THE HARMONIC MAP APPROACH TO TEICHMÜLLER SPACES

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Abstract. Notes for the last three lectures for a graduate course "Riemann Surfaces" at Nanjing University.

Contents

1. Hyperbolic geometry on a Riemann surface

Let M be a smooth oriented closed surface of genus $g \geq 2$. The two basic facts we need are:

- Every conformal class of a Riemannian structure on M determines uniquely a complex structure and hence making it into a Riemann surface. The Riemannian metric is called a "conformal metric" on this Riemann surface.
- By uniformization theorem of Koebe-Poincaré, every complex structure determines uniquely a conformal hyperbolic metric via the standard hyperbolic metric on the unit disk $\mathbb{D}$ or the upper half plane $\mathbb{H}$.

We explain briefly the uniqueness in the second fact: locally $ds^2 = \lambda |dz|^2$, then the Gauss curvature is

$$K(\lambda) = -\frac{2}{\lambda} \frac{\partial^2}{\partial z \partial \bar{z}} \log \lambda.$$

If $K(\lambda) = -1$ and $K(e^\rho \lambda) = -1$, where ρ is a globally defined function, then

$$\Delta_\lambda \rho = 2(e^\rho - 1),$$

where $\Delta_\lambda = \frac{4}{\lambda} \frac{\partial^2}{\partial z \partial \bar{z}}$. Since M is closed, its maximum and minimum are both achieved. At its maximum point, we have

$$0 \geq \Delta_\lambda \rho = 2(e^{\rho_{\max}} - 1),$$

so $\rho_{\max} \leq 0$. Similarly, at its minimum point, we have

$$0 \leq \Delta_\lambda \rho = 2(e^{\rho_{\min}} - 1),$$

so $\rho_{\min} \geq 0$. Together we get $\rho \equiv 0$.

By Gauss-Bonnet, if ds_M^2 is a conformal hyperbolic metric, then

$$-Area(M) = \int_M K_M dV_M = 2\pi \chi_M = 2\pi(2 - 2g),$$

so we conclude that $Area(M) = 4\pi(g-1)$.

Definition 1 (Teichmüller space). *Let $Diff_0$ be the group consists of diffeomorphisms on M homotopic to id_M. The the Teichmüller space of M is defined to be*

$$T_g := \mathcal{M}_{-1} / Diff_0,$$

where an element $\phi \in Diff_0$ acts on $\mathcal{M}_{-1}$ by pulling back.

At present, T_g is just a set, we shall later endow it with a "canonical" topology and making it a differentiable manifold (even a complex analytic manifold).

2. Harmonic maps between Riemann surfaces with conformal metrics

Let (M, ds^2_M) and (N, ds^2_N) be two closed Riemann surfaces with conformal metrics. Locally, we have

$$ds^2_M = \sigma|dz|^2, \quad ds^2_N = \rho|dw|^2.$$

We define the energy density and the total energy of a smooth map $u : M \to N$ to be

$$e(u) := \frac{\rho(u(z))}{\sigma(z)}(|u_z|^2 + |u_{\bar{z}}|^2),$$

$$E(u) := \int_M e(u)dV_M = \int_M \rho(u(z))(|u_z|^2 + |u_{\bar{z}}|^2)dxdy.$$

From this expression, it is easy to see that the total energy depends only on the complex structure of M, but depends on the metric of the target manifold N.

Definition 2 (Definition and Lemma)**.** *The critical point of $E(u)$ is called a harmonic map. $uM \to N$ is harmonic if and only if*

$$u_{z\bar{z}} + \frac{\rho_w}{\rho}u_z u_{\bar{z}} = 0.$$

From the equation, it is easy to see that holomorphic and anti-holomorphic maps are harmonic.

The key tool to use harmonic maps to study the Teichmüller space is the Hopf differential.

Definition 3. *Let $u : M \to N$ be a smooth map, we define the Hopf differntial of u to be*

$$\Phi_u := \rho(u)u_z\bar{u}_z dz^{\otimes 2}.$$

The motivation of introducing this Hopf differential is the following simple computation:

$$u^* ds^2_N = \rho(u(z))(|u_z|^2 + |u_{\bar{z}}|^2)|dz|^2 + 2Re\Phi_u.$$

So the Hopf differential is the obstruction of being conformal.

Lemma 1. *If u is harmonic, then Φ_u is a holomorphic quadratic differential.*

Proof. This is just simple computation:

$$\partial_{\bar{z}}(\rho(u)u_z\bar{u}_z) \quad = \quad (\rho_w u_{\bar{z}} + \rho_{\bar{w}}\bar{u}_{\bar{z}})u_z\bar{u}_z + \rho u_{z\bar{z}}\bar{u}_z + \rho u_z\bar{u}_{z\bar{z}} = 0.$$

The last equality is just plug the harmonic map equation and its conjugate in. □

In the following of this section, we shall derive some Bochner-type identities for the harmonic map u. For this, we define

$$\mathcal{H} := |\partial u|^2 := \frac{\rho(u(z))}{\sigma(z)}|u_z|^2, \quad \mathcal{L} := |\bar{\partial} u|^2 := \frac{\rho(u(z))}{\sigma(z)}|u_{\bar{z}}|^2.$$

Then obvious $e(u) = \mathcal{H} + \mathcal{L}$, and $\mathcal{HL} = |\Phi_u|^2$. We also have $\mathcal{J} := \mathcal{H} - \mathcal{L}$ is essentially the Jacobian of u, i.e

$$\mathcal{J} = \frac{u^* dV_N}{dV_M}.$$

In particular

$$\int_M \mathcal{J} dV_M = \int_M u^* dV_N = \deg(u) Area(N).$$

Proposition 1 (Bochner identity)**.** *If $u : M \to N$ is harmonic, then at the points where $\mathcal{H}$ (resp. $\mathcal{L}$) is positive, we have*

$$\Delta_M \log \mathcal{H} = -2K_N\mathcal{J} + 2K_M, \quad \Delta_M \log \mathcal{L} = 2K_N\mathcal{J} + 2K_M.$$

Proof. Just compute:

$$\begin{aligned}
\Delta_M \log \mathcal{H} &= \frac{4}{\sigma}\frac{\partial^2}{\partial z\partial \bar z}[\log\rho - \log\sigma + \log|u_z|^2] \\
&= 2K_M + \frac{4}{\sigma}\frac{\partial}{\partial z}\Big(\frac{\rho_w}{\rho}u_{\bar z} + \frac{\rho_{\bar w}}{\rho}\bar u_{\bar z} + \frac{u_{z\bar z}}{u_z} + \frac{\bar u_{\bar z\bar z}}{\bar u_{\bar z}}\Big) \\
&= 2K_M + \frac{4}{\sigma}\frac{\partial}{\partial z}\Big(\frac{\rho_{\bar w}}{\rho}\bar u_{\bar z} + \frac{\bar u_{\bar z\bar z}}{\bar u_{\bar z}}\Big) \\
&= 2K_M + \frac{4}{\sigma}\Big((\log\rho)_{w\bar w}|u_z|^2 + (\log\rho)_{\bar w\bar w}\bar u_z\bar u_{\bar z} + \frac{\rho_{\bar w}}{\rho}\bar u_{z\bar z} + \frac{\partial}{\partial \bar z}\frac{\bar u_{z\bar z}}{\bar u_{\bar z}}\Big) \\
&= 2K_M + \frac{4}{\sigma}\Big((\log\rho)_{w\bar w}|u_z|^2 + (\log\rho)_{\bar w\bar w}\bar u_z\bar u_{\bar z} + \frac{\rho_{\bar w}}{\rho}\bar u_{z\bar z} + \frac{\partial}{\partial \bar z}(-\frac{\rho_{\bar w}}{\rho}\bar u_z)\Big) \\
&= 2K_M + \frac{4}{\sigma}(\log\rho)_{w\bar w}(|u_z|^2 - |u_{\bar z}|^2) \\
&= 2K_M - 2K_N\mathcal{J}.
\end{aligned}$$

The computation for $\mathcal{L}$ is similar, we leave it as an exercise. □

Lemma 2. *If $u : M \to N$ is harmonic, then either $\mathcal{H}$ vanishes identically, or $\mathcal{H}$ has only isolated zero points and in this case we have a well-defined notion of vanishing order n_p for ∂u at $p \in M$. Similar conclusion holds for $\mathcal{L}$.*

Proof. Let $h := u_z$ be a local function. Then by harmonic map equation, we have

$$h_{\bar z} = -\frac{\rho_w}{\rho}u_{\bar z}h.$$

Let $\zeta(z)$ be a local function solving

$$\partial_{\bar z}\zeta = \frac{\rho_w}{\rho}u_{\bar z},$$

then $\partial_{\bar z}(he^{\zeta}) = 0$, so locally u_z equals a holomorphic function times a nowhere vanishing function, the vanishing order is well defined if it is not identically zero. □

Theorem 1 (Generalized Riemann-Hurwitz theorem)**.** *If $u : M \to N$ is harmonic and $\mathcal{H}$ is not identically zero, then we have*

$$\sum_{p\in M, \partial u(p)=0} n_p = -\deg(u)(2g_N - 2) + (2g_M - 2).$$

Or equivalently,

$$\chi_M = \deg(u)\chi_N - \deg B_u,$$

where B_u is the branching divisor. Similar conclusion holds for $\mathcal{L}$, we omit it.

Proof. Let $p_1, \dots, p_k$ be all the zeros of ∂u, of order $n_1, \dots n_k$. For each p_i, we choose a small coordinate disk $B_{i,\epsilon}$such that under the coordinate map, $z_i(B_{i,\epsilon}) = \mathbb{D}_\epsilon$, and that $B_{i,\epsilon} \cap B_{j,\epsilon} = \emptyset$. Then we compute

$$\begin{aligned}
\int_M (-2K_N\mathcal{J} + 2K_M)dV_M &= -2\deg(u)\int_N K_N dV_N + 2\int_M K_M dV_M \\
&= -4\pi\deg(u)\chi_N + 4\pi\chi_M.
\end{aligned}$$

On the other hand, by Bochner identity

$$\begin{aligned}\int_M (-2K_N\mathcal{J}+2K_M)dV_M &= \int_M \Delta_M \log \mathcal{H} dV_M \\ &= \lim_{\epsilon\to 0}\int_{M\setminus\cup_i B_{i,\epsilon}} \Delta_M \log\mathcal{H} dV_M \\ &= -\lim_{\epsilon\to 0}\sum_i \int_{\partial B_{i,\epsilon}} \nu\cdot\nabla^M \log\mathcal{H} dS \\ &= -\lim_{\epsilon\to 0}\sum_i \int_0^{2\pi} \epsilon\partial_r(\log\mathcal{H}(r,\theta))|_{r=\epsilon} d\theta.\end{aligned}$$

However, locally we have $\mathcal{H}=|z|^{2n_i}g$ for a non-vanishing g, so

$$\epsilon\partial_r(\log\mathcal{H}(r,\theta))|_{r=\epsilon}=2n_i+O(\epsilon).$$

So we get

$$-4\pi\sum_i n_i = -4\pi\deg(u)\chi_N+4\pi\chi_M.$$

This is precisely what we want. □

Using this, we have the following:

Theorem 2. *If $g_M=g_N>1$, $K_N<0$ and $u:M\to N$ is harmonic with $\deg(u)=1$, then u is a diffeomorphism.*

Proof. First, we can not have $\mathcal{H}\equiv 0$, for otherwise $\mathcal{J}\le 0$, we have $\deg(u)\le 0$, which contradicts the assumption $\deg(u)=1$. Then we can use the generalized Riemann-Hurwitz theorem to conclude that $\sum_{p\in M,\partial u(p)=0} n_p=0$, which means $\mathcal{H}>0$ everywhere.

Now we claim that $\mathcal{J}\ge 0$.

If not, then $\mathcal{L}>\mathcal{H}>0$ some where, so $\log\frac{\mathcal{H}}{\mathcal{L}}$ achieves its negative minimum at some point $p\in M$. Then at this point, by Bochner identity,

$$0\le \Delta_M\log\frac{\mathcal{H}}{\mathcal{L}}=-4K_N\mathcal{J}.$$

Since $K_N<0$, we conclude that $\mathcal{J}(p)>0$, contradicts $\mathcal{L}(p)>\mathcal{H}(p)>0$.

Now we have $\mathcal{J}\ge 0$, so $\log\frac{\mathcal{H}}{\mathcal{L}}\ge 0$, and again by Bochner identity

$$\Delta_M\log\frac{\mathcal{H}}{\mathcal{L}}=-4K_N\mathcal{J}=-4K_N\mathcal{H}(1-\frac{\mathcal{L}}{\mathcal{H}})\le C\log\frac{\mathcal{H}}{\mathcal{L}}.$$

Here we use the elementary inequality $e^{-t}\ge 1-t$ for any t. By Strong Maximum Principle, if $\mathcal{J}=0$ somewhere, we will have $\log\frac{\mathcal{H}}{\mathcal{L}}\equiv 0$, so $\mathcal{J}\equiv 0$, which implies $\deg(u)=0$. Contradiction! So in fact $\mathcal{J}>0$ everywhere.

From this, we know that u is locally a diffeomorphism, so it is a covering map. Since $\deg(u)=1$, it is in fact a diffeomorphism. □

3. Homeomorphism from T_g to $QD(M,\sigma)$

The logic is the following: We fix a complex structure with a conformal hyperbolic metric $(M,\sigma|dz|^2)$, for any other hyperbolic metric $(M,\rho|dw|^2)$, the identity map is in general not holomorphic. However, by Eells-Sampson, there is always a harmonic u homotopic to the identity map, and this harmonic map is unique by a theorem of Hartman. By our previous theorem, it is in fact a diffeomorphism. Also, it gives rise to a holomorphic quadratic differential $\Phi_u\in QD(M,\sigma)$.

Theorem 3. *Let $\rho|dw|^2$ and $\rho'|dw'|^2$ be two hyperbolic metrics on M, such that the harmonic maps from $(M,\sigma|dz|^2)$, u,u' induce the same Hopf differential, then $u'\circ u^{-1}$ is an isometry from $\rho|dw|^2$ to $\rho'|dw'|^2$.*

Proof. Since $\Phi_u = \Phi_{u'}$, we have $\mathcal{H}\mathcal{L} = \mathcal{H}'\mathcal{L}'$. Combining with Bochner identity, we have

$$\Delta_M \log \frac{\mathcal{H}'}{\mathcal{H}} = 2(\mathcal{J}' - \mathcal{J}) = (\mathcal{H}' - \mathcal{H})(1 + \frac{\mathcal{L}'}{\mathcal{H}}).$$

Suppose $\mathcal{H}' > \mathcal{H}$ somewhere, then at the maximum point of $\log \frac{\mathcal{H}'}{\mathcal{H}}$, we have

$$0 \geq (\mathcal{H}' - \mathcal{H})(1 + \frac{\mathcal{L}'}{\mathcal{H}}),$$

so $\mathcal{H}' \leq \mathcal{H}$ at this point, a contradiction! Switch $\mathcal{H}'$ and $\mathcal{H}$, we know that in fact $\mathcal{H}' = \mathcal{H}$. Since $\mathcal{H}\mathcal{L} = \mathcal{H}'\mathcal{L}'$, we also have $\mathcal{L}' = \mathcal{L}$. From this, by our previous discussion for introducing Hopf differential, we know that

$$u^* ds_M^2 = u'^* ds_M'^2.$$

Then it is direct to check that $u' \circ u^{-1}$ is an isometry. □

From the above theorem, we know that the mapping from T_g to $QD(M, \sigma)$ is an injection.

Theorem 4. *The mapping sending the equivalent class of $\rho|dw|^2$ to Φ_u is a bijection from T_g to $QD(\sigma)$.*

Proof. We denote Φ the mapping. The above theorem shows that Φ is injective. We need to show that it is surjective. That is, given $\Phi_1 \in QD(\sigma)$, we want to find a hyperbolic metric $\rho_1|dw_0|^2$ and a harmonic map u_1 such that $\Phi_{u_1} = \Phi_1$.

We shall use the continuity method. Namely, we consider the ray $\Phi_t := t\Phi_1$ in $QD(\sigma)$. We want to find a family of hyperbolic metrics h_t and a family of associated harmonic maps u_t. We need to show that the subset of $I = [0, 1]$ that such a u_t and an h_t exist is both open and closed.

Suppose it is OK at time t, then we have $u_t : M \to M$ w.r.t. metrics $\sigma|dz|^2$ and $\rho^t|dw^t|^2$. We get the corresponding functions $\mathcal{H}(t)$ and $\mathcal{L}(t)$ as before. Also, by definition,

$$\mathcal{H}(t)\mathcal{L}(t) = t^2|\Phi_1|^2, \tag{1}$$

$$(\rho^t|dw^t|^2) = t\Phi_1 + t\bar{\Phi}_1 + \sigma(\mathcal{H}(t) + \mathcal{L}(t))|dz|^2. \tag{2}$$

We also have

$$\Delta_M \log \mathcal{H}(t) = -2 + 2(\mathcal{H}(t) - \mathcal{L}(t)). \tag{3}$$

By maximum principle, at the minimum point of $\mathcal{H}(t)$, we have

$$0 \leq \Delta_M \log \mathcal{H}(t) = -2 + 2(\min \mathcal{H}(t) - \mathcal{L}(t)),$$

which implies $\min \mathcal{H}(t) \geq 1$. Formally we compute the derivative with respect to t to get

$$\dot{\mathcal{H}}\mathcal{L} + \mathcal{H}\dot{\mathcal{L}} = 2t|\Phi_1|^2$$

and

$$\begin{aligned}
\Delta_M \frac{\dot{\mathcal{H}}}{\mathcal{H}} &= 2(\dot{\mathcal{H}} - \dot{\mathcal{L}}) \\
&= \frac{2}{\mathcal{H}}\Big(\mathcal{H}\dot{\mathcal{H}} + \dot{\mathcal{H}}\mathcal{L} - 2t|\Phi_1|^2\Big) \\
&= 2\frac{\dot{\mathcal{H}}}{\mathcal{H}}\Big(\mathcal{H} + \frac{t^2|\Phi_1|^2}{\mathcal{H}}\Big) - 4t\frac{|\Phi_1|^2}{\mathcal{H}}.
\end{aligned}$$

Again we apply the maximum principle to get

$$\max \frac{\dot{\mathcal{H}}}{\mathcal{H}} \leq \frac{2t|\Phi_1|^2}{\mathcal{H}^2 + t^2|\Phi_1|^2} \leq \max |\Phi_1|,$$

and

$$\min \frac{\dot{\mathcal{H}}}{\mathcal{H}} \geq 0.$$

So

$$0 \leq \frac{\dot{\mathcal{H}}}{\mathcal{H}} \leq C.$$

Now after these formal discussions, we come back to the real proof. Let $E \subset [0,1]$ be defined as

$$E := \{\tau \in [0,1] | \forall t \le \tau, (1) \text{ and } (3) \text{ are simutanuously solvable}\}.$$

It is nonempty since $0 \in E$. By implicit function theorem in Banach spaces, it is open. At the same time, we know that $\mathcal{H}(t)$ and $\mathcal{L}(t)$ are differentiable with respect to t, so our previous formal discussion is now rigorous. For the detail of openness, please read page 168 of [2].

The closedness part uses regularity theory for elliptic equations as well as our estimate of $\dot{\mathcal{H}}$. We also refer the reader to [2].

The last thing we need is to find a hyperbolic metric and a harmonic map for each of the $\mathcal{H}(t)$ and $\mathcal{L}(t)$. This is easy! The key is the formula (2). It is direct to check that

$$t\Phi_1 + t\bar{\Phi}_1 + \sigma(\mathcal{H}(t) + \mathcal{L}(t))|dz|^2$$

defines a hyperbolic metric and the identity map is the corresponding harmonic map. □

4. The Weil-Petersson metric on T_g

Please take a look at [1] Chapter 6 for a very good introduction to Weil-Petersson metrics on T_g. For recent development, see the papers of Kefeng Liu, Xiaofeng Sun and Shing-Tung Yau [3][4].

One can also use this harmonic map approach to compactify the Teichmüller space. This is closely related to W. Thurston's work. See, for example [6] and the original paper of Wolf [7].

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