Remark on the definition of pesudoconvexity:

We shall prove that the definition is independent of the choice of defining functions for C^2 domains. Let r, r' be two defining function near a boundary point $z_0 \in \partial\Omega$, then $r, r' \in C^2$ and $r' = rh$ for some C^1 function h. Since $r < 0$ if and only if $r' < 0$, we know that $h > 0$ inside Ω. On the other hand,

$$dr'|_{\partial\Omega} = h|_{\partial\Omega} dr|_{\partial\Omega}.$$

Since both dr' and dr are non-vanishing on $\partial\Omega$, we conclude that h is non-zero on $\partial\Omega$. By continuity, h is positive in a neighborhood of z_0.

We have

$$\frac{\partial r'}{\partial z_i} = r\frac{\partial h}{\partial z_i} + h\frac{\partial r}{\partial z_i}.$$

In particular

$$\frac{\partial r'}{\partial z_i}(z_0) = h(z_0)\frac{\partial r}{\partial z_i}(z_0). \tag{1}$$

This implies that the linear space $T_{z_0}\partial\Omega := \{\xi \in \mathbb{C}^n | \ \sum_i \frac{\partial r}{\partial z_i}(z_0)\xi_i = 0\}$ is independent of the choice of r.

Note that $h\frac{\partial r}{\partial z_i}$ is differentiable, and

$$\frac{\partial}{\partial \bar{z}_j}\Big(h\frac{\partial r}{\partial z_i}\Big) = h\frac{\partial^2 r}{\partial z_i \partial \bar{z}_j} + \frac{\partial r}{\partial z_i}\frac{\partial h}{\partial \bar{z}_j}.$$

Moreover, $r\frac{\partial h}{\partial z_i}$ is in fact also differentiable at z_0:

$$\frac{\partial}{\partial x_j}\Big(r\frac{\partial h}{\partial z_i}\Big)(z_0) = \lim_{\Delta x_j \to 0}\frac{1}{\Delta x_j} r(z_0 + \Delta x_j)\frac{\partial h}{\partial z_i}(z_0 + \Delta x_j) = \frac{\partial r}{\partial x_j}(z_0)\frac{\partial h}{\partial z_i}(z_0),$$

so we conclude that

$$\frac{\partial^2 r'}{\partial z_i \partial \bar{z}_j}(z_0) = h(z_0)\frac{\partial^2 r}{\partial z_i \partial \bar{z}_j}(z_0) + \frac{\partial r}{\partial z_i}(z_0)\frac{\partial h}{\partial \bar{z}_j}(z_0) + \frac{\partial r}{\partial \bar{z}_j}(z_0)\frac{\partial h}{\partial z_i}(z_0).$$

For any $\xi \in T_{z_0}\partial\Omega$, we have

$$L_{r'}(z_0, \xi) := \sum_{i,j}\frac{\partial^2 r'}{\partial z_i \partial \bar{z}_j}(z_0)\xi_i\bar{\xi}_j = h(z_0)L_r(z_0, \xi).$$

Since $h(z_0) > 0$, the positivity of the Levi form on $T_{z_0}\partial\Omega$ is independent of the choice of r.