Introduction to complex geometry
(Chapter 6)

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Abstract

Notes for 2020 BICMR Summer School for Differential Geometry.

Contents

6 Calabi-Yau theorem

6.1 Calabi's problem and Aubin-Yau, Calabi-Yau theorem

Recall that $\Lambda^n T^{1,0}X =: K_X^{-1}$ is the anticanonical line bundle, and g induced an Hermitian metric on K_X^{-1}, with $|\frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_n}|_g^2 := \det(g_{i\bar{j}})$, its curvature form is exactly $-\bar{\partial}\partial\log\det(g_{i\bar{j}})$. So we get

$$\sqrt{-1}\Theta(K_X^{-1}, \det g) = Ric(\omega_g),$$

and by Chern's theorem,

$$[Ric(\omega_g)] = 2\pi c_1(K_X^{-1}) =: 2\pi c_1(X).$$

Calabi asked the following questions:

1. Given a Kähler metric g and a closed (1,1)-form η such that its cohomology class in $H^2_{dR}(X)$ is $[\eta] = 2\pi c_1(X)$, can we find another Kähler metric g' within the same Kähler class $[\omega_g]$ such that $Ric(\omega_{g'}) = \eta$?

2. When can we find a Kähler metric which is at the same time an Einstein metric? That is, $Ric(\omega_g) = \lambda\omega_g$ for a constant $\lambda \in \mathbb{R}$. We call such a metric an Kähler-Einstein metric.

Recall that by $\partial\bar{\partial}$-lemma, different Kähler metrics in the same Kähler class differ by $\sqrt{-1}\partial\bar{\partial}\varphi$ for a $\mathbb{R}$-valued function φ. So Calabi's problems actually ask whether we can find smooth function φ satisfying a specific equation.

Also recall that for a real $(1,1)$-form $\eta = \sqrt{-1}\eta_{i\bar{j}}dz_i \wedge d\bar{z}_j$, we say it is positive (write $\eta > 0$), if the matrix $(\eta_{i\bar{j}})$ is positive definite everywhere. And we say a real (1,1)-class $\alpha \in H^2_{dR}(X)$ is positive if we can find a closed $\eta > 0$ such that $[\eta] = \alpha$.

First, observe that:

Lemma 6.1. *If the compact Kähler manifold (X, J, g) is Einstein, then either $c_1(X) > 0$ or $c_1(X) < 0$ or $c_1(X) = 0$.*

Also observe that the Ricci form is invariant under rescaling, so for the Kähler-Einstein problem, we can assume $\lambda = 1, -1$ or 0.

The results we discuss in this chapter are:

Theorem 6.2 (Aubin-Yau)**.** *If X is compact Kähler manifold with $c_1(X) < 0$, then there is a unique Kähler metric g satisfying*

$$Ric(\omega_g) = -\omega_g.$$

Theorem 6.3 (Calabi-Yau theorem)**.** *If X is compact Kähler manifold with a given Kähler metric g_0, then given any closed $(1,1)$-form η such that $[\eta] = 2\pi c_1(X)$, there is a unique Kähler metric g with $[\omega_g] = [\omega_{g_0}]$ satisfying*

$$Ric(\omega_g) = \eta.$$

In particular, if $c_1(X) = 0$, then for any Kähler class α, there is a unique Ricci-flat Kähler metric in the class α. A Ricci-flat Kähler metric is usually called a "Calabi-Yau metric" in the literature.

However, when $c_1(X) > 0$ (then we say "X is a Fano manifold" in honor of the Italian algebraic geometer Fano), in general we can not find Kähler-Einstein metrics, due to various obstructions, like the vanishing of Futaki invariant and the reductiveness of the automorphism group of X. The ultimate result is:

Theorem 6.4 (Chen-Donaldson-Sun, Tian)**.** *Let X be a compact Kähler manifold with $c_1(X) > 0$. Then X admits a Kähler-Einstein metric if and only if X is K-polystable.*

I won't explain the meaning of K-stability here. For the original definition, we refer the readers to Tian's 1997 Invent. Math. paper. The uniqueness problem of positive Kähler-Einstein metrics is also very difficult, and first solved by Bando-Mabuchi. There is a recent proof by B. Berndtsson, using ideas from complex Brunn-Minkowski inequalities.

Now we derive the equation and prove the uniqueness part.

For Aubin-Yau theorem, we start with a g_0 such that its Kähler form $\omega \in -2\pi c_1(X) = -[Ric(\omega)]$, so we can apply the $\partial\bar\partial$-lemma to get a smooth function h satisfying $Ric(\omega) + \omega = \sqrt{-1}\partial\bar\partial h$, and h is unique if we require $\int_X e^h\omega^n = \int_X \omega^n$. We want to find $\varphi \in C^2(M;\mathbb{R})$ s.t. $\omega_\varphi := \omega + \sqrt{-1}\partial\bar\partial\varphi > 0$ and $Ric(\omega_\varphi) + \omega_\varphi = 0$, i.e.,

$$0 = -\partial_i\partial_{\bar j}\log\det(g_{p,\bar q} + \varphi_{p\bar q}) + g_{i\bar j} + \varphi_{i\bar j} = -\partial_i\partial_{\bar j}\Big(\log\frac{\det(g_{p,\bar q} + \varphi_{p\bar q})}{\det(g_{p\bar q})} - h - \varphi\Big).$$

So we get the equation

$$(\omega + \sqrt{-1}\partial\bar\partial\varphi)^n = e^{h+\varphi}\omega^n. \tag{6.1}$$

For Calabi-Yau theorem, we have a unique h satisfying $Ric(\omega) - \eta = \sqrt{-1}\partial\bar\partial h$ and $\int_X e^h\omega^n = \int_X \omega^n$. We want to find φ such that $\omega_\varphi > 0$ and $Ric(\omega_\varphi) = \eta$, i.e.

$$-\partial_i\partial_{\bar j}\log\det(g_{p,\bar q} + \varphi_{p\bar q}) = -\partial_i\partial_{\bar j}\log\det(g_{p,\bar q}) - h_{i\bar j}.$$

So the equation is

$$(\omega + \sqrt{-1}\partial\bar\partial\varphi)^n = e^h\omega^n. \tag{6.2}$$

Lemma 6.5 (Calabi)**.** *The solutions to* (6.1) *and* (6.2) *are both unique.*

Proof. If both φ_1 and φ_2 solve (6.1), set $\psi := \varphi_2 - \varphi_1$. Then ψ satisfies $(\omega_1 + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{\psi}\omega_1^n$. At the maximum point of ψ, we have $e^{\psi}\omega_1^n \leq \omega_1^n$, so $\psi \leq 0$. Similarly, we get $\psi \geq 0$, hence $\psi \equiv 0$.

If both φ_1 and φ_2 solve (6.2), set $\psi := \varphi_2 - \varphi_1$. Then ψ satisfies an elliptic equation of the form $L\psi = 0$, with $L = A^{i\bar{j}}(z, \partial^2\varphi_1, \partial^2\varphi_2)\partial_i\partial_{\bar{j}}$. Since ψ must achieve its maximum and minimum somewhere, by strong maximum principle, ψ is a constant, and the corresponding metrics are the same. $\square$

6.2 Proof of (Aubin-)Calabi-Yau theorem

We start with the Aubin-Yau theorem. The idea of proof is to use the so called "continuity method", introduced in the first half of 20th century by H. Weyl.

We introduce an extra parameter t into (6.1):

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{th+\varphi}\omega^n. \tag{6.3}$$

Then we study the set $S := \{t \in I = [0,1] \mid (6.3) \text{ is solvable in } C^{k,\alpha}(X)\}$. Obviously $0 \in S$, since in this case $\varphi \equiv 0$ is a solution. Then we try to show S is both open and closed. By connectness of I, we will get $1 \in S$, i.e. (6.1) is solvable.

To show the openness, we shall use the implicit function theorem in Banach spaces. We consider the operator $\Psi : I \times C^{k,\alpha}(X) \to C^{k-2,\alpha}$, where

$$\Psi(t,\varphi) := \log \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega^n} - \varphi - th.$$

Then we have

$$D_\varphi\Psi(\psi) = g_\varphi^{\bar{j}i}\partial_i\partial_{\bar{j}}\psi - \psi = (\Delta_\varphi - 1)\psi.$$

This is invertible by Fredholm alternative, since we can easily prove its injectivity, either use maximum principle or integration by parts. So we get the openness of S.

To prove the closedness, we shall derive *a priori* estimates: if $t_i \in S$ with solution $\varphi_i \in C^{k,\alpha}(X)$ and $t_i \to t_0 \in I$, we need to show that $\|\varphi_i\|_{k,\alpha} \le C$ with a uniform constant C. Then we can find a converging subsequence in $C^{k,\alpha}$. If $k \ge 2$, then we will get a solution for t_0 and S must be closed.

The C^0 estimate of φ is rather direct: if

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{th+\varphi}\omega^n$$

and φ achieves its maximum at $p \in X$. Then

$$e^{th(p)+\max\varphi}\omega^n(p) \le \omega^n(p),$$

so $\varphi \le \|h\|_\infty$. Similarly, we get $\varphi \ge -\|h\|_\infty$, so $\|\varphi\|_\infty \le \|h\|_\infty$. This is already known to Calabi.

We shall not prove C^1 estimate directly, (which is not simple, and first proved directly by Blocki, more than 30 years later than Yau's work) but use C^2 estimates.

The C^2 estimate is due independently to Aubin and Yau, with slightly different calculations.

We denote by $\Delta := g^{\bar{j}i}\partial_i\partial_{\bar{j}}$ and $\Delta_\varphi := g_\varphi^{\bar{j}i}\partial_i\partial_{\bar{j}}$. Since $(g_{i\bar{j}} + \varphi_{i\bar{j}})$ is positive definite, taking trace with respect to ω, we have $0 < g^{\bar{j}i}(g_{i\bar{j}} + \varphi_{i\bar{j}}) =: tr_\omega\omega_\varphi = n + \Delta\varphi$. Now we compute $\Delta_\varphi tr_\omega\omega_\varphi$ at a point p, using Kähler normal coordinates of g at p. Note that at this point, we have $R_{i\bar{j}k\bar{l}} = -\partial_i\partial_{\bar{j}}g_{k\bar{l}}$, so we have

$$\begin{aligned}
\Delta_\varphi tr_\omega\omega_\varphi &= g_\varphi^{\bar{j}i}\partial_i\partial_{\bar{j}}(g^{\bar{l}k}g_{\varphi,k\bar{l}}) = g_\varphi^{\bar{j}i}\partial_i(g^{\bar{l}k}\frac{\partial g_{\varphi,k\bar{l}}}{\partial\bar{z}_j} - g^{\bar{l}p}g^{\bar{q}k}\frac{\partial g_{p\bar{q}}}{\partial\bar{z}_j}g_{\varphi,k\bar{l}}) \\
&= g_\varphi^{\bar{j}i}g^{\bar{l}k}\frac{\partial^2 g_{\varphi,k\bar{l}}}{\partial z_i\partial\bar{z}_j} + g_\varphi^{\bar{j}i}g^{\bar{l}p}g^{\bar{q}k}R_{i\bar{j}p\bar{q}}g_{\varphi,k\bar{l}} \\
&= g_\varphi^{\bar{j}i}g^{\bar{l}k}\big(-R(g_\varphi)_{i\bar{j}k\bar{l}} + g_\varphi^{\bar{q}p}\varphi_{p\bar{l}\bar{j}}\varphi_{k\bar{q}i}\big) + g_\varphi^{\bar{j}i}g^{\bar{l}p}g^{\bar{q}k}R_{i\bar{j}p\bar{q}}g_{\varphi,k\bar{l}} \\
&= -tr_\omega Ric(\omega_\varphi) + g_\varphi^{\bar{j}i}g^{\bar{l}k}g_\varphi^{\bar{q}p}\varphi_{p\bar{l}\bar{j}}\varphi_{k\bar{q}i} + g_\varphi^{\bar{j}i}g^{\bar{l}p}g^{\bar{q}k}R_{i\bar{j}p\bar{q}}g_{\varphi,k\bar{l}}.
\end{aligned}$$

So we get

$$\begin{aligned}
\Delta_\varphi \log tr_\omega\omega_\varphi &= g_\varphi^{\bar{j}i}\partial_i\frac{\partial_{\bar{j}}tr_\omega\omega_\varphi}{tr_\omega\omega_\varphi} = \frac{\Delta_\varphi tr_\omega\omega_\varphi}{tr_\omega\omega_\varphi} - \frac{|\partial tr_\omega\omega_\varphi|_\varphi^2}{(tr_\omega\omega_\varphi)^2} \\
&= \frac{1}{tr_\omega\omega_\varphi}\Big(-tr_\omega Ric(\omega_\varphi) + g_\varphi^{\bar{j}i}g^{\bar{l}p}g^{\bar{q}k}R_{i\bar{j}p\bar{q}}g_{\varphi,k\bar{l}}\Big) \\
&\quad + \frac{(tr_\omega\omega_\varphi)g_\varphi^{\bar{j}i}g^{\bar{l}k}g_\varphi^{\bar{q}p}\varphi_{p\bar{l}\bar{j}}\varphi_{k\bar{q}i} - |\partial tr_\omega\omega_\varphi|_\varphi^2}{(tr_\omega\omega_\varphi)^2}.
\end{aligned}$$

Claim: We always have $(tr_\omega\omega_\varphi)g_\varphi^{\bar{j}i}g^{\bar{l}k}g_\varphi^{\bar{q}p}\varphi_{p\bar{l}\bar{j}}\varphi_{k\bar{q}i} - |\partial tr_\omega\omega_\varphi|_\varphi^2 \geq 0$.

To see this, recall that we work under a Kähler normal coordinate system. By an extra linear coordinate change, we can further assume that $\varphi_{i\bar{j}} = \lambda_i\delta_{ij}$, with $\lambda_i \in \mathbb{R}$ and $1+\lambda_i > 0$. So at this point, we have $g_{\varphi,i\bar{j}} = (1+\lambda_i)\delta_{ij}$ and $g_\varphi^{\bar{j}i} = \frac{\delta_{ij}}{1+\lambda_i}$, and so $tr_\omega\omega_\varphi = \sum_i(1+\lambda_i)$, and $g_\varphi^{\bar{j}i}g^{\bar{l}k}g_\varphi^{\bar{q}p}\varphi_{p\bar{l}\bar{j}}\varphi_{k\bar{q}i} = \sum_{i,p,k}\frac{1}{1+\lambda_i}\frac{1}{1+\lambda_p}|\varphi_{i\bar{p}k}|^2$. So we have

$$\begin{aligned}
|\partial tr_\omega\omega_\varphi|_\varphi^2 &= \sum_i \frac{1}{1+\lambda_i}|\partial_i(g^{\bar{l}k}g_{\varphi,k\bar{l}})|^2 = \sum_i\frac{1}{1+\lambda_i}|g^{\bar{l}k}\partial_i g_{\varphi,k\bar{l}}|^2 \\
&= \sum_i\frac{1}{1+\lambda_i}|\sum_k\varphi_{k\bar{k}i}|^2 = \sum_i\frac{1}{1+\lambda_i}|\sum_k\sqrt{1+\lambda_k}\frac{\varphi_{k\bar{k}i}}{\sqrt{1+\lambda_k}}|^2 \\
&\leq \sum_i\frac{1}{1+\lambda_i}\Big(\sum_k(1+\lambda_k)\Big)\Big(\sum_p\frac{|\varphi_{p\bar{p}i}|^2}{1+\lambda_p}\Big) = (tr_\omega\omega_\varphi)\sum_{i,p}\frac{1}{1+\lambda_i}\frac{|\varphi_{p\bar{p}i}|^2}{1+\lambda_p} \\
&\leq (tr_\omega\omega_\varphi)\sum_{i,p,k}\frac{1}{1+\lambda_i}\frac{1}{1+\lambda_p}|\varphi_{k\bar{p}i}|^2.
\end{aligned}$$

Lemma 6.6. *Let ω be a Kähler metric on a compact Kähler manifold X and $\varphi \in C^4(X;\mathbb{R})$ satisfies $\omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0$, then*

$$\Delta_\varphi \log tr_\omega\omega_\varphi \geq \frac{-tr_\omega Ric(\omega_\varphi)}{tr_\omega\omega_\varphi} - C tr_{\omega_\varphi}\omega. \tag{6.4}$$

Proof. By the above discussions, we have

$$\begin{aligned}\Delta_\varphi \log tr_\omega\omega_\varphi &\geq \frac{1}{tr_\omega\omega_\varphi}\Big(- tr_\omega Ric(\omega_\varphi) + g_\varphi^{\bar{j}i} g^{\bar{l}p} g^{\bar{q}k} R_{i\bar{j}p\bar{q}} g_{\varphi,k\bar{l}}\Big)\\ &= \frac{-tr_\omega Ric(\omega_\varphi)}{tr_\omega\omega_\varphi} + \frac{1}{tr_\omega\omega_\varphi}\sum_{i,k}\frac{1+\lambda_k}{1+\lambda_i}R_{i\bar{i}k\bar{k}}\\ &\geq \frac{-tr_\omega Ric(\omega_\varphi)}{tr_\omega\omega_\varphi} + \frac{\inf_{i,k} R_{i\bar{i}k\bar{k}}}{tr_\omega\omega_\varphi}\sum_{i,k}\frac{1+\lambda_k}{1+\lambda_i}\\ &= \frac{-tr_\omega Ric(\omega_\varphi)}{tr_\omega\omega_\varphi} + \inf_{i,k} R_{i\bar{i}k\bar{k}} tr_{\omega_\varphi}\omega.\end{aligned}$$

Since X is compact, we can find $C > 0$ such that $\inf_{i,k} R_{i\bar{i}k\bar{k}} \geq -C$. □

Note that we haven't use the equation! So the above computation applies to other situations.

Now we rewrite the equation (6.3) as

$$\begin{aligned}Ric(\omega_\varphi) &= Ric(\omega) - t\sqrt{-1}\partial\bar{\partial}h - \sqrt{-1}\partial\bar{\partial}\varphi\\ &= Ric(\omega) - t(Ric(\omega)+\omega) - (\omega_\varphi - \omega)\\ &= (1-t)(Ric(\omega)+\omega) - \omega_\varphi.\end{aligned}$$

So $-tr_\omega Ric(\omega_\varphi) \geq tr_\omega\omega_\varphi - C$. So we conclude that

$$\Delta_\varphi \log tr_\omega\omega_\varphi \geq 1 - C\Big(\frac{1}{tr_\omega\omega_\varphi} + tr_{\omega_\varphi}\omega\Big) \geq 1 - C' tr_{\omega_\varphi}\omega.$$

The last step used the fact $\frac{1}{tr_\omega\omega_\varphi} = \frac{1}{\sum_i(1+\lambda_i)} \leq \frac{1}{1+\lambda_1} \leq tr_{\omega_\varphi}\omega$.

On the other hand, we have

$$\Delta_\varphi\varphi = tr_{\omega_\varphi}(\omega_\varphi - \omega) = n - tr_{\omega_\varphi}\omega,$$

and so we get

$$\Delta_\varphi\big(\log tr_\omega\omega_\varphi - (C'+1)\varphi\big) \geq -C'' + tr_{\omega_\varphi}\omega.$$

At the maximum point of $\log tr_\omega\omega_\varphi - (C'+1)\varphi$, we have $tr_{\omega_\varphi}\omega \leq C''$. Use Kähler normal coordinates at that point and assume g_φ is diagonal as before, we get $\frac{1}{1+\lambda_i} \leq C''$ for each i. By (6.3), we have $\Pi_i(1+\lambda_i) = e^{th+\varphi} \leq C_0$, which implies $1+\lambda_i \leq C_0(C'')^{n-1}$. So $tr_\omega\omega_\varphi \leq nC_0(C'')^{n-1}$. This implies at this point $\log tr_\omega\omega_\varphi - (C'+1)\varphi$ is uniformly bounded from above (use $|\varphi| \leq ||h||_{C^0}$). This in turn implies $tr_\omega\omega_\varphi \leq C$ for a uniform constant C.

Since we have L^∞ control of $\Delta\varphi$, using L^p theory for linear elliptic equations, we get uniform control of C^1-norm for φ.

Also a direct consequence of the $\Delta\varphi$ estimate is that there is a uniform constant $C > 0$ such that $\frac{1}{C}\omega \leq \omega_\varphi \leq C\omega$.

After obtaining C^2 estimates, there are two ways to get higher order estimates. The original approach of Aubin and Yau used Calabi's 3rd order estimates, and then use Schauder estimates and then bootstrapping. Later, Evans and Krylov independently discovered that the $C^{2,\alpha}$ estimate follows directly from the C^2 estimate. The basic idea is that if we differentiate the equation in the tangent direction γ 2-times, we will get an elliptic equation for $u_{\gamma\gamma}$. The above estimate implies that we have uniform control for the ellipticity constants. Then we can get Harnack inequality for $u_{\gamma\gamma}$ by exploring the concavity structure of the complex Monge-Ampère operator. One can find the proof for real fully nonlinear equations in chapter 17 of Gilbarg-Trudinger's book. The adaptation to the complex Monge-Ampère equation has been carried out by Siu in his book [2].

After obtaining $C^{2,\alpha}$ control of φ, we can differentiate the equation once, then the coefficients have uniform Hölder norm, so we can use Schauder estimates and then bootstrapping. This finishes the proof to Theorem 6.2.

Now we study the Calabi-Yau equation.

First, we need a continuity path for the equation (6.2):

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{th+c_t}\omega^n, \tag{6.5}$$

where c_t is a constant defined by $\int_X e^{th+c_t}\omega^n = \int_X \omega^n$. Again let

$$S := \{t \in I \mid (6.5) \text{ is solvable in } C_0^{k,\alpha}\},$$

where we define $C_0^{k,\alpha} := \{\varphi \in C^{k,\alpha}(X) \mid \int_X \varphi\omega^n = 0\}$. When $t = 0$, $\varphi \equiv 0$ is the solution. So $S \neq \emptyset$. To show S is open, we use the implicit function theorem. However, there is additional difficulty caused by the change of c_t, so we modify the function spaces in Aubin-Yau's theorem.

We define the affine subspace of $C^{k-2,\alpha}$:

$$C_V^{k-2,\alpha} := \{f \in C^{k,\alpha}(X) \mid \int_X e^f\omega^n = \int_X \omega^n\}.$$

Then we define the operator $\Phi : C_0^{k,\alpha} \to C_V^{k-2,\alpha}$,

$$\Phi(\varphi) := \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega^n}.$$

The linearization at φ_{t_0} is $D\Phi_{\varphi_{t_0}} : C_0^{k,\alpha} \to C_0^{k-2,\alpha}$

$$D\Phi_{\varphi_{t_0}}(\psi) = \frac{\omega^n_{\varphi_{t_0}}}{\omega^n}\Delta_{\varphi_{t_0}}\psi.$$

This operator is invertible since $\Delta_{\varphi_{t_0}}\psi = f$ is solvable if and only if $\int_X f\omega^n_{\varphi_{t_0}} = 0$. This proves the openness.

For closedness, as before, we need to derive *a priori* estimates. Only the C^0 estimate is different, other parts are almost identical.

We will basically follow Yau's original proof using Moser iteration. Later there are other proofs, e.g. S. Kolodziej's approach using pluripotential theory and Z. Blocki's proof using Alexandrov's maximum principles. Our exposition follows [1].

Rewrite the equation as $(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = F\omega^n$ with $F = e^{th+c_t}$. Note that F has uniform positive upper and lower bounds, independent of t. Set $\psi := \sup_X \varphi - \varphi + 1 \geq 1$. Since

$$(F-1)\omega^n = (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n - \omega^n = \sqrt{-1}\partial\bar{\partial}\varphi \wedge \sum_{j=0}^{n-1} \omega_\varphi^{n-j-1} \wedge \omega^j,$$

we multiply $\psi^{\alpha+1}$ on both sides for some $\alpha \geq 0$, and integrate over X:

$$\begin{aligned}
\int_X \psi^{\alpha+1}(F-1)\omega^n &= (\alpha+1)\sum_{j=0}^{n-1}\int_X \psi^\alpha \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \wedge \omega_\varphi^{n-j-1} \wedge \omega^j \\
&\geq (\alpha+1)\int_X \psi^\alpha \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \wedge \omega^{n-1} \\
&= \frac{\alpha+1}{(\frac{\alpha}{2}+1)^2}\int_X \sqrt{-1}\partial\psi^{\frac{\alpha}{2}+1} \wedge \bar{\partial}\psi^{\frac{\alpha}{2}+1} \wedge \omega^{n-1} \\
&= \frac{\alpha+1}{(\frac{\alpha}{2}+1)^2}\|\nabla\psi^{\frac{\alpha}{2}+1}\|^2.
\end{aligned}$$

So we get

$$\|\nabla\psi^{\frac{\alpha}{2}+1}\|^2 \leq C_1\frac{(\frac{\alpha}{2}+1)^2}{\alpha+1}\int_X \psi^{\alpha+1}\omega^n,$$

where C_1 depends only on $\|F\|_{L^\infty}$.

On the other hand, we have Sobolev inequality

$$\|u\|^2_{L^{\frac{2n}{n-1}}} \leq C_2(\|\nabla u\|^2_{L^2} + \|u\|^2_{L^2}).$$

We apply this to $u := \psi^{\frac{p}{2}}$:

$$\|\psi\|^p_{L^{p\beta}} \leq C_2(\|\nabla\psi^{\frac{p}{2}}\|^2_{L^2} + \|\psi\|^p_{L^p}),$$

where $\beta := \frac{n}{n-1} > 1$. Then we choose $p = \alpha + 2$, to get

$$\|\psi\|_{L^{p\beta}} \leq (C_3 p)^{\frac{1}{p}}\|\psi\|_{L^p}, \quad p \geq 2.$$

Then we can iterate $p \to p\beta \to p\beta^2 \to \cdots \to p\beta^k \to \ldots$. Using the fact that $\lim_{k\to\infty}\|\psi\|_{L^{p\beta^k}} = \|\psi\|_{L^\infty}$, we conclude that once we have a uniform L^p bound for ψ for some $p \geq 2$, then we will have uniform L^∞ estimate for ψ.

To get this L^p bound, one can use, for example, the following result of G. Tian: Given a Kähler form ω, we can find a positive number $c > 0$, depending only on the Kähler class, such that we can find another uniform constant $C > 0$ such that

$$\int_X e^{-c(\varphi - \sup_X \varphi)} \omega^n \leq C,$$

$\forall \varphi \in C^\infty(X; \mathbb{R})$ such that $\omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0$. From this, we get uniform estimate of $\|\psi\|_{L^k}$ for any $k \in \mathbb{N}$.

References

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