{l}} &= \langle(\nabla_i\nabla_{\bar{j}} - \nabla_{\bar{j}}\nabla_i)\partial_k, \partial_{\bar{l}}\rangle = -\langle\nabla_{\bar{j}}(\Gamma^p_{ik}\partial_p), \partial_{\bar{l}}\rangle \\
&= -\partial_{\bar{j}}\Gamma^p_{ik}g_{p\bar{l}} = -\partial_{\bar{j}}(g^{\bar{q}p}\frac{\partial g_{k\bar{q}}}{\partial z_i})g_{p\bar{l}} \\
&= -g^{\bar{q}p}\frac{\partial^2 g_{k\bar{q}}}{\partial z_i \partial z_{\bar{j}}}g_{p\bar{l}} + g^{\bar{q}s}g^{\bar{t}p}\frac{\partial g_{s\bar{t}}}{\partial z_{\bar{j}}}\frac{\partial g_{k\bar{q}}}{\partial z_i}g_{p\bar{l}} \\
&= -\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial z_{\bar{j}}} + g^{\bar{q}s}\frac{\partial g_{s\bar{l}}}{\partial z_{\bar{j}}}\frac{\partial g_{k\bar{q}}}{\partial z_i}.
\end{aligned}$$

The first conclusion follows by Kähler metric's special symmetry.

For Ricci curvature, we choose a local orthonormal frame $\{e_i\}_{i=1}^{2n}$ to compute:

$$\begin{aligned}
Rc(JX, JY) &= \sum_{i=1}^{2n}\langle R(JX, e_i)e_i, JY\rangle = \sum_{i=1}^{2n}\langle JR(JX, e_i)e_i, J^2Y\rangle \\
&= -\sum_{i=1}^{2n}\langle R(JX, e_i)Je_i, Y\rangle = -\sum_{i=1}^{2n}\langle R(J^2X, Je_i)Je_i, Y\rangle \\
&= \sum_{i=1}^{2n}\langle R(X, Je_i)Je_i, Y\rangle = Rc(X, Y),
\end{aligned}$$

since $\{Je_i\}_{i=1}^{2n}$ is also an orthonormal frame. As the computation for ω_g, we easily get the formula

$$Ric(\omega_g) = \sqrt{-1}R_{i\bar{j}}dz_i \wedge d\bar{z}_j.$$

Finally, we calculate $R_{i\bar{j}}$: Choose a local orthonormal frame of the form $\{e_\alpha, Je_\alpha\}_{\alpha=1}^n$ at one point, and write $Z_\alpha := e_\alpha - \sqrt{-1}Je_\alpha$. Then we have

$$\begin{aligned}
R_{i\bar{j}} = Rc(\partial_i, \partial_{\bar{j}}) &= \sum_\alpha \langle R(\partial_i, e_\alpha)e_\alpha, \partial_{\bar{j}}\rangle + \sum_\alpha \langle R(\partial_i, Je_\alpha)Je_\alpha, \partial_{\bar{j}}\rangle \\
&= \sum_\alpha \langle R(\partial_i, e_\alpha)e_\alpha, \partial_{\bar{j}}\rangle + \sqrt{-1}\sum_\alpha \langle R(\partial_i, Je_\alpha)e_\alpha, \partial_{\bar{j}}\rangle \\
&= \sum_\alpha \langle R(\partial_i, \bar{Z}_\alpha)e_\alpha, \partial_{\bar{j}}\rangle \\
&= \frac{1}{2}\sum_\alpha \langle R(\partial_i, \bar{Z}_\alpha)e_\alpha, \partial_{\bar{j}}\rangle - \frac{\sqrt{-1}}{2}\sum_\alpha \langle R(\partial_i, \bar{Z}_\alpha)Je_\alpha, \partial_{\bar{j}}\rangle \\
&= \frac{1}{2}\sum_\alpha \langle R(\partial_i, \bar{Z}_\alpha)Z_\alpha, \partial_{\bar{j}}\rangle.
\end{aligned}$$

On the other hand, we have $Z_\alpha = a_\alpha^\mu \partial_\mu$ and $\partial_\mu = b_\mu^\alpha Z_\alpha$, with $a_\alpha^\mu b_\mu^\beta = \delta_\alpha^\beta$, so at the given point, we have

$$2\delta_{\alpha\beta} = g(Z_\alpha, \bar{Z}_\beta) = a_\alpha^\mu \bar{a}_\beta^\nu g_{\mu\bar{\nu}},$$

which implies that $g^{\bar{\beta}\alpha} = \frac{1}{2}\bar{a}_\mu^\beta a_\mu^\alpha$, and so

$$\begin{aligned}
R_{i\bar{j}} &= \frac{1}{2}\bar{a}_\alpha^\nu a_\alpha^\mu R_{i\bar{\nu}\mu\bar{j}} = g^{\bar{l}k}R_{i\bar{l}k\bar{j}} = g^{\bar{l}k}R_{i\bar{j}k\bar{l}} \\
&= -g^{\bar{l}k}\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + g^{\bar{l}k}g^{\bar{q}p}\frac{\partial g_{k\bar{q}}}{\partial z_i}\frac{\partial g_{p\bar{l}}}{\partial \bar{z}_j} = \frac{\partial}{\partial z_i}\Big(-g^{\bar{l}k}\frac{\partial g_{k\bar{l}}}{\partial \bar{z}_j}\Big) \\
&= -\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\log\det(g_{p\bar{q}}).
\end{aligned}$$

□

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