HIGHER KURTZ RANDOMNESS

BJØRN KJOS-HANSEN, ANDRÉ NIES, FRANK STEPHAN, AND LIANG YU

Abstract. A real $x$ is $\Delta^1_1$-Kurtz random ($\Pi^1_1$-Kurtz random) if it is in no closed null $\Delta^1_1$ set ($\Pi^1_1$ set). We show that there is a cone of $\Pi^1_1$-Kurtz random hyperdegrees. We characterize lowness for $\Delta^1_1$-Kurtz randomness as being $\Delta^1_1$-dominated and $\Delta^1_1$-semi-traceable.

1. Introduction
Traditionally one uses tools from recursion theory to obtain mathematical notions corresponding to our intuitive idea of randomness for reals. However, already Martin-Löf [10] suggested to use tools from higher recursion (or equivalently, effective descriptive set theory) when he introduced the notion of $\Delta^1_1$-randomness. This approach was pursued to greater depths by Hjorth and Nies [8] and Chong, Nies and Yu [1]. Hjorth and Nies investigated a higher analog of the usual Martin-Löf randomness, and a new notion with no direct analog in (lower) recursion theory: a real is $\Pi^1_1$-random if it avoids each null $\Pi^1_1$ set. Chong, Nies and Yu [1] studied $\Delta^1_1$-randomness in more detail, viewing it as a higher analog of both Schnorr and recursive randomness. By now a classical result is the characterization of lowness for Schnorr randomness by $\Delta^1_1$ traceability (see, for instance, Nies’ textbook [12]). Chong, Nies and Yu [1] proved a higher analog of this result, characterizing lowness for $\Delta^1_1$ randomness by $\Delta^1_1$ traceability.

Our goal is to carry out similar investigations for higher analogs of Kurtz randomness [3]. A real $x$ is Kurtz random if avoids each $\Pi^0_1$ null class. This is quite a weak notion of randomness: each weakly 1-generic set is Kurtz random, so for instance the law of large numbers can fail badly.

It is essential for Kurtz randomness that the tests are closed null sets. For higher analogs of Kurtz randomness one can require that these tests are closed and belong to a more permissive class such as $\Delta^1_1$, $\Pi^1_1$, or $\Sigma^1_1$.

Restrictions on the computational complexity of a real have been used successfully to analyze randomness notions. For instance, a Martin-Löf random real is weakly 2-random iff it forms a minimal pair with $\emptyset'$ (see [12]). We prove a result of that kind in the present setting. Chong, Nies, and Yu [1] studied a property restricting the

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complexity of a real: being $\Delta^1_1$-dominated. This is the higher analog of being recursively dominated (or of hyperimmune-free degree). We show that a $\Delta^1_1$-Kurtz random $\Delta^1_1$ dominated set is already $\Pi^1_1$-random. Thus $\Delta^1_1$-Kurtz randomness is equivalent to a proper randomness notion on a conull set. We also study the distribution of higher Kurtz random reals in the hyperdegrees. For instance, there is a cone of $\Pi^1_1$-Kurtz random hyperdegrees. However, its base is very complex, having the largest hyperdegree among all $\Sigma^1_3$ reals.

Thereafter we turn to lowness for higher Kurtz randomness. Recursive traceability of a real $x$ is easily seen to be equivalent to the condition that for each function $f \leq_T x$ there is a recursive functional $\hat{f}$ that agrees with $f$ on at least one input in each interval of the form $[2^n, 2^{n+1} - 1]$ (see [12, 8.2.21]). Following Kjos-Hanssen, Merkle, and Stephan [?] one says that $x$ is recursively semi-traceable (or infinitely often traceable) if for each $f \leq_T x$ there is a recursive function $\hat{f}$ that agrees with $f$ on infinitely many inputs. It is straightforward to define the higher analog of this notion, $\Delta^1_1$-semi-traceability. Our main result is that lowness for $\Delta^1_1$-Kurtz randomness is equivalent to being $\Delta^1_1$-dominated and $\Delta^1_1$-semi-traceable. We also show using forcing that being $\Delta^1_1$-dominated and $\Delta^1_1$-semi-traceable is strictly weaker than being $\Delta^1_1$-traceable. Thus, lowness for $\Delta^1_1$ Kurtz randomness is strictly weaker than lowness for $\Delta^1_1$-randomness.

2. Preliminaries

We assume that the reader is familiar with elements of higher recursion theory, as presented, for instance, in Sacks [15]. See [12, Ch. 9] for a summary.

A real is an element in $2^\omega$. Sometimes we write $n \in \omega$ to mean $x(n) = 1$. Fix a standard $\Pi^0_2$ set $H \subseteq \omega \times 2^\omega \times 2^\omega$ so that for all $x$ and $n \in O$, there is a unique real $y$ satisfying $H(n,x,y)$. Moreover, if $\omega^*_1 = \omega^*_1^{CK}$, then each real $z \leq_h x$ is Turing reducible to some $y$ so that $H(n,x,y)$ holds for some $n \in O$. Roughly speaking, $y$ is the $|n|$-th Turing jump of $x$. These $y$’s are called $H^*_x$ sets and denoted by $H^*_x$. For each $n \in O$, let $O_n = \{m \in O \mid |m| < |n|\}$. $O_n$ is a $\Delta^1_1$ set.

We use the Cantor pairing function, the bijection $p : \omega^2 \to \omega$ given by $p(n,s) = \frac{(n+s)^2+3n+s}{2}$, and write $\langle n, s \rangle = p(n,s)$. For a finite string $\sigma$, $[\sigma] = \{ x \mid \sigma \upharpoonright x \in 2^\omega \}$. For an open set $U$, there is a presentation $\hat{U} \subseteq 2^{<\omega}$ so that $\sigma \in \hat{U}$ if and only if $[\sigma] \subseteq U$. We sometimes identify $U$ with $\hat{U}$. For a recursive functional $\Phi$, we use $\Phi^s$ to denote the computation state of $\Phi^s$ at stage $s$. For a tree $T$, we use $[T]$ to denote the set of infinite paths in $T$. Some times we identify a finite string $\sigma \in \omega^{<\omega}$ with a natural number without confusion.

The following results will be used in later sections.

**Theorem 2.1** (Gandy). *If $A \subseteq 2^\omega$ is a nonempty $\Sigma^1_1$ set, then there is a real $x \in A$ so that $\sigma^x \leq_h \sigma^A$.*

**Theorem 2.2** (Spector [16] and Gandy [6]). *$A \subseteq 2^\omega$ is $\Pi^1_1$ if and only if there is an arithmetical predicate $P(x,y)$ such that $y \in A \leftrightarrow \exists x \leq_h yP(x,y)$.*
Theorem 2.3 (Sacks [13]). If \( x \) is non-hyperarithmetical, then \( \mu(\{y | y \geq_h x\}) = 0 \).

Theorem 2.4 (Sacks [15]). The set \( \{x | x \geq_h o\} \) is \( \Pi^1_1 \). Moreover, \( x \geq_h o \) if and only if \( \omega^x_1 > \omega^{CK}_1 \).

A consequence of the last two theorems above is that the set \( \{x | \omega^x_1 > \omega^{CK}_1\} \) is a \( \Pi^1_1 \) null set.

Given a class \( \Gamma \), an element \( x \in \omega^\omega \) is called a \( \Gamma \)-singleton if \( \{x\} \) is a \( \Gamma \) set. Note that if \( x \in \omega^\omega \) is a \( \Pi^1_1 \)-singleton, then too is \( x_0 = \{(n, m) | x(n) = m\} \equiv_T x \). Hence we do not distinguish \( \Pi^1_1 \)-singletons between Baire space and Cantor space.

A subset of \( 2^\omega \) is \( \Pi^0_0 \) if it is clopen. We can define \( \Pi^0_0 \) sets by a transfinite induction for all countable \( \gamma \). Every such set can be coded by a real (for more details see [15]). Given a class \( \Gamma \) (for example, \( \Gamma = \Delta^1_1 \)) of subsets of \( 2^\omega \), a set \( A \) is \( \Pi^0_0(\Gamma) \) if \( A \) is \( \Pi^0_0 \) and can be coded by a real in \( \Gamma \).

In the case \( \gamma = 1 \), every hyperarithmetic closed subset of reals is \( \Pi^0_0(\Delta^1_1) \). We also have the following result with an easy proof.

Proposition 2.5. If \( A \subset 2^\omega \) is \( \Sigma^1_1 \) and \( \Pi^0_0 \), then \( A \) is \( \Pi^0_0(\Sigma^1_1) \).

Proof. Let \( z = \{\sigma | \exists x (x \in A \land x > \sigma)\} \). Then \( x \in A \) if and only if \( \forall n(x | n \in z) \). So \( A \) is \( \Pi^0_0(z) \). Obviously \( z \) is \( \Sigma^1_1 \). \( \square \)

Note that Proposition 2.5 fails if we replace \( \Sigma^1_1 \) with \( \Pi^1_1 \) since \( O^O \) is a \( \Pi^1_1 \) singleton of hyperdegree greater than \( O \).

The ramified analytical hierarchy was introduced by Kleene, and applied by Feferman [4] and Cohen [2] to study forcing, a tool that turns out to be powerful in the investigation of higher randomness theory. We recall some basic facts from Sacks [15] whose notations we mostly follow:

The ramified analytic hierarchy language \( L(\omega^{CK}_1, \dot{x}) \) contains the following symbols:

1. Number variables: \( j, k, m, n, \ldots \);
2. Numerals: \( 0, 1, 2, \ldots \);
3. Constant: \( \dot{x} \);
4. Ranked set variables: \( x^\alpha, y^\alpha, \ldots \) where \( \alpha < \omega^{CK}_1 \);
5. Unranked set variables: \( x, y, \ldots \);
6. Others symbols include: \( +, \cdot \) (times), \( \prime \) (successor) and \( \varepsilon \).

Formulas are built in the usual way. A formula \( \varphi \) is ranked if all of its set variables are ranked. Due to its complexity, the language is not codable in a recursive set but rather in the countable admissible set \( L_{\omega^{CK}_1} \).

To code the language in a uniform way, we fix a \( \Pi^1_1 \) path \( O_1 \) through \( O \) (by [5] such a path exists). Then a ranked set variable \( x^\alpha \) is coded by the number \( (2, n) \) where \( n \in O_1 \) and \( |n| = \alpha \). Other symbols and formulas are coded recursively. With such a coding, the set of G"odel number of formulas is \( \Pi^1_1 \). Moreover, the set of G"odel numbers of ranked formulas of rank less than \( \alpha \) is r.e. uniformly in the unique notation for \( \alpha \) in \( O_1 \). Hence there is a recursive function \( f \) so that \( W_{f(n)} \) is the set of G"odel numbers of the ranked formula of rank less than \( |n| \) when \( n \in O_1 \) (\( \{W_e\}_e \) is, as usual, an effective enumeration of r.e. sets).
One now defines a structure $\mathcal{A}(\omega^\text{CK}_1, x)$, where $x$ is a real, analogous to the way Gödel’s $L$ is defined, by induction on the recursive ordinals. Only at successor stages are new sets defined in the structure. The reals constructed at a successor stage are arithmetically definable from the reals constructed at earlier stages. The details may be found in [15]. We define $\mathcal{A}(\omega^\text{CK}_1, x) \models \varphi$ for a formula $\varphi$ of $\mathcal{L}(\omega^\text{CK}_1, x)$ by allowing the unranked set variables to range over $\mathcal{A}(\omega^\text{CK}_1, x)$, while the symbol $x^\alpha$ will be interpreted as the reals built before stage $\alpha$. In fact, the domain of $\mathcal{A}(\omega^\text{CK}_1, x)$ is the set $\{y \mid y \leq_h x\}$ if and only if $\omega^\alpha = \omega^\text{CK}_1$ (see [15]).

A sentence $\varphi$ of $\mathcal{L}(\omega^\text{CK}_1, x)$ is said to be $\Sigma^1_n$ if it is ranked, or of the form $\exists x_1, \ldots, \exists x_n \psi$ for some formula $\psi$ with no unranked set variables bounded by a quantifier.

The following result is a model-theoretic version of the Gandy-Spector Theorem.

**Theorem 2.6** (Sacks [15]). The set $\{(n_\varphi, x) \mid \varphi \in \Sigma^1_n \land \mathcal{A}(\omega^\text{CK}_1, x) \models \varphi\}$ is $\Pi^1_1$, where $n_\varphi$ is the Gödel number of $\varphi$. Moreover, for each $\Pi^1_1$ set $A \subseteq 2^\omega$, there is a formula $\varphi \in \Sigma^1_1$ so that

1. $\mathcal{A}(\omega^\text{CK}_1, x) \models \varphi \iff x \in A$;
2. if $\omega^\alpha = \omega^\text{CK}_1$, then $\mathcal{A}(\omega^\text{CK}_1, x) \models \varphi \iff x \in A$.

Note that if $\varphi$ is ranked, then both the sets $\{x \mid \mathcal{A}(\omega^\text{CK}_1, x) \models \varphi\}$ (the Gödel number of $\varphi$ is omitted) and $\{x \mid \mathcal{A}(\omega^\text{CK}_1, x) \models \neg \varphi\}$ are $\Pi^1_1$. So both sets are $\Delta^1_1$. Moreover, if $A \subseteq 2^\omega$ is $\Delta^1_1$, then there is a ranked formula $\varphi$ so that $x \in A \iff \mathcal{A}(\omega^\text{CK}_1, x) \models \varphi$ (see Sacks [15]).

**Theorem 2.7** (Sacks [13]). The set

$\{(n_\varphi, p) \mid \mu(\{x \mid \mathcal{A}(\omega^\text{CK}_1, x) \models \varphi\}) > p \land \varphi \in \Sigma^1_1 \land p \text{ is a rational number}\}$

is $\Pi^1_1$ where $n_\varphi$ is the Gödel number of $\varphi$.

**Theorem 2.8** (Sacks [13]). There is a recursive function $f : \omega \times \omega \to \omega$ so that for all $n$ which is Gödel number of a ranked formula:

1. $f(n, p)$ is Gödel number of a ranked formula;
2. the set $\{x \mid \mathcal{A}(\omega^\text{CK}_1, x) \models \varphi_{f(n, p)}\}$ is open and
3. $\mu(\{x \mid \mathcal{A}(\omega^\text{CK}_1, x) \models \varphi_{f(n, p)}\}) < \frac{1}{p}$.

**Theorem 2.9** (Sacks [13] and Tanaka [17]). If $A$ is a $\Pi^1_1$ set of positive measure, then $A$ contains a hyperarithmetical real.

We also remind the reader of the higher analog of ML-randomness first studied by [8].

**Definition 2.10.** A $\Pi^1_1$-ML-test is a sequence $(G_m)_{m \in \omega}$ of open sets such that for each $m$, we have $\mu(G_m) \leq 2^{-m}$, and the relation $\{\langle m, \sigma \rangle \mid [\sigma] \subseteq G_m\}$ is $\Pi^1_1$. A real $x$ is $\Pi^1_1$-ML-random if $x \not\in \cap_m G_m$ for each $\Pi^1_1$-ML-test $(G_m)_{m \in \omega}$.

### 3. Higher Kurtz random reals and their distribution

**Definition 3.1.** Suppose we are given a point class $\Gamma$ (i.e., a class of sets of reals). A real $x$ is $\Gamma$-Kurtz random if $x \not\in A$ for every closed null set $A \in \Gamma$. Further, $x$ is said to be Kurtz random (y-Kurtz random) if $\Gamma = \Pi^0_1(y)$.
We focus on $\Delta_1^1$, $\Sigma_1^1$ and $\Pi_1^1$-Kurtz randomness. By the proof of Proposition 2.5, it is not difficult to see that a real $x$ is $\Delta_1^1$-Kurtz random if and only if $x$ does not belong to any $\Pi_0^1(\Delta_1^1)$ null set.

**Theorem 3.2.** $\Pi_1^1$-Kurtz randomness $\subseteq \Sigma_1^1$-Kurtz randomness $= \Delta_1^1$-Kurtz randomness.

**Proof.** It is obvious that $\Pi_1^1$-Kurtz randomness $\subseteq \Delta_1^1$-Kurtz randomness and $\Sigma_1^1$-Kurtz randomness $\subseteq \Delta_1^1$-Kurtz randomness. It suffices to prove that $\Pi_1^1$-Kurtz randomness $= \Delta_1^1$-Kurtz randomness and $\Pi_1^1$-Kurtz randomness $\subseteq \Delta_1^1$-Kurtz randomness.

Note that every $\Pi_1^1$-ML-random is $\Delta_1^1$-Kurtz random and there is a $\Pi_1^1$-ML-random real $x \equiv_K \mathcal{O}$ (see [8] and [1]). But $\{x\}$ is a $\Pi_1^1$ closed set. So $x$ is not $\Pi_1^1$-Kurtz random. Hence $\Pi_1^1$-Kurtz randomness $\subseteq \Delta_1^1$-Kurtz randomness.

Suppose we are given a $\Pi_1^1$ open set $A$ of measure 1. Define

$$x = \{\sigma \in 2^{<\omega} \mid \forall y(y > \sigma \implies y \in A)\}.$$ 

Then $x$ is a $\Pi_1^1$ real coding $A$ (i.e., $y \in A$ if and only if there is a $\sigma \in x$ for which $y > \sigma$, or $y \in [\sigma]$). So there is a recursive function $f : 2^{<\omega} \to \omega$ so that $\sigma \in x$ if and only if $f(\sigma) \in \mathcal{O}$. Define a $\Pi_1^1$ relation $R \subseteq \omega \times \omega$ so that $(k, n) \in R$ if and only if $n \in \mathcal{O}$ and $\mu(\{\sigma \mid \exists m \in \mathcal{O}_n(f(\sigma) = m)\}) > 1 - \frac{1}{k}$. Obviously $R$ is a $\Pi_1^1$ relation which can be uniformized by a $\Pi_1^1$ function $f^*$ (see [11]). Since $\mu(A) = 1$, $f^*$ is a total function. So the range of $f^*$ is bounded by a notation $n \in \mathcal{O}$. Define $B = \{y \mid \exists \sigma(y > \sigma \land f(\sigma) \in \mathcal{O}_n)\}$. Then $B \subseteq A$ is a $\Delta_1^1$ open set with measure 1. So every $\Pi_1^1$ open conull set has a $\Delta_1^1$ open conull subset. Hence $\Sigma_1^1$-Kurtz randomness equals $\Delta_1^1$-Kurtz randomness.

It should be pointed out that, by the proof of Theorem 3.2, not every $\Pi_1^1$-ML-random real is $\Pi_1^1$-Kurtz random.

The following result clarifies the relationship between $\Delta_1^1$- and $\Pi_1^1$-Kurtz randomness.

**Proposition 3.3.** If $\omega_1^f = \omega_1^{\text{CK}}$, then $x$ is $\Pi_1^1$-Kurtz random if and only if $x$ is $\Delta_1^1$-Kurtz random.

**Proof.** Suppose that $\omega_1^f = \omega_1^{\text{CK}}$ and $x$ is $\Delta_1^1$-Kurtz random. If $A$ is a $\Pi_1^1$ closed null set so that $x \in A$, then by Theorem 2.6, there is a formula $\varphi(z, y)$ whose only unranked set variables are $z$ and $y$ so that the formula $\exists z \varphi(z, y)$ defines $A$. Since $\omega_1^f = \omega_1^{\text{CK}}$, $x \in B = \{y \mid \exists \sigma(\omega(z^{\alpha}, y) \models \exists z^\alpha \varphi(z^\alpha, y)) \subseteq A\}$ for some recursive ordinal $\alpha$. Define $T = \{\sigma \in 2^{<\omega} \mid \exists y \in B(y > \sigma)\}$. Obviously $B \subseteq [T]$. Since $B$ is $\Delta_1^1$, $[T]$ is $\Sigma_1^1$. Since $A$ is closed, $B \subseteq A$, and $[T]$ is the closure of $B$, we have $[T] \subseteq A$. Hence since $A$ is null, so is $[T]$. By the proof of Theorem 3.2, there is a $\Delta_1^1$ closed null set $C \supseteq [T]$. Hence $x \in C$, a contradiction.

From the proof of Theorem 3.2, one sees that every hyperdegree above $\mathcal{O}$ contains a $\Delta_1^1$-Kurtz random real. But this fails for $\Pi_1^1$-Kurtz randomness. We say that a hyperdegree $d$ is a *base for a cone of $\Gamma$-Kurtz randoms* if for every hyperarithmetic degree $h \geq d$, $h$ contains a $\Gamma$-Kurtz random real.
The hyperdegree of $O$ is a base for a cone of $\Delta^1_1$-Kurtz randoms as proved in Theorem 3.2. In Corollary 5.3 we will show that not every nonzero hyperdegree is a base of a cone of $\Delta^1_1$-Kurtz randoms.

Is there a base for a cone of $\Pi^1_1$-Kurtz randoms? If such a base $b$ exists, then $b$ is not hyperarithmetically reducible to any $\Pi^1_1$ singleton. Intuitively, this means that such bases must be complex.

To obtain such a base we need a lemma.

**Lemma 3.4.** For any reals $x$ and $z \geq_T x'$, there is an $x$-Kurtz random real $y \equiv_T z$.

**Proof.** Fix an enumeration of the $x$-r.e. open sets $\{U_n^x\}_{n \in \omega}$.

We inductively define an increasing sequence of binary strings $\{\sigma_s\}_{s < \omega}$.

Stage 0. Let $\sigma_0$ be the empty string.

Stage $s + 1$. Let $l_0 = 0$, $l_1 = |\sigma_s|$, and $l_{n+1} = 2^{l_n}$ for all $n > 1$. For every $n > 1$, let

$$A_n = \{\sigma \in 2^{l_{n-1}} \mid \exists m < n \forall i \forall j (l_m \leq i, j < l_{m+1} \implies \sigma(i) = \sigma(j))\}.$$  

Then

$$|A_n| \leq 2 \cdot 2^{l_{n-1}}.$$  

In other words,

$$\mu(\bigcup\{[\sigma] \mid \sigma \geq \sigma_s \land \sigma \notin A_n\}) \geq 2^{-l_1} \cdot (1 - 2^{l_n+1-l_{n+1}}).$$

Case(1): There is some $m > l_1 + 1$ so that $|\{\sigma \geq \sigma_s \mid \sigma \in 2^m \land [\sigma] \subseteq U_s^x\}| > 2^{m-l_1-1}$. Let $n = m + 1$. Then $l_{n+1} - 1 - l_n > 2$ and $l_n > m$. So there must be some \( \sigma \in 2^{l_{n-1}} - A_n \) so that there is a $\tau \preceq \sigma$ for which $[\tau] \subseteq U_s^x$ and $\tau \in 2^m$.

Let $\sigma_{s+1} = \sigma^\tau(z(s))^{l_n-1}$.

Case(2): Otherwise. Let $\sigma_{s+1} = \sigma^\tau(z(s))^{l_1-1}$.

This finishes the construction at stage $s + 1$.

Let $y = \bigcup_s \sigma_s$.

Obviously the construction is recursive in $z$. So $y \leq_T z$. Moreover, if $U_s^x$ is of measure 1, then Case (1) happens at the stage $n + 1$. So $y$ is $x$-Kurtz random.

Let $l_0 = 0, l_{n+1} = 2^{l_n}$ for all $n \in \omega$. To compute $z(n)$ from $y$, we $y$-recursively find the $n$-th $l_m$ for which for all $i, j$ with $l_m \leq i < j < l_{m+1}$, $y(i) = y(j)$. Then

$$z(n) = y(l_m).$$

Let $Q \subseteq \omega \times 2^\omega$ be a universal $\Pi^1_1$ set. In other words, $Q$ is a $\Pi^1_1$ set so that every $\Pi^1_1$ set is some $Q_n = \{x \mid (n, x) \in Q\}$. By Theorem 2.2.3 in [9], the real $x_0 = \{n \mid \mu(Q_n) = 0\}$ is $\Sigma^1_1$. Let

$$c = \{(n, \sigma) \mid n \in x_0 \land \exists x((n, x) \in Q \land \sigma \prec x)\} \subseteq \omega \times 2^{2^\omega}.$$  

Then $c$ can be viewed as a $\Sigma^1_2$ real. Since every $\Pi^1_1$ null closed set is $\Pi^0_1(c)$, every $c$-Kurtz random real is $\Pi^1_1$-Kurtz random.

**Theorem 3.5.** $c$ is a base for a cone of $\Pi^1_1$-Kurtz randoms.
Proof. For every real \( y_0 \geq_h c \), there is a real \( y_1 \equiv_h y_0 \) so that \( y_1 \geq_T c' \), the Turing jump of \( c \). By Lemma 3.4, there is a real \( z \equiv_T y_1 \) for which \( z \) is \( c \)-Kurtz random and so \( \Pi_*^1 \)-Kurtz random. \( \Box 
\)

Recall that every \( \Sigma_*^1 \) real is constructible (see e.g. the last chapter of Moschovakis [11]). In the following we will determine the position of \( c \) within the constructible hierarchy. A real is called constructible if it belongs to some level \( L_\alpha \) of Gödel’s hierarchy of constructible sets

\[
L = \bigcup \{ L_\beta : \beta \text{ is an ordinal} \}.
\]

More generally, for each real \( x \) we have the hierarchy

\[
L[x] = \bigcup \{ L_\beta[x] : \beta \text{ is an ordinal} \}
\]

of sets constructible from \( x \).

Let

\[
\delta_*^1 = \sup\{ \alpha : \alpha \text{ is an ordinal isomorphic to a } \Delta_*^1 \text{ wellordering of } \omega \},
\]

and

\[
\delta = \min\{ \alpha \mid L \setminus L_\alpha \text{ contains no } \Pi_*^1 \text{ singleton} \}.
\]

**Proposition 3.6** (Forklore). \( \delta = \delta_*^1 \).

**Proof.** If \( \alpha < \delta \), then there is a \( \Pi_*^1 \) singleton \( x \in L_\delta \setminus L_\alpha \). Since \( x \in L_\omega_*^1 \) and \( \omega_*^1 \) is a \( \Pi_*^1(x) \) wellordering, it must be that \( \alpha < \omega_*^1 < \delta_*^1 \). So \( \delta 
\delta_*^1 \).

If \( \alpha < \delta_*^1 \), there is a \( \Delta_*^1 \) wellordering relation \( R \subseteq \omega \times \omega \) of order type \( \alpha \). So there are two recursive relations \( S, T \subseteq (\omega^\omega)^2 \times \omega^3 \) so that

\[
R(n, m) \iff \exists f \forall g \exists k S(f, g, n, m, k), \text{ and }
\]

\[
-R(n, m) \iff \exists f \forall g \exists k T(f, g, n, m, k).
\]

Define a \( \Pi_*^1 \) set \( R_0 = \{(f, n, m) \mid \forall g \exists k S(f, g, n, m, k)\} \). By the Gandy-Spector Theorem 2.2, there is an arithmetical relation \( S' \) so that \( R_0 = \{(f, n, m) \mid \exists g \leq_h f(S'(f, g, n, m))\} \). Recall that every nonempty \( \Pi_*^1 \) set contains a \( \Pi_*^1 \)-singleton (Kondo-Addison [15]). Then

\[
R(n, m) \iff \exists f \in L_\delta \exists g \in L_\omega_*^1[f](S'(f, g, n, m)).
\]

In other words, \( R \) is \( \Sigma_*^1 \)-definable over \( L_\delta \). By the same method, the complement of \( R \) is \( \Sigma_*^1 \)-definable over \( L_\delta \) too. So \( R \) is \( \Delta_*^1 \)-definable over \( L_\delta \). It is clear that \( L_\delta \) is admissible. So \( R \in L_\delta \). Hence \( \alpha < \delta \). Thus \( \delta_*^1 = \delta \). \( \Box 
\)

Note that if \( x \) is a \( \Delta_*^1 \)-real, then \( \omega_*^1 \) is isomorphic to a \( \Delta_*^1 \) wellordering of \( \omega \). So

\[
\sup\{ \omega_*^1 \mid x \text{ is a } \Pi_*^1 \text{-singleton} \} \leq \delta_*^1.
\]

Since \( x \in L_\omega_*^1 \) for every \( \Pi_*^1 \)-singleton \( x \),

\[
\sup\{ \omega_*^1 \mid x \text{ is a } \Pi_*^1 \text{-singleton} \} \geq \delta = \delta_*^1.
\]

Thus

\[
\sup\{ \omega_*^1 \mid x \text{ is a } \Pi_*^1 \text{-singleton} \} = \delta = \delta_*^1.
\]

Since every \( \Pi_*^1 \) singleton is recursive in \( c \), we have \( c \not\in L_\delta_*^1 \) and \( \omega_*^1 \geq \delta_*^1 \).
Proposition 4.2.

A real result is a higher analog of the result that Kurtz randomness coincides with weak 2-

Recall that a real is \( \Pi^1_1 \) if and only if there is a recursive function \( f \) such that, for infinitely many \( n \), \( f(n) \) is \( \Pi^1_1 \)-null set. The following result is a higher analog of the result that Kurtz randomness coincides with weak 2-

randomness for reals of hyperimmune-free degree.

Definition 4.3.

(i) Let \( h : \omega \to \omega \) be a nondecreasing unbounded function that is hyperarithmetical. A \( \Delta^1_1 \) trace with bound \( h \) is a uniformly \( \Delta^1_1 \) sequence \((T_e)_{e \in \omega}\) such that \( |T_e| \leq h(e) \) for each \( e \).

(ii) \( x \in 2^\omega \) is a \( \Delta^1_1 \)-traceable [1] if there is \( h \in \Delta^1_1 \) such that, for each \( f \leq h \), there is a \( \Delta^1_1 \) trace with bound \( h \) such that, for each \( e \), \( f(e) \in T_e \).

(iii) \( x \in 2^\omega \) is a \( \Delta^1_1 \)-semi-traceable if for each \( f \leq h \), there is a \( \Delta^1_1 \) function \( g \) so that, for infinitely many \( n \), \( f(n) = g(n) \). We say that \( g \) semi-traces \( f \).

(iv) \( x \in 2^\omega \) is \( \Pi^1_1 \)-semi-traceable if for each \( f \leq h \), there is a partial \( \Pi^1_1 \) function \( p \) so that, for infinitely many \( n \) we have \( f(n) = p(n) \).

4. \( \Delta^1_1 \)-traceability and dominability

We begin with the characterization of \( \Pi^1_1 \)-randomness within \( \Delta^1_1 \)-Kurtz randomness.

Definition 4.4. A real \( x \) is hyp-dominated if for all functions \( f : \omega \to \omega \) with \( f \leq h \), there is a hyperarithmetic function \( g \) so that \( g(n) > f(n) \) for all \( n \).

Recall that a real is \( \Pi^1_1 \)-random if it does not belong to any \( \Pi^1_1 \)-null set. The following result is a higher analog of the result that Kurtz randomness coincides with weak 2-

randomness for reals of hyperimmune-free degree.

Proposition 4.2. A real \( x \) is \( \Pi^1_1 \)-random if and only if \( x \) is hyp-dominated and \( \Delta^1_1 \)-Kurtz random.

Proof. Every \( \Pi^1_1 \)-random real is \( \Delta^1_1 \)-Kurtz random and also hyp-dominated (see [1]). We prove the other direction.

Suppose \( x \) is hyp-dominated and \( \Delta^1_1 \)-Kurtz random. We show that \( x \) is \( \Pi^1_1 \)-Martin-Löf random. If not, then fix a universal \( \Pi^1_1 \)-Martin-Löf test \( \{U_n\}_{n \in \omega} \) (see [8]). Then there is a recursive function \( f : \omega \times 2^{\omega} \to \omega \) so that for any pair \( (n, \sigma) \), \( \sigma \in U_n \) if and only if \( f(n, \sigma) \in \mathcal{O} \). Since \( x \) is hyp-dominated, \( \omega^x_1 = \omega^CK \) (see [1]). Then we define a \( \Pi^1_1 \)(\( x \)) relation \( R \subseteq \omega \times \omega \) so that \( R(n, m) \) if and only if there is a \( \sigma \) so that \( m \in \mathcal{O} \), \( f(n, \sigma) \in \mathcal{O}_m = \{i \in \mathcal{O} | |i| < |m| \} \) and \( \sigma \prec x \). Then by the \( \Pi^1_1 \)-uniformization relativized to \( x \), there is a partial function \( p \) uniformizing \( R \). Since \( x \in \bigcap_n U_n \), \( p \) is a total function. Since \( \omega^x_1 = \omega^CK \), there must be some \( m_0 \in \mathcal{O} \) so that \( p(n) \in \mathcal{O}_{m_0} \) for every \( n \). Then define a \( \Delta^1_1 \)-Martin-Löf test \( \{\hat{U}_n\}_{n \in \omega} \) so that \( \sigma \in \hat{U}_n \) if and only if \( f(n, \sigma) \in \mathcal{O}_{m_0} \). So \( x \in \bigcap_n \hat{U}_n \). Let \( \hat{f}(n) = \min\{l \ | \ \exists \sigma \in 2^l (\sigma \in \hat{U}_n \land x \in [\sigma])\} \) be a \( \Delta^1_1 \) function \( f \) dominating \( \hat{f} \). Define \( V_n = \{\sigma \ | \ \sigma \in 2^{\leq \hat{f}(n)} \land \sigma \in \hat{U}_n\} \) for every \( n \). Then \( P = \bigcap_n V_n \) is a \( \Delta^1_1 \) closed set and \( x \in P \). So \( x \) is not \( \Delta^1_1 \)-Kurtz random, a contradiction.

Since is \( \Pi^1_1 \)-Martin-Löf random and \( \omega^x_1 = \omega^CK \), \( x \) is already \( \Pi^1_1 \)-random (see [1]).  

Next we proceed to traceability.

Definition 4.3. (i) Let \( h : \omega \to \omega \) be a nondecreasing unbounded function that is hyperarithmetical. A \( \Delta^1_1 \) trace with bound \( h \) is a uniformly \( \Delta^1_1 \) sequence \((T_e)_{e \in \omega}\) such that \( |T_e| \leq h(e) \) for each \( e \).

(ii) \( x \in 2^\omega \) is a \( \Delta^1_1 \)-traceable [1] if there is \( h \in \Delta^1_1 \) such that, for each \( f \leq h \), there is a \( \Delta^1_1 \) trace with bound \( h \) such that, for each \( e \), \( f(e) \in T_e \).

(iii) \( x \in 2^\omega \) is a \( \Delta^1_1 \)-semi-traceable if for each \( f \leq h \), there is a \( \Delta^1_1 \) function \( g \) so that, for infinitely many \( n \), \( f(n) = g(n) \). We say that \( g \) semi-traces \( f \).

(iv) \( x \in 2^\omega \) is a \( \Pi^1_1 \)-semi-traceable if for each \( f \leq h \), there is a partial \( \Pi^1_1 \) function \( p \) so that, for infinitely many \( n \) we have \( f(n) = p(n) \).
Note that, if \((T_e)_{e \in \omega}\) is a uniformly \(\Delta^1_1\) sequence of finite sets, then there is \(g \in \Delta^1_1\) such that for each \(e\), \(D_{g(e)} = T_e\) (where \(D_n\) is the \(n\)th finite set according to some recursive ordering). Thus

\[
g(e) = \mu n \forall u \in D_n \Leftrightarrow u \in T_e.
\]

In this formulation, the definition of \(\Delta^1_1\) traceability is very close to that of recursive traceability.

Also notice that the choice of a bound as a witness for traceability is immaterial:

**Proposition 4.4** (As in Terwijn and Zambella [18]). Let \(A\) be a real that is \(\Delta^1_1\) traceable with bound \(h\). Then \(A\) is \(\Delta^1_1\) traceable with bound \(h'\) for any monotone and unbounded \(\Delta^1_1\) function \(h'\).

**Lemma 4.5.** \(x\) is \(\Pi^1_1\)-semi-traceable if and only if \(x\) is \(\Delta^1_1\)-semi-traceable.

**Proof.** It is not difficult to see that if \(x\) is \(\Pi^1_1\)-semi-traceable, then \(\omega^x_1 = \omega^\text{CK}_{\Pi^1_1}\). For otherwise, \(x \geq^* \Omega\). So it suffices to show that \(\Omega\) is not \(\Pi^1_1\)-semi-traceable. Let \(\{\phi_i\}_{i \in \omega}\) be an effective enumeration of partial recursive functions. Define a function \(g \leq^r \Omega'\) so that \(g(i) = \sum_{j \leq i} m_j^i + 1\) where \(m_j^i\) is the least number \(k\) so that \(p_j(i, k) \in \Omega\); if there is no such \(k\), then \(m_j^i = 0\). Note that for any \(\Pi^1_1\) partial function \(p\), there must be some partial recursive function \(p_j\) so that for every pair \(n, m, p(n) = m\) if and only if \(p_j(n, m) \in \Omega\). Then by the definition of \(g\), for any \(i > j\), \(g(k) \neq p(i)\). So \(g\) cannot be traced by \(p\).

Suppose that \(x\) is \(\Pi^1_1\)-semi-traceable, \(\omega^x_1 = \omega^\text{CK}_{\Pi^1_1}\), and \(f \leq^s h\). Fix a \(\Pi^1_1\) partial function \(p\) for \(f\). Since \(p\) is a \(\Pi^1_1\) function, there must be some recursive injection \(h\) so that \(p(n) = m \Leftrightarrow h(n, m) \in \Omega\).

Let \(R(n, m)\) be a \(\Pi^1_1(x)\) relation so that \(R(n, m)\) iff there exists \(m > k \geq n\) for which \(f(k) = p(k)\). Then some total function \(g\) uniformizes \(R\) such that \(g \in \Pi^1_1(x)\), and so \(\Delta^1_1(x)\). Thus, for every \(n\), there is some \(m \in \{g(n), g(g(n))\} \) so that \(f(m) = p(m)\). Let \(g'(0) = g(0)\), and \(g'(n + 1) = g(g'(n))\) for all \(n \in \omega\). Define a \(\Pi^1_1(x)\) relation \(S(n, m)\) so that \(S(n, m)\) if and only if \(m \in \{g'(n), g'(n + 1)\}\) and \(p(m) = f(m)\). Uniformizing \(S\) we obtain a \(\Delta^1_1(x)\) function \(g''\).

Define a \(\Delta^1_1(x)\) set by \(H = \{h(m, k) \mid \exists n(g''(n) = m \land f(m) = k)\}\). Since \(\omega^x_1 = \omega^\text{CK}_{\Pi^1_1}\), \(H \subseteq \Omega_n\) for some \(n \in \omega\). Since \(\Omega_n\) is a \(\Delta^1_1\) set, we can define a \(\Delta^1_1\) function \(\hat{f}\) by: \(\hat{f}(i) = j\) if \(h(i, j) \in \Omega_n\); \(\hat{f}(i) = 1\), otherwise. Then there are infinitely many \(i\) so that \(f(i) = \hat{f}(i)\).

Note that the \(\Delta^1_1\)-dominated reals form a measure 1 set [1] but the set of \(\Delta^1_1\)-semi-traceable reals is null. Chong, Nies and Yu [1] constructed a non-hyperarithmetic \(\Delta^1_1\)-traceable real.

**Proposition 4.6.** Every \(\Delta^1_1\)-traceable real is \(\Delta^1_1\)-dominated and \(\Delta^1_1\)-semi-traceable.

**Proof.** Obviously every \(\Delta^1_1\)-traceable real is \(\Delta^1_1\)-dominated.

Suppose we are given a \(\Delta^1_1\)-traceable real \(x\) and \(\Delta^1_1(x)\) function \(f\). Let \(g(n) = (f(2^n), f(2^n + 2), \ldots, f(2^{n+1} - 1))\) for all \(n \in \omega\). Then there is a \(\Delta^1_1\) trace \(T\) for \(g\) so that \(|T_n| \leq n\) for all \(n\).
Then for all $2^n + 1 \leq m \leq 2^{n+1}$, let $\hat{f}(m) = (m - 2^n)$-th entry of the tuple of $(m - 2^n)$-th element of $T_m$ if there exists such an $m$; otherwise, let $\hat{f}(m) = 1$. It is not difficult to see that for every $n$ there is at least one $m \in [2^n, 2^{n+1})$ so that $f(m) = \hat{f}(m)$.

From the proof above, one can see the following corollary.

**Corollary 4.7.** A real $x$ is $\Delta^1_1$-traceable if and only if for every $x$-hyperarithmetic $\hat{f}$, there is a hyperarithmetic function $f$ so that for every $n$, there is some $m \in [2^n, 2^{n+1})$ so that $f(m) = \hat{f}(m)$.

The following proposition will be used in Theorem 4.13 to disprove the converse of Proposition 4.6.

**Proposition 4.8.** For any real $x$, the following are equivalent.

1. $x$ is $\Delta^1_1$-semi-traceable and $\Delta^1_1$-dominated.

2. For every function $\hat{g} \leq_h x$, there exist an increasing $\Delta^1_1$ function $f$ and a $\Delta^1_1$ function $F : \omega \to [\omega]^{<\omega}$ with $|F(n)| \leq n$ so that for every $n$, there exists some $m \in [f(n), f(n + 1))$ with $g(m) \in F(m)$.

**Proof.** (1) $\implies$ (2): Immediate because $1 \leq n$.

(2) $\implies$ (1). Suppose we are given a function $\hat{g} \leq_h x$. Without loss of generality, $\hat{g}$ is nondecreasing. Let $f$ and $F$ be the corresponding $\Delta^1_1$ functions. Let $j(n) = \sum_{i \leq f(n + 1)} \sum_{k \in F(i)} k$ and note that $j$ is a $\Delta^1_1$ function dominating $\hat{g}$.

To show that $x$ is $\Delta^1_1$-traceable, suppose we are given a function $\hat{g} \leq_h x$. Let $h(n) = (g(2^n + 1), g(2^n + 2), \ldots, g(2^{n+1} - 1))$. Then by assumption there are corresponding $\Delta^1_1$ functions $f_h$ and $F_h$. For every $n$ and $m \in [2^n, 2^{n+1})$, let $g(m) = (m - 2^n)$th column of the $(m - 2^n)$th element in $F_h(n)$ if such an $m$ exists; let $g(m) = 1$ otherwise. Then $g$ is a $\Delta^1_1$ function semi-tracing $\hat{g}$. □

To separate $\Delta^1_1$-traceability from the conjunction of $\Delta^1_1$-semi-traceability and $\Delta^1_1$-dominability, we have to modify Sacks’ perfect set forcing.

**Definition 4.9.**

1. A $\Delta^1_1$ perfect tree $T \subseteq 2^{<\omega}$ is fat at $n$ if for every $\sigma \in T$ with $|\sigma| \in [2^n, 2^{n+1})$, we have $\sigma \preceq 0 \in T$ and $\sigma \preceq 1 \in T$. Then we also say that $n$ is a fat number of $T$.

2. A $\Delta^1_1$ perfect tree $T \subseteq 2^{<\omega}$ is clumpy if there are infinitely many $n$ so that $T$ is fat at $n$.

3. Let $F = (F, \subseteq)$ be a partial order of which the domain $F$ is the collection of clumpy trees, ordered by inclusion.

Let $\varphi$ be a sentence of $\mathcal{L}(\omega_1^{CK}, x)$. Then we can define the forcing relation, $T \Vdash \varphi$, as done by Sacks in Section 4, IV [15].

1. $\varphi$ is ranked and $\forall x \in T(\mathfrak{A}(\omega_1^{CK}, x) \models \varphi)$, then $T \Vdash \varphi$.

2. If $\varphi(y)$ is unranked and $T \Vdash \varphi(\psi(n))$ for some $\psi(n)$ of rank at most $\alpha$, then $T \Vdash \exists y^\alpha \varphi(\psi^n)$.

3. If $T \Vdash \exists y \varphi(\psi^n)$, then $T \Vdash \exists y \varphi(y)$.

4. If $\varphi(n)$ is unranked and $T \Vdash \varphi(m)$ for some number $m$, then $T \Vdash \exists n \varphi(n)$. 

(5) If \( \varphi \) and \( \psi \) are unranked, \( T \models \varphi \) and \( T \models \psi \), then \( T \models \varphi \land \psi \).

(6) If \( \varphi \) is unranked and \( \forall P(P \subseteq T \implies P \not\models \varphi) \), then \( T \not\models \varphi \).

The following lemma can be deduced as done in [15].

**Lemma 4.10.** The relation \( T \models \varphi \), restricted to \( \Sigma^1_1 \) formulas \( \varphi \), is \( \Pi^1_1 \).

**Lemma 4.11.** (1) Let \( \{ \varphi_i \}_{i \in \omega} \) be a hyperarithmetic sequence of \( \Sigma^1_1 \) sentences.

Suppose for every \( i \) and \( Q \subseteq T \), there exists some \( R \subseteq Q \) so that \( R \models \varphi_i \).

Then there exists some \( Q \subseteq T \) so that for every \( i \), \( Q \models \varphi_i \).

(2) \( \forall \varphi \forall T \exists Q \subseteq T( Q \models \varphi \lor Q \not\models \neg \varphi ) \).

**Proof.** Using the notation \( P \upharpoonright n = \{ \tau \in 2^{\leq n} \mid \tau \in P \} \), define \( R \) by

\[ R(R, i, \sigma, P) \iff (\sigma \in R, P \subseteq R, P \models \varphi_i, P \upharpoonright \sigma = \{ \tau \mid \tau < \sigma \}, \text{ and } \log |\sigma| = i \text{ is the } i^{th} \text{ fat number of } R) \]

Note that \( R \) is a \( \Pi^1_1 \) relation. Then \( R \) can be uniformized by a partial \( \Pi^1_1 \) function \( F : F \times \omega \times 2^{<\omega} \to F \). Using \( F \), a hyperarithmetic family \( \{ P_\sigma \mid \sigma \in 2^{<\omega} \} \) can be defined by recursion on \( \sigma \).

\[ P_0 = T \]

If \( \log |\sigma| = 1 \) is not a fat number of \( P_\sigma \), then \( P_{\sigma-0}, P_{\sigma-1} = P_\sigma \).

Otherwise: If \( \sigma \not\in P_\sigma \), then \( P_{\sigma-0} = P_{\sigma-1} = \emptyset \).

Otherwise: \( P_{\sigma-0} \cap P_{\sigma-1} = \emptyset, P_{\sigma-0} \cup P_{\sigma-1} \subseteq P_\sigma \),

\[ P_{\sigma-0} \upharpoonright \sigma, P_{\sigma-1} \upharpoonright \sigma = \{ \tau \mid \tau < \sigma \} \text{ and } \]

\[ P_{\sigma-0}, P_{\sigma-1} \models \land_{j \leq i}(\varphi_j) \text{ where } \]

\( i \) is the number so that \( \log |\sigma| = i \) is the \( i \)-th fat number of \( T \).

Let \( Q = \bigcap_i \bigcup_{|\sigma|=n} P_\sigma \). Then \( Q \in F \). It is routine to check that for every \( i \), \( Q \models \varphi_i \).

The proof of (2) is the same as the proof of Lemma 4.4 IV [15].

We say that a real \( x \) is generic if it is the union of roots of trees in a generic filter; equivalently, for each \( \Sigma^1_1 \) sentence \( \varphi \), there is a condition \( T \) such that \( x \in T \) and either \( T \models \varphi \) or \( T \models \neg \varphi \). One can check (Lemma 4.8, IV [15]) that for every \( \Sigma^1_1 \)-sentence \( \varphi \),

\[ \mathfrak{A}(\omega^C_1, x) \models \varphi \iff \exists P(x \in P \land P \models \varphi) \]

**Lemma 4.12.** If \( x \) is a generic real, then

(1) \( \mathfrak{A}(\omega^C_1, x) \) satisfies \( \Delta^1_1 \)-comprehension. So \( \omega^*_1 = \omega^C_1 \).

(2) \( x \) is \( \Delta^1_1 \)-dominated and \( \Delta^1_1 \)-semi-traceable.

(3) \( x \) is not \( \Delta^1_1 \)-traceable.

**Proof.** (1) The proof of (1) is exactly same as the proof of Theorem 5.4 IV, [15].

(2) By Proposition 4.8, it suffices to show that for every function \( g \leq h \) \( x \), there are an increasing \( \Delta^1_1 \) function \( f \) and a \( \Delta^1_1 \) function \( F : \omega \to \omega^{<\omega} \) with \( |F(n)| \leq n \) so that for every \( n \), there exists some \( m \in [f(n), f(n+1)] \) so that \( g(m) \in F(m) \). Since \( g \leq h \) \( x \) and \( \omega^*_1 = \omega^C_1 \), there is a ranked formula \( \varphi \) so that for every \( n \), \( g(n) = m \) if and only if \( \mathfrak{A}(\omega^C_1, x) \models \varphi(n, m) \). So there is a condition \( S \models \forall n \exists m \varphi(n, m) \). Fix a condition \( T \subseteq S \). As in the proof of Lemma 4.11, we can build a hyperarithmetic sequence of conditions \( \{ P_\sigma \}_{\sigma \in 2^{<\omega}} \) so that

\[ P_{\sigma-0} \models \varphi(\sigma, m_{\sigma-0}) \text{ for } i \leq 1 \]
if $\log |\sigma| - 1$ is a fat number of $P_\sigma$ and $\sigma \in P_\sigma$. Let $Q$ be as defined in the proof of Lemma 4.11. Let $f$ be the $\Delta^1_1$ function such that $f(0) = 0$, and $f(n + 1)$ is the least number $k > f(n)$ so that $m_\sigma$ is defined for some $\sigma$ with $f(n) < |\sigma| < k$. Let $F(n) = \{0\} \cup \{m_\sigma \mid |\sigma| = n\}$, and note that $F$ is a $\Delta^1_1$ function. Then

$$Q \Vdash \forall n |F(n)| \leq n \land \forall n \exists m \in [f(n), f(n + 1)) \exists i \in F(m)(\varphi(m, i)).$$

So

$$Q \Vdash \exists F \exists f(\forall n |F(n)| \leq n \land \forall n \exists m \in [f(n), f(n + 1)) \exists i \in F(m)(\varphi(m, i))).$$

Since $T$ is an arbitrary condition stronger than $S$, this means

$$S \Vdash \exists F \exists f(\forall n |F(n)| \leq n \land \forall n \exists m \in [f(n), f(n + 1)) \exists i \in F(m)(\varphi(m, i))).$$

Since $x \in S$,

$$\mathcal{A}(\omega^CK, x) \models \exists F \exists f(\forall n |F(n)| \leq n \land \forall n \exists m \in [f(n), f(n + 1)) \exists i \in F(m)(\varphi(m, i))).$$

So $x$ is $\Delta^1_1$-dominated and $\Delta^1_1$-semi-traceable.

(3). Suppose $f : \omega \to \omega$ is a $\Delta^1_1$ function so that for every $n$, there is a number $m \in [2^n, 2^{n+1})$ with $f(m) = x(m)$. Then there is a ranked formula $\varphi$ so that $f(n) = m \iff \mathcal{A}(\omega^CK, x) \models \varphi(n, m)$. Moreover, $\mathcal{A}(\omega^CK, x) \models \forall n \exists m \in [2^n, 2^{n+1}) (\varphi(m, x(m)))$. So there is a condition $T \Vdash \forall n \exists m \in [2^n, 2^{n+1}) (\varphi(m, x(m)))$ and $x \in T$. Let $n$ be a number so that $T$ is fat at $n$ and $\sigma \in 2^{2^n-1}$ be a finite string in $T$. Let $\mu$ be a finite string so that $\mu(m) = 1 - f(m + 2^n - 1)$. Define $S = \{\sigma \cup \mu \cdot \tau \mid \sigma \cup \mu \cdot \tau \in T\} \subset T$. Then $S \Vdash \forall m \in [2^n, 2^{n+1}) (\neg \varphi(m, x(m)))$. But $S$ is stronger than $T$, a contradiction. By Corollary 4.7, $x$ is not $\Delta^1_1$-traceable.

We may now separate $\Delta^1_1$-traceability from the conjunction of $\Delta^1_1$-semi-traceability and $\Delta^1_1$-dominability.

**Theorem 4.13.** There are $2^{2^{\omega_1}}$ many $\Delta^1_1$-dominated and $\Delta^1_1$-semi-traceable reals which are not $\Delta^1_1$-traceable.

**Proof.** This is immediate from Lemma 4.12. Note that there are $2^{2^{\omega_1}}$ many generic reals. □

5. Lowness for higher Kurtz randomness

Given a relativizable class of reals $\mathcal{C}$ (for instance, the class of random reals), we call a real $x$ low for $\mathcal{C}$ if $\mathcal{C} = \mathcal{C}^x$. We shall prove that lowness for $\Delta^1_1$-randomness is different from lowness for $\Delta^1_1$-Kurtz randomness. A real $x$ is low for $\Delta^1_1$-Kurtz tests if every $\Delta^1_1(x)$ open set with measure 1 has a $\Delta^1_1$ open subset of measure 1. Clearly, lowness for $\Delta^1_1$-Kurtz tests implies lowness for $\Delta^1_1$-Kurtz randomness.

**Theorem 5.1.** If $x$ is $\Delta^1_1$-dominated and $\Delta^1_1$-semi-traceable, then $x$ is low for $\Delta^1_1$-Kurtz tests.
Proof. Suppose $x$ is $\Delta^1_1$-dominated and $\Delta^1_1$-semi-traceable and $U$ is a $\Delta^1_1(x)$ open set with measure $1$. Then there is a real $y \leq_h x$ so that $U$ is $\Sigma^0_1(y)$. Hence for some Turing reduction $\Phi$, if for all $z$ we write $U^z$ for the domain of $\Phi^z$, then we have $U = U^y$.

Define a $\Delta^1_1(x)$ function $\hat{f}$ by: $\hat{f}(n)$ is the shortest string $\sigma < y$ so that $\mu(U^\sigma[\sigma]) > 1 - 2^{-n}$. By the assumptions of the Theorem, there are an increasing $\Delta^1_1$ function $g$ and a $\Delta^1_1$ function $f$ so that for every $n$, there is an $m \in [g(n), g(n + 1) \cup 1)$ so that $f(m) = f(m)$. Without loss of generality, we can assume that $\mu(U^{f(m)[m]}) > 1 - 2^{-m}$ for every $m$.

Define a $\Delta^1_1$ open set $V$ so that $\sigma \in V$ if and only if there exists some $n$ so that $[\sigma] \subseteq \bigcap_{g(n) \leq m < g(n + 1)} U^{f(m)[m]}$. By the property of $f$ and $g$, $V \subseteq U^y = U$. But for every $n$,

$$\mu(\bigcap_{g(n) \leq m < g(n + 1)} U^{f(m)[m]}) > 1 - \sum_{g(n) \leq m < g(n + 1)} 2^{-m} \geq 1 - 2^{-g(n)+1}.$$ 

So

$$\mu(V) \geq \lim_{n} \mu \left( \bigcap_{g(n) \leq m < g(n + 1)} U^{f(m)[m]} \right) = 1. $$

Hence $x$ is low for $\Delta^1_1$-Kurtz tests.

Corollary 5.2. Lowness for $\Delta^1_1$-randomness differs from lowness for $\Delta^1_1$-Kurtz randomness.

Proof. By Theorem 4.13, there is a real $x$ that is $\Delta^1_1$-dominated and $\Delta^1_1$-semi-traceable but not $\Delta^1_1$-traceable. By Theorem 5.1, $x$ is low for $\Delta^1_1$-Kurtz randomness. Chong, Nies and Yu [1] proved that lowness for $\Delta^1_1$-randomness is the same as $\Delta^1_1$-traceability. Thus $x$ is not low for $\Delta^1_1$-randomness.

Corollary 5.3. There is a non-zero hyperdegree below $O$ which is not a base for a cone of $\Delta^1_1$-Kurtz randoms.

Proof. Clearly there is a real $x <_h O$ which is $\Delta^1_1$-dominated and $\Delta^1_1$-semi-traceable. Then the hyperdegree of $x$ is not a base for a cone of $\Delta^1_1$-Kurtz randoms.

Actually the converse of Theorem 5.1 is also true.

Lemma 5.4. If $x$ is low for $\Delta^1_1$-Kurtz randomness, then $x$ is $\Delta^1_1$-dominated.

Proof. Firstly we show that if $x$ is low for $\Delta^1_1$-Kurtz tests, then $x$ is $\Delta^1_1$-dominated.

Suppose $f \leq_h x$ is an increasing function. Let $S_f = \{ z \mid \forall n(z(f(n)) = 0) \}$. Obviously $S_f$ is a $\Delta^1_1(x)$ closed null set. So there is a $\Delta^1_1$ closed null set $[T] \supseteq S_f$ where $T \subseteq 2^\omega$ is a $\Delta^1_1$ tree. Define

$$g(n) = \min\{ m \mid |\{ \sigma \in 2^m \mid \sigma \in T\}| < 2^{-n} \} + 1.$$ 

Since $\mu([T]) = 0$, $g$ is a well defined $\Delta^1_1$ function. We claim that $g$ dominates $f$. 


For every \( n \), \( S_{f(n)} = \{ \sigma \in 2^{f(n)} | \forall i \leq n (\sigma(f(i)) = 0) \} \) has cardinality \( 2^{f(n)-n} \). But if \( g(n) \leq f(n) \), then since \( S \subseteq [T] \), we have
\[
|S_{f(n)}| \leq 2^{f(n)-g(n)} \cdot |\{ \sigma \in 2^{g(n)} | \sigma \in T \}| < 2^{f(n)-g(n)} \cdot 2^{g(n)-n} = 2^{f(n)-n}.
\]
This is a contradiction. So \( x \) is \( \Delta^1_1 \)-dominated.

Now suppose \( x \) is not \( \Delta^1_1 \)-dominated witnessed by some \( f \leq_h x \). Then \( S_f \) is not contained in any \( \Delta^1_1 \) closed null set. Actually, it is not difficult to see that for any \( \sigma \) with \( [\sigma] \cap S_f \neq \emptyset \), \([\sigma] \cap S_f \) is not contained in any \( \Delta^1_1 \) closed null set (otherwise, one can show that \( f \) is dominated by some \( \Delta^1_1 \) function). Then, by induction, we can construct a \( \Delta^1_1 \)-Kurtz random real \( z \in S_f \) as follows:

Fix an enumeration \( P_0, P_1, \ldots \) of the \( \Delta^1_1 \) closed null sets.

At stage \( n + 1 \), we have constructed some \( z \upharpoonright l_n \) so that \( [z] \upharpoonright l_n \cap S_f \neq \emptyset \). Then there is a \( \tau > z \upharpoonright l_n \) so that \( [\tau] \cap S_f \neq \emptyset \) but \( [\tau] \cap S_f \cap P_n = \emptyset \). Fix such a \( \tau \), let \( l_{n+1} = |\tau| \) and \( z \upharpoonright l_{n+1} = \tau \).

Then \( z \in S_f \) is \( \Delta^1_1 \)-Kurtz random.

So \( x \) is not low for \( \Delta^1_1 \)-Kurtz randomness.

\( \square \)

**Lemma 5.5.** If \( x \) is low for \( \Delta^1_1 \)-Kurtz randomness, then \( x \) is \( \Delta^1_1 \)-semi-traceable.

**Proof.** The proof is analogous to that of the main result in [7].

Firstly we show that if \( x \) is low for \( \Delta^1_1 \)-Kurtz tests, then \( x \) is \( \Delta^1_1 \)-semi-traceable.

Suppose that \( x \) is low for \( \Delta^1_1 \)-Kurtz tests and \( f \leq_h x \). Partition \( \omega \) into finite intervals \( D_{m,k} \) for \( 0 < k < m \) so that \( |D_{m,k}| = 2^{m-k-1} \). Moreover, if \( m < m' \), then \( \max D_{m,k} < \min D_{m',k'} \) for any \( k < m \) and \( k' < m' \). Let \( n_m = \max\{i | i \in D_{m,k} \land k < m\} \) for every \( m \in \omega \). Note that \( \{n_m\}_{m \in \omega} \) is a recursive increasing sequence.

For every function \( h \), let
\[
P^h = \{ x \in 2^\omega | \forall m (x(h \upharpoonright n_m) = 0) \}
\]
be a closed null set. Obviously \( P^f \) is a \( \Delta^1_1(x) \) closed null set. Then there is a \( \Delta^1_1 \) closed null set \( Q \supseteq P^f \). We define a \( \Delta^1_1 \) function \( g \) as follows.

For each \( k \in \omega \), let \( d_k \) be the least number \( d \) so that
\[
|\{ \sigma \in 2^d | \exists x \in Q(x \succ \sigma) \}| \leq 2^{d-k-1}.
\]
Note that \( \{d_k\}_{k \in \omega} \) is a \( \Delta^1_1 \) sequence. Define
\[
Q_k = \{ \sigma | \sigma \in 2^{d_k} \land \exists x \in Q(x \succ \sigma) \}.
\]
Then \( \{Q_k\}_{k \in \omega} \) is a \( \Delta^1_1 \) sequence of clopen sets and \( |Q_k| \leq 2^{d_k-k-1} \) for each \( k < d_k \).

Then Greenberg and Miller [7] constructed a finite tree \( S \subseteq \omega^{<\omega} \) and a finite sequence \( \{S_m\}_{k \leq m \leq l} \) for some \( l \) with the following properties:

1. \( [S] = \{ h \in \omega^\omega | P^h \subseteq [Q_k] \} \);
2. \( S_m \subseteq S \cap \omega^{n_m} \);
3. \( |S_m| \leq 2^{m-k-1} \);
4. every leaf of \( S \) extends some string in \( \bigcup_{k < m \leq l} S_m \).

Moreover, both the finite tree $S$ and sequence $\{S_m\}_{k \leq m \leq l}$ can be obtained uniformly from $Q_k$.

Now for each $m$ with $k < m \leq l$ and $\sigma \in S_m$, we pick a distinct $i \in D_{m,k}$ and define $g(i) = \sigma(i)$. For the other undefined $i \in D_{m,k}$, let $g(i) = 0$.

So $g$ is a well-defined $\Delta_1^1$ function.

For each $k$, $P^f \subseteq Q \subseteq [Q_k]$. So $f \in [S]$. Hence there must be some $i > n_k$ so that $f(i) = g(i)$.

Thus $x$ is $\Delta_1^1$-semi-traceable.

Now suppose $x$ is not $\Delta_1^1$-semi-traceable as witnessed by $f \leq_h x$. Then $P^f$ is not contained in any $\Delta_1^1$ closed null set. It is shown in [7] that for any $\sigma$, assuming that $[\sigma] \cap P^f \neq \emptyset$, $[\sigma] \cap P^f$ is not contained in any $\Delta_1^1$ closed null set. Then by an easy induction, one can construct a $\Delta_1^1$-Kurtz random real in $P^f$.

So $x$ is not low for $\Delta_1^1$-Kurtz randomness.

So we have the following theorem.

**Theorem 5.6.** For any real $x \in 2^\omega$, the following are equivalent:

1. $x$ is low for $\Delta_1^1$-Kurtz tests;
2. $x$ is low for $\Delta_1^1$-Kurtz randomness;
3. $x$ is $\Delta_1^1$-dominated and $\Delta_1^1$-semi-traceable.

It is unknown whether there exists a nonhyperarithmetic real which is low for $\Pi_1^1$-Kurtz randomness. However, we can prove the following containment.

**Proposition 5.7.** If $x$ is low for $\Pi_1^1$-Kurtz randomness, then $x$ is low for $\Delta_1^1$-Kurtz randomness.

**Proof.** Assume that $x$ is low for $\Pi_1^1$-Kurtz randomness, $y$ is $\Delta_1^1$-Kurtz random and there is a $\Delta_1^1(x)$ closed null set $A$ with $y \in A$. By Theorem 2.7, the set

$$B = \bigcup \{C \mid C \text{ is a } \Delta_1^1 \text{ closed null set}\}$$

is a $\Pi_1^1$ null set. So $A - B$ is a $\Sigma_1^1(x)$ set. Since $y$ is $\Delta_1^1$-Kurtz random, $y \notin B$. Hence $y \in A - B$ and so $A - B$ is a $\Sigma_1^1(x)$ nonempty set. Thus there must be some real $z \in A - B$ with $\omega^z_1 = \omega_1^y = \omega_1^{CK}$. Since $z \notin B$, $z$ is $\Delta_1^1$-Kurtz random. So by Proposition 3.3, $z$ is $\Pi_1^1$-Kurtz random. This contradicts the fact that $x$ is low for $\Pi_1^1$-Kurtz randomness.

**References**


