A short note on the 1st Chern class of a line bundle

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Abstract
Notes for 2020 BICMR Summer School for Differential Geometry.

1 Review of two definitions

Let $X$ be a complex manifold, using the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp(2\pi \sqrt{-1})} \mathcal{O}^* \rightarrow 1$$

we get the exact sequence

$$\cdots \rightarrow H^1(X, \mathcal{O}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \rightarrow \cdots.$$ 

We call $\delta : H^1(X, \mathcal{O}^*) \rightarrow H^2(X, \mathbb{Z})$ the “first Chern class” map.

Instead of holomorphic line bundles, we can consider $C^\infty$ line bundles. These bundles are classified by $H^1(X, \mathcal{E}^*)$. Similarly, we have short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{E} \xrightarrow{\exp(2\pi \sqrt{-1})} \mathcal{E}^* \rightarrow 1,$$

and consequently a short exact sequence:

$$\cdots \rightarrow H^1(X, \mathcal{E}) \cdots \rightarrow H^1(X, \mathcal{E}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{E}) \rightarrow \cdots.$$ 

Since $\mathcal{E}$ is a fine sheaf, we have $H^p(X, \mathcal{E}) = 0$ whenever $p \geq 1$. So $\delta : H^1(X, \mathcal{E}^*) \rightarrow H^2(X, \mathbb{Z})$ is an isomorphism (also called “first Chern class map”). This means that complex line bundles are determined up to $C^\infty$ isomorphisms by their first Chern class.

On the other hand, we can use a connection on a given $C^\infty$ complex line bundle $L$, and use the curvature form $\Theta$ to define

$$c_1(L) := \left[ \frac{\sqrt{-1}}{2\pi} \Theta \right] \in H^2_{dR}(X; \mathbb{R}) \cong H^2(X, \mathbb{R}).$$
2 Relation between these two definitions

Since we have a natural homomorphism $\Phi : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$ using the sheaf morphism $\mathbb{Z} \to \mathbb{R}$. We shall explore the relation between $\Phi(\delta([L])) \in H^2(X, \mathbb{R})$ and $c_1(L) \in H^2_{dR}(X, \mathbb{R})$.

For simplicity, in the following we assume $L$ is a holomorphic line bundle with Hermitian metric $h$. We leave the necessary modification in the general complex line bundle case as an exercise. (hint: you need to replace the Chern connection by any connection on the bundle, use the transformation formula for connection 1-forms when you change a frame.)

First recall the construction of $\delta : H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$. Let $L$ be a complex line bundle. We use sufficiently fine locally finite trivializations $\mathcal{U} = \{U_a\}_{a \in \Lambda}$ such that each $U_a$ is simply connected and $\check{H}^*(X, \mathcal{O}^*)$ is isomorphic to $H^*(\mathcal{U}, \mathcal{O}^*)$. Then $[L] \in H^1(X, \mathcal{O}^*)$ is determined by the Čech cocycle $[\psi_{\alpha\beta}], \psi_{\alpha\beta} \in \mathcal{O}^*(U_a \cap U_\beta)$. We define $\phi_{\alpha\beta} := \frac{1}{2\pi i} \log \psi_{\alpha\beta}$.

Note that this is not a well-defined Čech cochain: $\log$ is a multi-valued function!

However, since $\psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha} = 1$ on $U_a \cap U_\beta \cap U_\gamma$, we get

$$z_{\alpha\beta\gamma} := \phi_{\alpha\beta} + \phi_{\beta\gamma} - \phi_{\gamma\alpha} \in \mathbb{Z}(U_a \cap U_\beta \cap U_\gamma).$$

This defines a Čech cocycle, whose cohomology class defines $\delta([L])$. Then $\Phi(\delta([L]))$ is also defined by $[z_{\alpha\beta\gamma}]$, just viewing $\mathbb{Z}$ as a subsheaf of $\mathbb{R}$.

To compare it with $c_1(L)$, we need a closer look at the de Rham isomorphism. We first break the resolution

$$0 \to \mathbb{R} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \ldots$$

into short exact sequences:

$$0 \to \mathbb{R} \to \mathcal{A}^0 \to \mathcal{K}_1 \to 0, \quad 0 \to \mathcal{K}_1 \to \mathcal{A}^1 \to \mathcal{K}_2 \to 0, \quad \ldots$$

where $\mathcal{K}_i$ is the sheaf of closed $i$-forms. We get exact sequence for cohomology:

$$0 \to H^1(X, \mathcal{K}_1) \to H^2(X, \mathbb{R}) \to 0, \quad A^1(X) \to \mathcal{K}_2(X) \to H^1(X, \mathcal{K}_1) \to 0.$$

The first one gives $\delta_2 : H^1(X, \mathcal{K}_1) \cong H^2(X, \mathbb{R})$ and the second gives $\delta_1 : H^2_{dR}(X) \cong H^1(X, \mathcal{K}_1)$.

First we study $\delta_1$: Our de Rham class is given by $\frac{-1}{2\pi} \Theta(h) \in \mathcal{K}_2(X)$. Locally, we have

$$\Theta = d\theta_a, \quad \theta_a = \partial \log h_a, \quad h_a = h(e_a, e_a), \quad e_a(p) = \varphi^{-1}_a(p, 1).$$

Then $\delta_1([\frac{-1}{2\pi} \Theta(h)])$ is given by $\{[\frac{-1}{2\pi} (\theta_\beta - \theta_\alpha)]\}$.

Now

$$e_\beta(p) = \varphi^{-1}_\beta(p, 1) = \varphi^{-1}_\alpha \circ (\varphi_\alpha \circ \varphi^{-1}_\beta)(p, 1) = \varphi^{-1}_\alpha(p, \psi_{\alpha\beta}(p)) = \psi_{\alpha\beta}(p)e_\alpha(p).$$
So we get \( h_\beta = h_\alpha |\psi_{\alpha\beta}|^2 \), and hence \( \log h_\beta = \log h_\alpha + \log |\psi_{\alpha\beta}|^2 \). So on \( U_\alpha \cap U_\beta \), we have

\[
\frac{\sqrt{-1}}{2\pi} (\theta_\beta - \theta_\alpha) = \frac{\sqrt{-1}}{2\pi} \partial \log |\psi_{\alpha\beta}|^2 = \frac{\sqrt{-1}}{2\pi} \partial \log \psi_{\alpha\beta} = \frac{\sqrt{-1}}{2\pi} d \log \psi_{\alpha\beta}.
\]

Then \( \delta_2([\frac{\sqrt{-1}}{2\pi} (\theta_\beta - \theta_\alpha)]) \) is represented by

\[
\left\{ \frac{\sqrt{-1}}{2\pi} \left( \log \psi_{\beta\gamma} - \log \psi_{\alpha\gamma} + \log \psi_{\alpha\beta} \right) \right\}.
\]

This is precisely our \( \{ z_{\alpha\beta\gamma} \} \).

References
