Introduction to complex geometry
(Chapter 5)

Yalong Shi*

Abstract

Notes for 2020 BICMR Summer School for Differential Geometry.

Contents

5 Kodaira's vanishing theorem and its applications

5.1 Kodaira vanishing theorem

Using Hodge theorem, we can prove an important cohomology vanishing theorem of Kodaira. To state the theorem, we recall the following positivity notions for real $(1,1)$-forms and for line bundles: We say a real $(1,1)$-form ω is "positive" if locally it can be written as $\omega = \sqrt{-1}\sum_{i,j} a_{i\bar{j}} dz_i \wedge d\bar{z}_j$ where $(a_{i\bar{j}})$ is positive definite everywhere. A line bundle L is called "positive" if there exists an Hermitian metric h on L such that $\sqrt{-1}\Theta(h)$ is positive.

Theorem 5.1 (Kodaira-Nakano)**.** *If $L \to X$ is a positive holomorphic line bundle on a compact Kähler manifold,*[1] *then we have*

$$H^q(X, \Omega^p(L)) = 0, \quad \text{for } p + q > n.$$

In particular, $H^q(X, \mathscr{O}(K_X \otimes L)) = 0$ for $q > 0$.

Proof. (due to Akizuki-Nakano) We use $\omega := \sqrt{-1}\Theta(h)$ as our reference Kähler metric. [2] The Hodge theorem ensures that $H^q(X, \Omega^p(L)) \cong \mathcal{H}^{p,q}(X, L)$. So we need to show that when $p + q > n$ each L-valued harmonic (p,q)-form must be zero.

We need the following lemma, whose proof is almost identical to the "un-twisted case" we proved before:

Lemma 5.2. *Let E be a holomorphic vector bundle over a compact Kähler manifold (X, ω) with Hermitian metric h. Introduce the operator $L : A^{p,q}(X,E) \to A^{p+1,q+1}(X,E)$ as before and define $\Lambda := L^*$. If we denote the $(1,0)$ and $(0,1)$ components of the Chern connection D by D' and $D''(= \bar{\partial})$, then we have*

$$[\Lambda, \bar{\partial}] = -\sqrt{-1}D'^*, \quad [\Lambda, D'] = \sqrt{-1}\bar{\partial}^*.$$

Assuming the lemma at present. Then the proof of Kodaira vanishing theorem essentially follows from the comparison of two "Laplacians", the so called "Bochner's technique":

$$\Delta_{\bar{\partial},E} - \Delta_{D',E} = [\sqrt{-1}\Theta(h), \Lambda],$$

where $\Delta_{D',E} := D'D'^* + D'^*D'$. The reason for this equality is:

$$\begin{aligned} -\sqrt{-1}\Delta_{D',E} &= D'[\Lambda, \bar{\partial}] + [\Lambda, \bar{\partial}]D' \\ &= D'\Lambda\bar{\partial} - D'\bar{\partial}\Lambda + \Lambda\bar{\partial}D' - \bar{\partial}\Lambda D', \end{aligned}$$

[1] We can just assume X is compact complex manifold. Then if $\sqrt{-1}\Theta(h) > 0$, then it is a Kähler form on X and so X is in fact Kähler. Later, by Kodaira's embedding theorem, X is in fact projective algebraic.

[2] In this case, $[\omega] = 2\pi c_1(L)$. In general, if we have a compact Kähler manifold (X, J, g) such that $[\omega_g] = 2\pi c_1(L)$ (or $c_1(L)$) for some holomorphic line bundle L, then we call the triple (X, L, g) a *"polarized manifold"*. L is called *"the polarizing line bundle"* or *"the polarization"*.

while

$$\begin{aligned}\sqrt{-1}\Delta_{\bar\partial,E} &= \bar\partial[\Lambda, D'] + [\Lambda, D']\bar\partial \\ &= \bar\partial\Lambda D' - \bar\partial D'\Lambda + \Lambda D'\bar\partial - D'\Lambda\bar\partial.\end{aligned}$$

So we get

$$\sqrt{-1}\Delta_{\bar\partial,E} - \sqrt{-1}\Delta_{D',E} = \Lambda(\bar\partial D' + D'\bar\partial) - (\bar\partial D' + D'\bar\partial)\Lambda.$$

Note that $\Theta(h)$ is of type $(1,1)$, we get $D'D' = 0$, $\bar\partial\bar\partial = 0$, so

$$\Theta(h) = D^2 = (D' + \bar\partial)(D' + \bar\partial) = D'\bar\partial + \bar\partial D'.$$

So we get

$$\Delta_{\bar\partial,E} - \Delta_{D',E} = -\sqrt{-1}[\Lambda, \Theta(h)] = [\sqrt{-1}\Theta(h), \Lambda].$$

Now back to the proof of Kodaira's vanishing theorem. We have $\sqrt{-1}\Theta(h) = \omega$, so the above Bochner formula reduces to

$$\Delta_{\bar\partial} - \Delta_{D'} = [L, \Lambda] = (p + q - n)id.$$

So if $s \in \mathcal{H}^{p,q}(X, L)$ is not identically zero, then we have

$$(\Delta_{\bar\partial}s - \Delta_{D'}s, s) = (p + q - n)\|s\|^2 > 0.$$

On the other hand,

$$(\Delta_{\bar\partial}s - \Delta_{D'}s, s) = -(\Delta_{D'}s, s) = -\|D's\|^2 - \|D'^*s\|^2 \leq 0.$$

This is a contradiction. □

5.2 The embedding theorem

One important application of the Kodaira vanishing theorem is the following embedding theorem of Kodaira:

Theorem 5.3. *If a compact complex manifold X has a positive line bundle, then it is projective algebraic.*

The basic construction we shall use is the following: Let $L \to X$ be a holomorphic line bundle, such that $H^0(X, \mathscr{O}(L)) \neq 0$. Then we can take a basis of $H^0(X, \mathscr{O}(L))$, $s_0, \dots, s_N$, and define a "map" from X to $\mathbb{C}P^N$:

$$x \mapsto [s_0(x), \dots, s_N(x)].$$

This is defined using a local trivialization, so that we can identify each s_i as a locally defined holomorphic function. This map is independent of the trivialization we choose, but it is un-defined on the "base locus" of L: [3]

$$Bs(L) := \{x \in X \mid s(x) = 0, \quad \forall s \in H^0(X, \mathscr{O}(L))\}.$$

What Kodaira actually proved is the following: If $L \to X$ is a positive line bundle on a compact complex manifold, then *we can find a large integer $m_0 > 0$ such that for all $m > m_0$:*

1. *$L^{\otimes m}$ is "base point free", i.e. $Bs(L^{\otimes m}) = \emptyset$;*
2. *Choose a basis of $H^0(X, \mathscr{O}(L^{\otimes m}))$, $s_0, \ldots, s_{N_m}$, then the "Kodaira map" $\iota_{L^m} : X \to \mathbb{C}P^{N_m}$ defined by*
$$x \mapsto [s_0(x), \ldots, s_{N_m}(x)]$$
is a holomorphic embedding.

Definition 5.4. *Let $L \to X$ be a holomorphic line bundle on a compact complex manifold.*

- *If there is an integer $m_0 > 0$ such that for all $m > m_0$, $L^{\otimes m}$ is base point free, then we say L is* semi-ample*;*
- *If L is base point free and the Kodaira map ι_L is a holomorphic embedding, then we say L is* very ample*;*
- *If there is an integer $m_0 > 0$ such that for all $m > m_0$, $L^{\otimes m}$ is very ample, then we say L is* ample*.*

A corollary of Kodaira's theorem is that on a compact complex manifold, a holomorphic line bundle is ample if and only if it is positive.

In fact, if L is positive, then it is ample by Kodaira's theorem. On the other hand, if L is ample, we can find $m \in \mathbb{N}$ such that ι_{L^m} is a holomorphic embedding. Then the pulling back of the hyperplane bundle is isomorphic to $L^{\otimes m}$, and the induced metric has positive curvature. The corresponding metric on L also has positive curvature.

Outline of the proof of Kodaira embedding theorem: For simplicity, we only prove that there is a sufficiently large m such that ι_{L^m} is an embedding. We need to prove the following 3 properties:

1. Prove that $L^{\otimes m}$ is base point free when m large enough. We only need to show that for any point $p \in X$, we can find a $m_p \in \mathbb{N}$ such that for all $m \geq m_p$, we can find a $s \in H^0(X, \mathscr{O}(L^{\otimes m}))$ such that $s(p) \neq 0$. That is, the linear map $r_p : H^0(X, \mathscr{O}(L^{\otimes m})) \to L_p^{\otimes m}$ is surjective.

[3] In fact, one can suitably extend the map to codimension one part of $Bs(L)$.

2. Prove that for m large, global sections of $L^{\otimes m}$ separate points. For this, we need to prove that for any two points $p \neq q$ in X, the linear map $r_{p,q} : H^0(X, \mathscr{O}(L^{\otimes m})) \to L_p^{\otimes m} \oplus L_q^{\otimes m}$ is surjective for m sufficiently large.

3. Prove that for m large, ι_{L^m} is an immersion. That is, for any point $p \in X$, global sections of $L^{\otimes m}$ separate tangent directions at p. We only need to show the linear map $r_{p,p} : H^0(X, \mathscr{O}(L^{\otimes m})) \to L_p^{\otimes m} \otimes (\mathscr{O}_p/\mathfrak{m}_p^2)$ is surjective for m sufficiently large.

Note that property 2 is stronger than property 1. So we only need to prove 2 and 3. Also note that if we denote by $\mathfrak{m}_p$ the ideal sheaf of holomorphic germs vanishing at p and $\mathfrak{m}_{p,q}$ the ideal sheaf of holomorphic germs vanishing at p and q, then what we need prove is that

$$H^0(X, \mathscr{O}(L^{\otimes m})) \to H^0(X, \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_{p,q})$$

and

$$H^0(X, \mathscr{O}(L^{\otimes m})) \to H^0(X, \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_p^2)$$

are both surjective when m is large enough.

For this, we use short exact sequences of sheaves:

$$0 \to \mathfrak{m}_{p,q} \to \mathscr{O} \to \mathscr{O}/\mathfrak{m}_{p,q} \to 0, \qquad 0 \to \mathfrak{m}_p^2 \to \mathscr{O} \to \mathscr{O}/\mathfrak{m}_p^2 \to 0.$$

Tensor with the locally free sheaf $\mathscr{O}(L^{\otimes m})$, we get exact sequences

$$0 \to \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_{p,q} \to \mathscr{O}(L^{\otimes m}) \to \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_{p,q} \to 0$$

and

$$0 \to \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_p^2 \to \mathscr{O}(L^{\otimes m}) \to \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_p^2 \to 0.$$

The induced long exact sequences give us:

$$H^0(X, \mathscr{O}(L^{\otimes m})) \to H^0(X, \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_{p,q}) \to H^1(X, \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_{p,q})$$

and

$$H^0(X, \mathscr{O}(L^{\otimes m})) \to H^0(X, \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_p^2) \to H^1(X, \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_p^2).$$

We need to prove the vanishing of $H^1(X, \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_{p,q})$ and $H^1(X, \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_p^2)$.

Comparing with Kodaira's vanishing theorem, we found that the main problem is that $\mathfrak{m}_{p,q}$ and $\mathfrak{m}_p^2$ are not sheaves of germs of holomorphic line bundles. They are examples of "coherent analytic sheaves". This "generalized Kodaira vanishing theorem" for coherent analytic sheaves is indeed true, but harder to prove. Kodaira's method (as appeared in Griffiths-Harris and Wells) is to replace X by its blown-up $\tilde{X}$ at p and q. Pulling everything back to $\tilde{X}$ we can work purely with line bundles, and then Kodaira's vanishing theorem works. Then one need to show that vanishing upstairs implies vanishing downstairs.

Finally, since both property 2 and 3 are "open" properties, we can use a "finite covering trick" to find a uniform m, independent of $p, q \in X$. □

In short, the proof says that positivity of a line bundle L implies $L^{\otimes m}$ has so many global sections that they can separate points and tangent directions. Here we use Kodaira's cohomology vanishing to prove the existence of global sections satisfying special properties. This is typical when applying vanishing theorems. Also, to prove the existence of global sections separating points and tangent directions, one can directly construct sections by solving $\bar{\partial}$-equations using Hörmander's L^2-method. It turns out that we also need a certain type of Bochner type identity, and the positivity of the line bundle is also crucial.

References

[1] Phillip Griffiths and Joseph Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, Inc., 1978.

[2] R.O. Wells, *Differential analysis on complex manifolds*, 2nd edition, Springer, 1980.