Introduction to complex geometry
(Chapter 4)

Yalong Shi*

Abstract

Notes for 2020 BICMR Summer School for Differential Geometry.

Contents

4 Hodge theorem **2**
4.1 Hodge theorem on compact Riemannian manifolds 2
4.2 The Hermitian case 7
4.3 Applications 10
4.4 The Kähler case 11
4.4.1 Hodge identities for Kähler metrics 11
4.4.2 Hodge decomposition for compact Kähler manifolds 13

4 Hodge theorem

4.1 Hodge theorem on compact Riemannian manifolds

Let (M^m, g) be a compact oriented Riemannian manifold. Then we can define inner product on the space of real differential forms: for $\omega, \eta \in A^p(M)$

$$(\omega, \eta) := \int_M \langle \omega, \eta \rangle_g dV_g.$$

The idea of Hodge theorem is to represent a de Rham cohomology class by a "best" closed form. Since we can define norm of a differential form, a natural idea is to find a closed form of minimal norm within its cohomology class.

To be precise, start with a closed p-form $\eta \in A^p(M)$, we want to minimize the functional:

$$\Phi(\xi) := \|\eta + d\xi\|^2, \quad \xi \in A^{p-1}(M).$$

We can solve this variational problem by considering the corresponding Euler-Lagrange equation, which is an elliptic system.

Suppose $\eta_0 = \eta + d\xi_0$ achieves the minimum of $\|\eta + d\xi\|^2$, then for any $\xi \in A^{p-1}(M)$,

$$\|\eta_0 + td\xi\|^2 = (\eta_0 + td\xi, \eta_0 + td\xi) = \|\eta_0\|^2 + 2t(\eta_0, d\xi) + t^2\|d\xi\|^2$$

achieves its minimum at $t = 0$. This happens if and only if $(\eta_0, d\xi) = 0$ for any $\xi \in A^{p-1}(M)$. We can define an operator d^*, the "formal adjoint" of d, such that $(\alpha, d\beta) = (d^*\alpha, \beta)$ for any $\alpha \in A^p(M)$ and $\beta \in A^{p-1}(M)$. Then $(\eta_0, d\xi) = 0$ for any $\xi \in A^{p-1}(M)$ if and only if $(d^*\eta_0, \xi) = 0$ for any $\xi \in A^{p-1}(M)$, which implies $d^*\eta_0 = 0$.

Definition 4.1. *Let (M^m, g) be a compact oriented Riemannian manifold. A smooth differential form $\omega \in A^p(M)$ is called a "harmonic p-form" if $d\omega = 0, d^*\omega = 0$.*

If we define the Laplacian operator to be $\Delta_d : A^p(M) \to A^p(M)$, $\Delta_d := dd^* + d^*d$, then for any smooth p-form $\omega \in A^p(M)$, we have

$$(\omega, \Delta_d\omega) = (\omega, dd^*\omega) + (\omega, d^*d\omega) = \|d^*\omega\|^2 + \|d\omega\|^2.$$

So we conclude that $\omega \in A^p(M)$ is harmonic if and only if $\Delta_d\omega = 0$.

To write down a precise formula for d^*, we introduce Hodge's "star"-operator: $* : A^p(M) \to A^{m-p}(M)$. If $\omega_1, \ldots, \omega_m$ is an orthonormal basis of T_x^*M, such that $\omega_1 \wedge \cdots \wedge \omega_m = dV_g$ gives the positive orientation, then we define

$$*\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} = \delta^{i_1,\ldots,i_p,j_1,\ldots,j_{m-p}}_{1,2,\ldots,m} \omega_{j_1} \wedge \cdots \wedge \omega_{j_{m-p}}.$$

(Note that this implies $\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \wedge *\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} = \omega_1 \wedge \cdots \wedge \omega_m$.) Then we extend $*$ linearly. It is direct to check that this is well-defined.

Moreover, if $\alpha = \sum_{i_1<\cdots<i_p} a_{i_1,\dots,i_p}\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}$, $\beta = \sum_{i_1<\cdots<i_p} b_{i_1,\dots,i_p}\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}$, we have

$$\begin{aligned}
\alpha \wedge *\beta &= \sum_{k_1<\cdots<k_p} \sum_{i_1<\cdots<i_p} a_{k_1,\dots,k_p} b_{i_1,\dots,i_p} \omega_{k_1} \wedge \cdots \wedge \omega_{k_p} \wedge *(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}) \\
&= \sum_{i_1<\cdots<i_p} a_{i_1,\dots,i_p} b_{i_1,\dots,i_p} \omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \wedge *(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}) \\
&= \langle \alpha, \beta \rangle_g dV_g = \beta \wedge *\alpha.
\end{aligned}$$

From the definition, it is easy to check that $** = (-1)^{p(m-p)} = (-1)^{pm+p}$ on $A^p(M)$. Also, we have

$$\langle *\alpha, *\beta \rangle_g dV_g = *\alpha \wedge **\beta = (-1)^{p(m-p)} * \alpha \wedge \beta = \beta \wedge *\alpha = \langle \beta, \alpha \rangle_g dV_g = \langle \alpha, \beta \rangle_g dV_g.$$

So $*$ is a point-wise isometry. Using $*$, we can also express d^* as:

Lemma 4.2. *We have $d^* = (-1)^{mp+m+1} * d*$ on $A^p(M)$.*

Proof. Let $\alpha \in A^p(M)$, $\beta \in A^{p-1}(M)$, then we have

$$\begin{aligned}
(d^*\alpha, \beta) = (\alpha, d\beta) &= \int_M \langle \alpha, d\beta \rangle_g dV_g = \int_M d\beta \wedge *\alpha \\
&= \int_M d(\beta \wedge *\alpha) + (-1)^p \beta \wedge d(*\alpha) \\
&= \int_M (-1)^p \beta \wedge (-1)^{(m-p+1)(p-1)} * * d(*\alpha) \\
&= (-1)^{mp+m+1} \int_M \langle \beta, *d(*\alpha) \rangle_g dV_g \\
&= \Big((-1)^{mp+m+1} * d * \alpha, \beta \Big).
\end{aligned}$$

□

From this concrete formula, we have on $A^p(M)$:

$$\Delta_d = dd^* + d^*d = (-1)^{mp+m+1} d*d* + (-1)^{m(p+1)+m+1} *d*d = (-1)^{mp+m+1} d*d* + (-1)^{mp+1} *d*d.$$

We have

$$\begin{aligned}
\Delta_d * &= (-1)^{m(m-p)+m+1} d * d * * + (-1)^{m(m-p)+1} * d * d * \\
&= (-1)^{mp+1}(-1)^{mp+p} d * d + (-1)^{mp+m+1} * d * d * \\
&= (-1)^{p+1} d * d + (-1)^{mp+m+1} * d * d * .
\end{aligned}$$

And

$$\begin{aligned} *\Delta_d &= (-1)^{mp+m+1} * d * d * + (-1)^{mp+1} * * d * d \\ &= (-1)^{mp+m+1} * d * d * + (-1)^{mp+1+p(m-p)} d * d \\ &= (-1)^{mp+m+1} * d * d * + (-1)^{p+1} d * d. \end{aligned}$$

So we get $*\Delta_d = \Delta_d *$. Similarly,

$$d\Delta_d = (-1)^{mp+1} d * d * d,$$

$$\Delta_d d = (-1)^{m(p+1)+m+1} d * d * d = (-1)^{mp+1} d * d * d = d\Delta_d.$$

Example 4.3. *In case of* $(\mathbb{R}^n, g_{Euc})$*, we can define* d^* *by the same formula, then we still have* $(\xi, d\eta) = (d^*\xi, \eta)$ *when one of them has compact support. Then we have*

$$\begin{aligned} d^* &\Big(\sum_{1\le i_1<\cdots<i_p\le m} f_{i_1\ldots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \Big) \\ = &\quad (-1)^{mp+m+1} * d\Big(\sum_{1\le i_1<\cdots<i_p\le m} f_{i_1\ldots i_p} * dx_{i_1} \wedge \cdots \wedge dx_{i_p} \Big) \\ = &\quad (-1)^{mp+m+1} * \Big(\sum_{1\le i_1<\cdots<i_p\le m} \sum_{i\in\{i_1,\ldots,i_p\}} \frac{\partial f_{i_1\ldots i_p}}{\partial x_i} dx_i \wedge * dx_{i_1} \wedge \cdots \wedge dx_{i_p} \Big) \\ = &\quad (-1)^{mp+m+1} \sum_{1\le i_1<\cdots<i_p\le m} \sum_{k=1}^{p} \frac{\partial f_{i_1\ldots i_p}}{\partial x_{i_k}} (-1)^{(p-1)(m-p)+k-1} dx_{i_1} \wedge \cdots \wedge d\hat{x}_{i_k} \wedge \cdots \wedge dx_{i_p} \\ = &\quad \sum_{1\le i_1<\cdots<i_p\le m} \sum_{k=1}^{p} \frac{\partial f_{i_1\ldots i_p}}{\partial x_{i_k}} (-1)^{k} dx_{i_1} \wedge \cdots \wedge d\hat{x}_{i_k} \wedge \cdots \wedge dx_{i_p}. \end{aligned}$$

From this we get

$$\begin{aligned} dd^* &\Big(\sum_{1\le i_1<\cdots<i_p\le m} f_{i_1\ldots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \Big) \\ = &\quad - \sum_{1\le i_1<\cdots<i_p\le m} \sum_{k=1}^{p} \frac{\partial^2 f_{i_1\ldots i_p}}{\partial x_{i_k}^2} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \\ &\quad + \sum_{1\le i_1<\cdots<i_p\le m} \sum_{k=1}^{p} \sum_{i\notin\{i_1,\ldots,i_p\}} \frac{\partial^2 f_{i_1\ldots i_p}}{\partial x_{i_k} \partial x_i} (-1)^k dx_i \wedge dx_{i_1} \wedge \cdots \wedge d\hat{x}_{i_k} \wedge \cdots \wedge dx_{i_p}, \end{aligned}$$

and

$$\begin{aligned}
d^*d\ &\Big(\sum_{1\le i_1<\cdots<i_p\le m} f_{i_1\dots i_p}dx_{i_1}\wedge\cdots\wedge dx_{i_p}\Big)\\
= \ & d^*\Big(\sum_{1\le i_1<\cdots<i_p\le m}\sum_{i\notin\{i_1,\dots,i_p\}}\frac{\partial f_{i_1\dots i_p}}{\partial x_i}dx_i\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_p}\Big)\\
= \ & -\sum_{1\le i_1<\cdots<i_p\le m}\sum_{i\notin\{i_1,\dots,i_p\}}\frac{\partial^2 f_{i_1\dots i_p}}{\partial x_i^2}dx_{i_1}\wedge\cdots\wedge dx_{i_p}\\
& -\sum_{1\le i_1<\cdots<i_p\le m}\sum_{i\notin\{i_1,\dots,i_p\}}\sum_{k=1}^{p}\frac{\partial^2 f_{i_1\dots i_p}}{\partial x_{i_k}\partial x_i}(-1)^k dx_i\wedge dx_{i_1}\wedge\cdots\wedge \widehat{dx_{i_k}}\wedge\cdots\wedge dx_{i_p}.
\end{aligned}$$

So we have

$$\Delta_d\Big(\sum_{1\le i_1<\cdots<i_p\le m} f_{i_1\dots i_p}dx_{i_1}\wedge\cdots\wedge dx_{i_p}\Big)=-\sum_{1\le i_1<\cdots<i_p\le m}\Big(\sum_i\frac{\partial^2 f_{i_1\dots i_p}}{\partial x_i^2}\Big)dx_{i_1}\wedge\cdots\wedge dx_{i_p}.$$

The main result is that harmonic forms exists in each cohomology class:

Theorem 4.4 (Hodge)**.** *Let (M^m, g) be a compact oriented Riemannian manifold. Then each de Rham cohomology class has a unique harmonic representative, so we have a linear isomorphism*

$$\mathcal{H}^p(M) := \{\omega\in A^p(M)\,\big|\,\Delta_d\omega=0\}\cong H^p_{dR}(M;\mathbb{R}),\quad p=0,\dots,m.$$

Moreover, $\mathcal{H}^p(M)$ is always a finite dimensional vector space,[1] *and we have a linear operator $G: A^p(M)\to A^p(M)$ such that for any $\omega\in A^p(M)$, if we denote its orthogonal projection to $\mathcal{H}^p(M)$ by ω_h, then we have the decomposition:*

$$\omega=\omega_h+\Delta_d G\omega=\omega_h+d(d^*G\omega)+d^*(dG\omega).$$

In fact, we have a orthogonal direct sum decomposition $A^p(M)=\mathcal{H}^p(M)\oplus Im\, d\oplus Im\, d^*$.

Remark 4.5. *G is usually called the "Green operator". It is constructed in the following way: suppose the eigenvalues of Δ_d on $A^p(M)$ are $0=\lambda_0<\lambda_1<\lambda_2<\dots$. The corresponding eigenspaces are $\mathcal{H}^p(M)$ and $E_1, E_2,\dots$. Then we define $G|_{\mathcal{H}^p(M)}\equiv 0$ and $G|_{E_i}:=\frac{1}{\lambda_i}id_{E_i}$.*

Proof of parts of the results: Uniqueness: Suppose ω_1 and ω_2 are both harmonic p-forms and $\omega_2=\omega_1+d\eta$ for some $\eta\in A^{p-1}(M)$. Then

$$(d\eta,d\eta)=(\omega_2-\omega_1,d\eta)=(d^*(\omega_2-\omega_1),\eta)=0.$$

[1] We can prove directly that $H^p_{dR}(M;\mathbb{R})$ is a finite dimensional vector space via the Mayer-Vietoris argument as in Bott-Tu's book.

So we necessarily have $d\eta = 0$ and $\omega_2 = \omega_1$.

$\mathcal{H}^p(M), Im\, d, Im\, d^*$ are orthogonal to each other: Let $\omega_h \in \mathcal{H}^p(M),\ \xi \in A^{p+1}(M), \eta \in A^{p-1}(M)$, then

$$\begin{aligned}(\omega_h, d^*\xi) = &\quad (d\omega_h, \xi) = 0\\ (\omega_h, d\eta) = &\quad (d^*\omega_h, \eta) = 0\\ (d^*\xi, d\eta) = &\quad (\xi, dd\eta) = 0.\end{aligned}$$

Rough idea about existence: One can show that Δ_d is a 2nd order elliptic operator, and we have a "basic estimate" of the form

$$\|\omega\|^2_{W^{1,2}} \leq C(\Delta_d\omega + \omega, \omega) = C(\|\omega\|^2 + \|d\omega\|^2 + \|d^*\omega\|^2).$$

(For general elliptic operator, this kind of estimates still hold, known as "Gårding's inequality".) We consider the quadratic form on $W^{1,2}(M, \Lambda^p T^*M)$:

$$\mathcal{D}(\xi, \eta) := (\xi, \eta) + (d\xi, d\eta) + (d^*\xi, d^*\eta).$$

Gårding's inequality implies that $\mathcal{D}(\omega)$ is an equivalent norm on $W^{1,2}(M, \Lambda^p T^*M)$. Given $\eta \in L^2(M, \Lambda^p T^*M)$, $\xi \mapsto (\xi, \eta)$ is a bounded linear functional on $A^p(M) \subset W^{1,2}(M, \Lambda^p T^*M)$:

$$|(\xi, \eta)| \leq \|\xi\| \cdot \|\eta\| \leq \|\eta\| \cdot \|\xi\|_{W^{1,2}} \leq C\sqrt{\mathcal{D}(\xi, \xi)}.$$

This extends to a bounded linear functional on $W^{1,2}(M, \Lambda^p T^*M)$, and we can use Riesz representation theorem to get a unique $\varphi \in W^{1,2}(M, \Lambda^p T^*M)$ such that for all $\xi \in A^p(M)$:

$$(\xi, \eta) = \mathcal{D}(\xi, \varphi).$$

Using this to define a linear map $T(\eta) := \varphi$. It is a bounded linear operator from $L^2(M, \Lambda^p T^*M)$ to $W^{1,2}(M, \Lambda^p T^*M)$. Its composition with the compact embedding $W^{1,2} \to L^2$ (also denoted by T) gives us a compact self-adjoint operator on $L^2(M, \Lambda^p T^*M)$. Intuitively, $T = (id + \Delta_d)^{-1}$.

By spectrum theorem and elliptic regularity, we have a Hilbert space direct sum decomposition $L^2(M, \Lambda^p T^*M) = \oplus_{m=0}^{\infty} E_m$, where each E_m is a finite dimensional space of smooth p-forms, satisfying $T\varphi = \rho_m\varphi, \forall\varphi \in E_m$, with $\rho_0 = 1 > \rho_1 > \rho_2 \dots$ and $\rho_m \to 0$. Then $E_0 = \mathcal{H}^p(M)$ and for $\varphi \in E_m$, we have $\Delta_d\varphi = \left(\frac{1}{\rho_m} - 1\right)\varphi =: \lambda_m\varphi$, $\lambda_m \nearrow \infty$. □

4.2 The Hermitian case

Now let X^n be a n-dimensional compact complex manifold, with almost complex structure J and Hermitian metric g. As before, we define $\omega_g := \sqrt{-1}\sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$. It is a real $(1,1)$-form. A direct computation shows that we always have

$$dV_g = \frac{\omega_g^n}{n!}.$$

In fact, we can choose coordinates around a given point p such that at p, $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\}_{i,j=1}^n$ is an orthonormal basis with $z_i = x_i + \sqrt{-1}y_i$ the complex coordinate function. [2] Then at p the left equals $dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ while on the other hand, we have at p: $\omega_g = \frac{\sqrt{-1}}{2}\sum_i dz_i \wedge d\bar{z}_i = \sum_i dx_i \wedge dy_i$ and hence $\frac{\omega_g^n}{n!} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n = dV_g$.

Exercise: *Show that under local coordinates, we have*

$$\frac{\omega_g^n}{n!} = \det(g_{i\bar{j}})(\sqrt{-1})^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.$$

In this case, we also extend Hodge's star operator complex linearly to complex differential forms. Then we also have $** = (-1)^{p(2n-p)} = (-1)^p$ on $A^p(X)$ and

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle_{\mathbb{C}} dV_g.$$

On the space of smooth complex differential forms, the correct Hermitian inner product should be

$$(\alpha, \beta) := \int_X \alpha \wedge *\bar{\beta}.$$

Lemma 4.6. *The $*$ operator maps $A^{p,q}(X)$ to $A^{n-q,n-p}(X)$.*

Proof. We compute at a given point x, and we choose complex coordinates such that $g_{i\bar{j}}(x) = \frac{1}{2}\delta_{ij}$. Then $dx_1, dy_1, \ldots, dx_n, dy_n$ is a positively oriented orthonormal basis of $T_x^{*\mathbb{R}}X$. For multi-index $I = (\mu_1, \ldots, \mu_p)$, we shall write

$$dz_I := dz_{\mu_1} \wedge \cdots \wedge dz_{\mu_p}, \quad dx_I := dx_{\mu_1} \wedge \cdots \wedge dx_{\mu_p}, \quad \ldots$$

Also for multi-index M, we define

$$w_M := \Pi_{\mu \in M} dz_\mu \wedge d\bar{z}_\mu = (-2\sqrt{-1})^{|M|}\Pi_{\mu \in M} dx_\mu \wedge dy_\mu.$$

A direct computation shows that for mutually disjoint increasing multi-indices A, B, M, we have

$$*(dz_A \wedge d\bar{z}_B \wedge w_M) = \gamma(a,b,m) dz_A \wedge d\bar{z}_B \wedge w_{M'},$$

[2] What we need to do is to use a complex linear coordinate transformation such that $g_p(\frac{\partial}{\partial z_i}|_p, \frac{\partial}{\partial \bar{z}_j}|_p) = \frac{1}{2}\delta_{ij}$.

where $a = |A|, b = |B|, m = |M|$, $M' = (1, 2, , \ldots, n) - (A \cup B \cup M)$, and $\gamma(a, b, m)$ is a non-vanishing constant. In fact, one can show that

$$\gamma(a, b, m) = (\sqrt{-1})^{a-b}(-1)^{\frac{k(k+1)}{2}+m}(-2\sqrt{-1})^{k-n},$$

where $k = a + b + 2m$ is the total degree.

If we write $p = a + m, q = b + m$, then all (p, q)-form is locally a linear combination of forms of the type $dz_A \wedge d\bar{z}_B \wedge w_M$. Since $dz_A \wedge d\bar{z}_B \wedge w_{M'}$ is a $(a + m', b + m') = (a+n-a-b-m, b+n-a-b-m) = (n-q, n-p)$-form, we get $*A^{p,q}(X) \subset A^{n-q,n-p}(X)$. $\square$

As in the real case, we consider the Hermitian inner product on $A^{p,q}(X)$, and define an operator $\bar{\partial}^*$ by

$$(\xi, \bar{\partial}\eta) = (\bar{\partial}^*\xi, \eta), \quad \forall \xi \in A^{p,q}(X), \eta \in A^{p,q-1}(X).$$

Then we get

$$\begin{aligned}
(\bar{\partial}^*\xi, \eta) &= \int_X \bar{\partial}^*\xi \wedge *\bar{\eta} \\
&= (\xi, \bar{\partial}\eta) = \overline{(\bar{\partial}\eta, \xi)} = \overline{\int_X \bar{\partial}\eta \wedge *\bar{\xi}} = \int_X \partial\bar{\eta} \wedge *\xi \\
&= \int_X \partial\big(\bar{\eta} \wedge *\xi\big) - (-1)^{p+q-1}\bar{\eta} \wedge \partial(*\xi) = (-1)^{p+q}\int_X \bar{\eta} \wedge \partial(*\xi) \\
&= -\int_X \partial(*\xi) \wedge \bar{\eta} = -\int_X *\partial(*\xi) \wedge *\bar{\eta}.
\end{aligned}$$

So we get:

Lemma 4.7. *On $A^{p,q}(X)$, we always have $\bar{\partial}^* = - * \partial *$.*

Exercise: Show that on the space of complex valued p-forms $A^p(X)$, we have $d^* = - * d *$.

We define the $\bar{\partial}$-Laplacian $\Delta_{\bar{\partial}} : A^{p,q}(X) \to A^{p,q}(X)$ by

$$\Delta_{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

We look for $\bar{\partial}$-closed form of minimal norm within a given Dolbeault cohomology class. Suppose $\xi \in A^{p,q}(X)$ is such a $\bar{\partial}$-closed form, then for any $\eta \in A^{p,q-1}(X)$, the quadratic function of $t \in \mathbb{R}$:

$$||\xi + t\bar{\partial}\eta||^2 = (\xi + t\bar{\partial}\eta, \xi + t\bar{\partial}\eta) = ||\xi||^2 + 2tRe(\xi, \bar{\partial}\eta) + t^2||\bar{\partial}\eta||^2$$

takes its minimum at $t = 0$. We get $Re(\xi, \bar{\partial}\eta) = 0$ for all $\eta \in A^{p,q-1}(X)$. Using $||\xi + t\sqrt{-1}\bar{\partial}\eta||^2$ instead, we get $Im(\xi, \bar{\partial}\eta) = 0$ for all $\eta \in A^{p,q-1}(X)$. So we get $(\xi, \bar{\partial}\eta) = (\bar{\partial}^*\xi, \eta) = 0$ for all $\eta \in A^{p,q-1}(X)$. This implies $\bar{\partial}^*\xi = 0$.

Definition 4.8. *If $\omega \in A^{p,q}(X)$ satisfies $\bar{\partial}\omega = 0$ and $\bar{\partial}^*\omega = 0$ (equivalently, $\Delta_{\bar{\partial}}\omega = 0$), then ω is called a "$\bar{\partial}$-harmonic (p,q)-form".*

The counterpart of Hodge theorem for Dolbeault cohomology is the following:

Theorem 4.9 (Hodge)**.** *Let (X^n, J, g) be a compact Hermitian manifold. Then each Dolbeault cohomology class has a unique $\bar{\partial}$-harmonic representative, so we have a complex linear isomorphism*

$$\mathcal{H}^{p,q}(X) := \{\omega \in A^{p,q}(X) \mid \Delta_{\bar{\partial}}\omega = 0\} \cong H^{p,q}_{\bar{\partial}}(X), \quad p, q = 0, \dots, n.$$

Moreover, $\mathcal{H}^{p,q}(X)$ is always a finite dimensional complex vector space, and we have a complex linear operator $G : A^{p,q}(X) \to A^{p,q}(X)$ such that for any $\omega \in A^{p,q}(X)$, if we denote its orthogonal projection to $\mathcal{H}^{p,q}(X)$ by ω_h, then we have the decomposition:

$$\omega = \omega_h + \Delta_{\bar{\partial}}G\omega = \omega_h + \bar{\partial}(\bar{\partial}^* G\omega) + \bar{\partial}^*(\bar{\partial}G\omega).$$

In fact, we have an orthogonal direct sum decomposition $A^{p,q}(X) = \mathcal{H}^{p,q}(X) \oplus Im\, \bar{\partial} \oplus Im\, \bar{\partial}^*$.

Generalization: Assume also that we have a holomorphic vector bundle $E \to X$ of rank r, with Hermitian metric h. X is compact. We define an Hermitian inner product on $C^\infty(X, \Lambda^{p,q}(X) \otimes E)$ by

$$(s, t) := \int_X \langle s, t\rangle_{g,h} dV_g,$$

where the pointwise Hermitian inner product $\langle , \rangle_{g,h}$ is induced from the Hermitian metric g on X and bundle metric h on E. We can define a $\bar{\partial}$-operator on $A^{p,q}(X, E)$, which we shall write $\bar{\partial}_E : A^{p,q}(X, E) \to A^{p,q+1}(X, E)$. We can also define a formal adjoint operator $\bar{\partial}^*_E : A^{p,q}(X, E) \to A^{p,q-1}(X, E)$ by requiring that

$$(s, \bar{\partial}_E t) = (\bar{\partial}^*_E s, t), \quad \forall s \in A^{p,q}(X, E), t \in A^{p,q-1}(X, E).$$

Then we define $\Delta_{\bar{\partial}_E} := \bar{\partial}^*_E\bar{\partial}_E + \bar{\partial}_E\bar{\partial}^*_E : A^{p,q}(X, E) \to A^{p,q}(X, E)$, and $\mathcal{H}^{p,q}(X, E) := Ker\Big(\Delta_{\bar{\partial}_E}|_{A^{p,q}(X,E)}\Big)$. The elements of $\mathcal{H}^{p,q}(X, E)$ are called "E-valued harmonic (p, q)-forms". In this case, we also have:

Theorem 4.10. *Let (X^n, J, g) be a compact Hermitian manifold. $E \to X$ be a holomorphic vector bundle of rank r, with Hermitian metric h. Then each cohomology class in $H^{p,q}_{\bar{\partial}}(X, E)$ has a unique harmonic representative, so we have a complex linear isomorphism*

$$\mathcal{H}^{p,q}(X, E) \cong H^{p,q}_{\bar{\partial}}(X, E), \quad p, q = 0, \dots, n.$$

Moreover, $\mathcal{H}^{p,q}(X, E)$ is always a finite dimensional complex vector space, and we have a complex linear operator $G_E : A^{p,q}(X, E) \to A^{p,q}(X, E)$ such that for any $\omega \in A^{p,q}(X, E)$, if we denote its orthogonal projection to $\mathcal{H}^{p,q}(X, E)$ by ω_h, then we have the decomposition:

$$\omega = \omega_h + \Delta_{\bar{\partial}_E}G_E\omega = \omega_h + \bar{\partial}_E(\bar{\partial}^*_E G_E\omega) + \bar{\partial}^*_E(\bar{\partial}_E G_E\omega).$$

In fact, we have an orthogonal direct sum decomposition $A^{p,q}(X, E) = \mathcal{H}^{p,q}(X, E) \oplus Im\, \bar{\partial}_E \oplus Im\, \bar{\partial}^*_E$.

4.3 Applications

Theorem 4.11 (Poincaré duality for de Rham cohomology)**.** *Let M^m be a compact oriented differentiable manifold. Then*

$$H^p_{dR}(M,\mathbb{R}) \cong H^{m-p}_{dR}(M,\mathbb{R}).$$

In particular, $b_p(M) = b_{m-p}(M)$.

Proof. Since $*$ commutes with Δ_d, and $** = \pm 1$, we conclude that $*$ induces a linear isomorphism between $\mathcal{H}^p(M)$ and $\mathcal{H}^{m-p}(M)$. Then the result follows from Hodge theorem. □

Theorem 4.12 (Kodaira-Serre duality)**.** *Let $E \to X$ be a holomorphic vector bundle over a compact complex manifold X of complex dimension n. Then we have a conjugate-linear isomorphism*

$$\sigma : H^r(X, \Omega^p(E)) \xrightarrow{\cong} H^{n-r}(X, \Omega^{n-p}(E^*)).$$

Proof. (Sketch) We introduce a conjugate-linear operator $\bar{*}_E$, constructing from $* : A^{p,q} \to A^{n-q,n-p}$ and the conjugate-linear isomorphism $\tau : E \to E^*$ via bundle metric h. To make everything conjugate-linear, we also define $\bar{*} : A^{p,q}(X) \to A^{n-p,n-q}(X)$ by $\bar{*}(\eta) := *\bar{\eta}$. Then $\bar{*}_E : A^{p,q}(X, E) \to A^{n-p,n-q}(X, E^*)$ is defined by

$$\bar{*}_E(\eta \otimes s) := \bar{*}(\eta) \otimes \tau(s).$$

Then we have $\bar{\partial}^*_E = -\bar{*}_{E^*} \circ \bar{\partial}_{E^*} \circ \bar{*}_E$ and hence $\bar{*}_E \Delta_{\bar{\partial}_E} = \Delta_{\bar{\partial}_{E^*}} \bar{*}_E$.

By Hodge theorem, we have

$$H^r(X, \Omega^p(E)) \cong H^{p,r}_{\bar{\partial}}(X, E), \quad H^{n-r}(X, \Omega^{n-p}(E^*)) \cong H^{n-p,n-r}_{\bar{\partial}}(X, E^*).$$

Then $\bar{*}_E$ induces a conjugate-linear map $\sigma : H^r(X, \Omega^p(E)) \to H^{n-r}(X, \Omega^{n-p}(E^*))$, and the Kodaira-Serre duality follows from the fact $\bar{*}_E \circ \bar{*}_{E^*} = \pm 1$. □

4.4 The Kähler case

Now we assume (X^n, J, g) is a compact Kähler manifold. Then we will have a better understanding of harmonic forms and Dolbeault cohomology. We shall begin by exploring the relation between Δ_d and $\Delta_{\bar\partial}$.

4.4.1 Hodge identities for Kähler metrics

We introduce some operators that will be useful in our discussion:

$$d^c := \sqrt{-1}(\bar\partial - \partial).$$

Here my notation is the same as Wells, but differs from Griffiths-Harris by a factor 4π. Then $dd^c = \sqrt{-1}(\partial + \bar\partial)(\bar\partial - \partial) = 2\sqrt{-1}\partial\bar\partial$. We define the *"Lefschetz operator"* $L : A^{p,q}(X) \to A^{p+1,q+1}(X)$ by:

$$L(\eta) := \omega_g \wedge \eta =: L\eta.$$

Its adjoint will be denoted by $\Lambda : A^{p+1,q+1}(X) \to A^{p,q}(X)$. We have

$$(\xi, L\eta) = (\Lambda\xi, \eta), \quad \forall \xi \in A^{p+1,q+1}(X), \eta \in A^{p,q}(X).$$

The basic equality in the Kähler case is:

Lemma 4.13. *On $A^{p,q}(X)$, we have $[\Lambda, \partial] = \sqrt{-1}\bar\partial^*$.*

Given this, since L is a real operator, so is Λ, and we have

$$[\Lambda, \bar\partial] = -\sqrt{-1}\partial^*.$$

Combining these two identities, we further get

$$[\Lambda, d] = -d^{c*}, \quad [\Lambda, d^c] = d^*.$$

Proof of Lemma 4.13 . We first prove the identity in $\mathbb{C}^n$. Let $\omega = \frac{\sqrt{-1}}{2}\sum_i dz_i \wedge d\bar z_i$ be the standard Kähler form on $\mathbb{C}^n$. Let $A_c^{p,q}(\mathbb{C}^n)$ be the space of smooth (p,q)-forms on $\mathbb{C}^n$ with compact support. Then $L : A_c^{p,q}(\mathbb{C}^n) \to A_c^{p+1,q+1}(\mathbb{C}^n), L\eta := \omega \wedge \eta$.

To derive a formula for $\Lambda = L^*$, we introduce operators $e_k, \bar e_k$ by

$$e_k(\eta) := dz_k \wedge \eta, \quad \bar e_k(\eta) := d\bar z_k \wedge \eta.$$

Their adjoints are denoted by i_k and $\bar i_k$ respectively. Recall that $\|dz_k\|^2 = \|dx\|^2 + \|dy\|^2 = 2$, so we conclude that $i_k = 2\iota_{\frac{\partial}{\partial z_k}}$, where $\iota_{\frac{\partial}{\partial z_k}}$ is the "interior product" operator, defined by $\iota_{\frac{\partial}{\partial z_k}}\eta = \eta(\frac{\partial}{\partial z_k}, \cdot, \ldots, \cdot)$. Similarly, $\bar i_k = 2\iota_{\frac{\partial}{\partial \bar z_k}}$. It is easy to check that

$$i_k e_k + e_k i_k = 2, \quad \bar i_k \bar e_k + \bar e_k \bar i_k = 2.$$

And for $k \neq l$,

$$e_k i_l + i_l e_k = 0, \quad \bar{e}_k \bar{i}_l + \bar{i}_l \bar{e}_k = 0.$$

We also define the degree-preserving linear maps $\partial_k, \bar{\partial}_k$ by

$$\partial_k\Big(\sum_{I,J} \eta_{I\bar{J}} dz_I \wedge d\bar{z}_J\Big) := \sum_{I,J} \frac{\eta_{I\bar{J}}}{\partial z_k} dz_I \wedge d\bar{z}_J,$$

$$\bar{\partial}_k\Big(\sum_{I,J} \eta_{I\bar{J}} dz_I \wedge d\bar{z}_J\Big) := \sum_{I,J} \frac{\eta_{I\bar{J}}}{\partial \bar{z}_k} dz_I \wedge d\bar{z}_J.$$

Note that ∂_k and $\bar{\partial}_k$ commute with $e_k, \bar{e}_k$ and hence also $i_k, \bar{i}_k$. Also an "integration by part" trick gives us the relation $\partial_k^* = -\bar{\partial}_k$, $\bar{\partial}_k^* = -\partial_k$.

Now we can express all the operators we care by $e_k, \bar{e}_k, i_k, \bar{i}_k$ and $\partial_k, \bar{\partial}_k$:

$$\partial = \sum_k \partial_k e_k = \sum_k e_k \partial_k, \quad \bar{\partial} = \sum_k \bar{\partial}_k \bar{e}_k = \sum_k \bar{e}_k \bar{\partial}_k.$$

Taking adjoints, we get

$$\partial^* = -\sum_k \bar{\partial}_k i_k = \sum_k i_k \bar{\partial}_k, \quad \bar{\partial}^* = -\sum_k \partial_k \bar{i}_k = -\sum_k \bar{i}_k \partial_k.$$

Also

$$L = \frac{\sqrt{-1}}{2} \sum_k e_k \bar{e}_k, \quad \Lambda = -\frac{\sqrt{-1}}{2} \sum_k \bar{i}_k i_k.$$

So we can compute

$$\begin{aligned}
\Lambda\partial &= -\frac{\sqrt{-1}}{2} \sum_{k,l} \bar{i}_k i_k \partial_l e_l = -\frac{\sqrt{-1}}{2} \sum_{k,l} \partial_l \bar{i}_k i_k e_l \\
&= -\frac{\sqrt{-1}}{2} \Big(\sum_k \partial_k \bar{i}_k i_k e_k + \sum_{k \neq l} \partial_l \bar{i}_k i_k e_l \Big).
\end{aligned}$$

We compute the last two summands seperately.

$$\begin{aligned}
-\frac{\sqrt{-1}}{2} \sum_k \partial_k \bar{i}_k i_k e_k &= -\frac{\sqrt{-1}}{2} \sum_k \partial_k \bar{i}_k (2 - e_k i_k) \\
&= -\sqrt{-1} \sum_k \partial_k \bar{i}_k - \frac{\sqrt{-1}}{2} \sum_k \partial_k e_k \bar{i}_k i_k \\
&= \sqrt{-1}\bar{\partial}^* - \frac{\sqrt{-1}}{2} \sum_k \partial_k e_k \bar{i}_k i_k,
\end{aligned}$$

and

$$-\frac{\sqrt{-1}}{2}\sum_{k\neq l}\partial_l\bar{i}_k i_k e_l \quad = \quad \frac{\sqrt{-1}}{2}\sum_{k\neq l}\partial_l\bar{i}_k e_l i_k = -\frac{\sqrt{-1}}{2}\sum_{k\neq l}\partial_l e_l\bar{i}_k i_k.$$

So we get

$$\Lambda\partial = \sqrt{-1}\bar{\partial}^* - \frac{\sqrt{-1}}{2}\sum_{k}\partial_k e_k\bar{i}_k i_k - \frac{\sqrt{-1}}{2}\sum_{k\neq l}\partial_l e_l\bar{i}_k i_k = \sqrt{-1}\bar{\partial}^* + \partial\Lambda.$$

For the general compact Kähler case, one can use Kähler normal coordinates to reduce the computations to our $\mathbb{C}^n$ case. The key point is that only first order derivatives are involved. $\square$

4.4.2 Hodge decomposition for compact Kähler manifolds

A direct consequence of Hodge identities is that Δ_d commutes with both L and Λ: Since ω_g is closed, we have $dL(\eta) = d(\omega_g \wedge \eta) = \omega_g \wedge d\eta$, so $[L, d] = 0$. Taking adjoints, we get $[\Lambda, d^*] = 0$. So using $[\Lambda, d] = -d^{c*}$, we get

$$\begin{aligned}
\Lambda\Delta_d &= \Lambda(dd^* + d^*d) = [\Lambda, d]d^* + d\Lambda d^* + d^*\Lambda d \\
&= -d^{c*}d^* + dd^*\Lambda + d^*[\Lambda, d] + d^*d\Lambda \\
&= -d^{c*}d^* - d^*d^{c*} + \Delta_d\Lambda = \Delta_d\Lambda.
\end{aligned}$$

Taking adjoints, we also get $[L, \Delta_d] = 0$.

Besides Δ_d and $\Delta_{\bar{\partial}}$, we can similarly define Δ_∂. For compact Kähler manifolds, we have the following:

Proposition 4.14. *In the Kähler case, we always have $\Delta_{\bar{\partial}} = \Delta_\partial = \frac{1}{2}\Delta_d$.*

Proof. Use $d = \partial + \bar{\partial}$ and $d^* = \partial^* + \bar{\partial}^*$ to compute:

$$\begin{aligned}
\Delta_d &= dd^* + d^*d = (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}) \\
&= (\partial\partial^* + \partial^*\partial) + (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) + \partial\bar{\partial}^* + \bar{\partial}\partial^* + \partial^*\bar{\partial} + \bar{\partial}^*\partial \\
&= \Delta_\partial + \Delta_{\bar{\partial}} + (\partial\bar{\partial}^* + \bar{\partial}^*\partial) + (\bar{\partial}\partial^* + \partial^*\bar{\partial}).
\end{aligned}$$

We need to prove:

- $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$, $\bar{\partial}\partial^* + \partial^*\bar{\partial} = 0$ (these two identities are equivalent by conjugation);
- $\Delta_\partial = \Delta_{\bar{\partial}}$.

To prove $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$, we use the Hodge identity $[\Lambda, \partial] = \sqrt{-1}\bar{\partial}^*$:

$$\begin{aligned}\sqrt{-1}(\partial\bar{\partial}^* + \bar{\partial}^*\partial) &= \partial[\Lambda, \partial] + [\Lambda, \partial]\partial \\ &= \partial\Lambda\partial - \partial^2\Lambda + \Lambda\partial^2 - \partial\Lambda\partial \\ &= 0.\end{aligned}$$

Now we compute Δ_∂ and $\Delta_{\bar{\partial}}$ separately, both using Hodge identities:

$$\begin{aligned}-\sqrt{-1}\Delta_\partial &= \partial[\Lambda, \bar{\partial}] + [\Lambda, \bar{\partial}]\partial \\ &= \partial\Lambda\bar{\partial} - \partial\bar{\partial}\Lambda + \Lambda\bar{\partial}\partial - \bar{\partial}\Lambda\partial.\end{aligned}$$

$$\begin{aligned}\sqrt{-1}\Delta_{\bar{\partial}} &= \bar{\partial}[\Lambda, \partial] + [\Lambda, \partial]\bar{\partial} \\ &= \bar{\partial}\Lambda\partial - \bar{\partial}\partial\Lambda + \Lambda\partial\bar{\partial} - \partial\Lambda\bar{\partial} \\ &= \bar{\partial}\Lambda\partial + \partial\bar{\partial}\Lambda - \Lambda\bar{\partial}\partial - \partial\Lambda\bar{\partial} \\ &= \sqrt{-1}\Delta_\partial.\end{aligned}$$

From the above computations, we conclude that $\Delta_d = \Delta_\partial + \Delta_{\bar{\partial}} = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$. □

From this we conclude that $\Delta_d : A^{p,q}(X) \to A^{p,q}(X)$, and

$$\mathcal{H}_d^{p+q}(X, \mathbb{C}) \cap A^{p,q}(X) = \mathcal{H}_{\bar{\partial}}^{p,q}(X).$$

Since $\mathcal{H}_d^r(X, \mathbb{C}) = \oplus_{p+q=r}\left(\mathcal{H}_d^{p+q}(X, \mathbb{C}) \cap A^{p,q}(X)\right) = \oplus_{p+q=r}\mathcal{H}_{\bar{\partial}}^{p,q}(X)$. Also note that $\overline{\mathcal{H}_{\bar{\partial}}^{p,q}(X)} = \mathcal{H}_{\bar{\partial}}^{q,p}(X)$. Applying Hodge theorem for compact Hermitian manifolds, we get:

Theorem 4.15 (Hodge decomposition for compact Kähler manifolds)**.** *Let (X^n, J, g) be a compact Kähler manifold, then we have isomorphisms*

$$H_{dR}^r(X, \mathbb{C}) \cong \oplus_{p+q=r}H_{\bar{\partial}}^{p,q}(X) \cong \oplus_{p+q=r}H^q(X, \Omega^p), \quad r = 0, 1, \ldots, 2n,$$

and

$$\overline{H_{\bar{\partial}}^{p,q}(X)} \cong H_{\bar{\partial}}^{q,p}(X).$$

In particular, we have

$$b_r = \sum_{p+q=r} h^{p,q}, \quad h^{p,q} = h^{q,p}.$$

For example, we always have

$$\mathcal{H}_{\bar{\partial}}^{p,0}(X) = H^0(X, \Omega^p),$$

since any $(p, 0)$-form is $\bar{\partial}^*$-closed and it is $\bar{\partial}$-closed if and only if it is holomorphic. Then we conclude that *any holomorphic p-form on a compact Kähler manifold is also d-closed and even d-harmonic.*

Exercise: Show that any holomorphic 1-form on a compact complex surface (not necessarily Kähler) is always d-closed. (Kodaira)

Corollary 4.16. *The odd Betti number b_{2k+1} of a compact Kähler manifold X^n is always even.*

Proof. We have

$$\begin{aligned} b_{2k+1} &= \sum_{0\le p,q\le n, p+q=2k+1} h^{p,q} \\ &= \sum_{p<q, p+q=2k+1} h^{p,q} + \sum_{p>q,p+q=2k+1} h^{p,q} \\ &= \sum_{p<q, p+q=2k+1} h^{p,q} + \sum_{p>q,p+q=2k+1} h^{q,p} \\ &= 2\sum_{p<q,p+q=2k+1} h^{p,q} \equiv 0 \mod 2. \end{aligned}$$

□

As a concrete application, let's compute the cohomologies of $\mathbb{C}P^n$: The topological structure is rather simple: we have $\mathbb{C}P^n = U_0 \cup \{z_0 = 0\}$, with $U_0 \cong \mathbb{C}^n$ and $\{z_0 = 0\} \cong \mathbb{C}P^{n-1}$. So we can construct $\mathbb{C}P^n$ in the following way: start with a point (a "0-cell"), glue a $\mathbb{C}^1$ (a "2-cell") to get $\mathbb{C}P^1$, then glue a $\mathbb{C}^2$ (a "4-cell") to get $\mathbb{C}P^2$, $\ldots$.... So the cellular cohomologies of $\mathbb{C}P^n$ are:

$$H^{2k+1}(\mathbb{C}P^n, \mathbb{Z}) = 0, \quad H^{2k}(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}, k = 0, \ldots, n.$$

Now ω_{FS} is a Kähler forms on $\mathbb{C}P^n$. Since $\omega_{FS}^k = L^k 1$ and $\Delta_d L = L\Delta_d$, each ω_{FS}^k is a harmonic (k,k)-form. So we conclude that $h^{p,p} \ge 1, p = 0, \ldots, n$. On the other hand, $1 = b_{2p} \ge h^{p,p}$, we must have $b_{2p} = h^{p,p}$. Also, $h^{p,q} = 0$ when $p+q$ is odd. So the only non-zero Dolbeault cohomologies of $\mathbb{C}P^n$ are $H_{\bar\partial}^{p,p}(X) \cong \mathbb{C}, p = 0, \ldots, n$. In particular, there are no non-zero holomorphic forms on $\mathbb{C}P^n$.

For another application, we state the so called "$\partial\bar\partial$-lemma", which is very useful in Kähler geometry:

Lemma 4.17. *If η is any d-closed (p,q)-form on a compact Kähler manifold X^n, and η is d- or ∂- or $\bar\partial$-exact, then*

$$\eta = \partial\bar\partial\gamma$$

for some $(p-1, q-1)$-form γ. When $p = q$ and η is real, then we can take $\gamma = \sqrt{-1}\xi$ for a real $(p-1, q-1)$-form ξ.

Proof. Recall that in the Kähler case we have $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$, they share the same kernel: harmonic forms. Since η is d- or ∂- or $\bar\partial$-exact, its harmonic projection must be zero. So we have

$$\eta = \Delta_{\bar\partial} G_{\bar\partial}\eta = \bar\partial\bar\partial^* G_{\bar\partial}\eta.$$

Here we use the fact that $\bar\partial$ commutes with $G_{\bar\partial}$ and that $d\eta = 0 \Rightarrow \bar\partial\eta = 0$.

Now we look at the form $\bar\partial^* G_{\bar\partial}\eta$, it is also orthogonal to harmonic forms. Also since $G_\partial = G_{\bar\partial}$, we have $\partial\bar\partial^* G_{\bar\partial}\eta = -\bar\partial^*\partial G_\partial\eta = -\bar\partial^* G_\partial\partial\eta = 0$. Then we can use Hodge decomposition for Δ_∂:

$$\bar\partial^* G_{\bar\partial}\eta = \Delta_\partial G_\partial \bar\partial^* G_{\bar\partial}\eta = \partial\partial^* G_\partial\bar\partial^* G_{\bar\partial}\eta.$$

So we get

$$\eta = \bar\partial\partial\partial^* G_\partial\bar\partial^* G_{\bar\partial}\eta = \partial\bar\partial\Big(- \partial^* G_\partial\bar\partial^* G_{\bar\partial}\eta\Big) = \partial\bar\partial\Big(- \partial^*\bar\partial^* G_{\bar\partial}^2\eta\Big).$$

□

The most often used case is about $(1,1)$-class. Let ω and $\tilde\omega$ be two Kähler forms on X such that $[\omega] = [\tilde\omega] \in H^2_{dR}(X)$. Then $\tilde\omega - \omega$ is a $d-$exact form, so by the $\partial\bar\partial$-lemma, we can find a smooth function $\varphi \in C^\infty(X;\mathbb{R})$ such that

$$\tilde\omega = \omega + \sqrt{-1}\partial\bar\partial\varphi.$$

φ is unique up to a constant. On the other hand, if $\varphi \in C^\infty(X;\mathbb{R})$ satisfies $\omega + \sqrt{-1}\partial\bar\partial\varphi > 0$, then it defines a Kähler metric with the same Kähler class. So we conclude that the space of Kähler metrics within the same cohomology class $[\omega]$ is isomorphic to

$$\{\varphi \in C^\infty(X;\mathbb{R}) \mid \omega + \sqrt{-1}\partial\bar\partial\varphi > 0\}\Big/\mathbb{R}.$$

One of the most important problem in Kähler geometry is the existence of canonical metrics in a given Kähler class. Through the $\partial\bar\partial$-lemma, we can reduce the problem to a (usually non-linear) partial differential equation for φ. This is the starting point of using non-linear PDEs to solve problems in Kähler geometry.

Remark 4.18. *If we further introduce the operator $h : A^*(X) \to A^*(X)$ by $h = \sum_{p=0}^{2n}(n - p)\Pi_p$, then we will have*

$$[\Lambda, L] = h, \quad [h,\Lambda] = 2\Lambda, \quad [h, L] = -2L.$$

Recall the 3-dimensional complex Lie algebra sl_2, generated by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

They satisfy

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

So $H \mapsto h, X \mapsto \Lambda, Y \mapsto L$ gives a representation of sl_2 on $\mathcal{H}^*(X,\mathbb{C})$. Using elementary representation theory, we can get a finer decomposition result, due to S. Lefschetz.

References

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