Σ_1 INDISCERNIBLES IN C.E. DEGREES

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ABSTRACT. We study indiscernibles in the upper semi-lattice of computably enumerable Turing degrees.

1. INTRODUCTION

In model theory and set theory, indiscernibles play important roles in finding elementary embeddings and substructures (see Marker [2]). However, for the upper semi-lattice of computably enumerable degrees $\mathcal{R} = (\mathbf{R}, \leq)$, little was known. For example, the question below is open.

Question 1.1. *Does there exist an infinite class of indiscernibles in $\mathcal{R}$?*

By indiscernibles in $\mathcal{R}$ we just consider the language of partial orders, $\mathcal{L} = \{\leq\}$.

In this paper, we are not so ambitious to seek an answer of the question above, but just to study a strongly restricted variance of the notion indiscernibles. (It remains standing for long time whether every c.e. degrees is definable in $\mathcal{R}$. This may be a good excuse of our being 'humble'.)

We call $\mathbf{I} \subseteq \mathbf{R}$ a class of *n(-order)-indiscernibles in $\mathcal{R}$* iff for any Σ_n formula $\varphi(x_1, \dots, x_k)$ of $\mathcal{L}$ with all free variables displayed and any ($\leq_{\mathrm{T}}$-ascending) k-tuples $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{b}}$ of $\mathbf{I}$,

$$\mathcal{R} \models \varphi[\overrightarrow{\mathbf{a}}] \Leftrightarrow \mathcal{R} \models \varphi[\overrightarrow{\mathbf{b}}].$$

It is obvious that every class of n-indiscernibles form an antichain, and every class of n-order-indiscernibles is either a chain or an antichain of n-indiscernibles.

We also need some notions of partial orders. Given a partial order $\langle P, \leq_P \rangle$, if P has a least element, denote it by 0_P. From now on we always *assume that given partial orders are finite and have a least element* unless otherwise indicated. An element of P is an *atom* iff it is not 0_P but bounds nothing other than 0_P. Let $\mathcal{A}(P)$ denote the set of atoms of P, and $\mathcal{A}(x)$ denote the atoms bounded by $x \in P$. We say *$x, y \in P$ form a minimal pair* iff $\mathcal{A}(x) \cap \mathcal{A}(y) = \emptyset$, and call a subset $\overrightarrow{x} \subseteq P$ an *antichain* iff its elements are pairwise $\leq_P$-incomparable.

2. THE EXISTENCE OF 1-INDISCERNIBLES

Lempp observed that the following is directly implied by Theorem 2.2.

Proposition 2.1 (Lempp). *Every chain in $\mathcal{R}$ not containing $\mathbf{0}$ and $\mathbf{0}'$ is an infinite class of order 1-indiscernibles.*

Let Q be partial orders with 0_Q and 1_Q, P a sub partial order of Q, and $S \subset Q$. Define

(1) $U(S) = \{x \in P : \forall y \in S(x \geq y)\}$, and $U(y) = U(\{y\})$;
(2) $V(S) = \{x \in P : \forall y \in S(x \leq y)\}$, and $V(y) = V(\{y\})$;

(3) $Z(y) = \{z \in Q \backslash P : z < y \text{ and } V(y) \not\subseteq V(U(z))\}$.

For notational simplicity, we write $VU(S)$ for $V(U(S))$ and etc.

Theorem 2.2 (Slaman and Soare [3]). *Every embedding of P into $\mathcal{R}$ can be extended to an embedding of Q into $\mathcal{R}$ iff both conditions below* fail

(i) $\exists x, y \in Q(x \not\geq y$ *and* $VU(y) \subseteq VUV(x))$;
(ii) $\exists x \in Q \backslash P(Z(x) \neq \emptyset$ *and* $VU(Z(x) \cup V(x)) \not\subseteq V(x))$.

Now fix $f : P \to \mathcal{R}$ an embedding of a sub partial order of Q onto a chain. Then P is a chain in Q, let $x_0 < x_1 < \ldots < x_n$ enumerate P. We may in addition assume that $x_0 = 0_Q$ and $x_n = 1_Q$.

It follows from easy calculations that for $x, y \in Q$

(1) $VUV(x) = V(x) = \{x_i : x_i \leq v_x\}$ where v_x is the greatest x_j such that $x_j \leq x$;
(2) $VU(y) = \{x_i : x_i \leq u_y\}$ where u_y is the least x_j such that $x_j \geq y$.

If $x \not\geq y$ then $u_y \notin VUV(x)$. Thus Theorem 2.2(i) fails.

If $Z(x) \neq \emptyset$, then $v_x > u_z$ since $V(x) \not\subseteq VU(z)$ for any $z \in Z(x)$. Thus $VU(Z(x) \cup V(x)) = VUV(x) = V(x)$ and Theorem 2.2(ii) fails too.

By Theorem 2.2, there exists an embedding extending f. Hence any embedding of a finite chain is extendable to an embedding of an arbitrary finite partial order respecting the chain.

To determine Σ_1 types realized by chains, we need some notions.

Given $\psi(\overrightarrow{x})$ a conjunction of atomic formulas (in $\mathcal{L}$), we call it a *diagram* iff there is a partial order $P \models \psi[\overrightarrow{x}]$ (we identify the elements of P with the variables of ψ). A diagram ψ is a *full diagram* iff for any different variables x and y of ψ, $x \neq y \in \psi$ (we identify a diagram with a finite set of atomic formulas) and either $x \leq y \in \psi$ or $x \not\leq y \in \psi$.

It then suffices to consider $\varphi \in \Sigma_1$ of the form $\exists \overrightarrow{y} \psi(\overrightarrow{x}, \overrightarrow{y})$ where ψ is a full diagram. Suppose such a φ is given, let P be the unique partial order with the number of elements same as the numbers of variables of ψ and modeling ψ. The question whether a finite tuple $\overrightarrow{\mathbf{c}}$ realizes φ can be reformulated as: *given an embedding $f : \langle \overrightarrow{x}, \leq_P \rangle \to \langle \overrightarrow{\mathbf{c}}, \leq \rangle$, is there an extension $g : P \to \mathcal{R}$ of f?*

In the case where $\overrightarrow{\mathbf{c}}$ is a chain not containing $\mathbf{0}$ and $\mathbf{0}'$, such an extension always exists by the preceding arguments. Thus all finite chains not containing $\mathbf{0}$ and $\mathbf{0}'$ realize same Σ_1 types and Proposition 2.1 holds.

Now we turn to 1-indiscernibles and prove the following.

Theorem 2.3. *There exists an infinite class of 1-indiscernibles.*

An instance is constructed by the technical lemma below.

Lemma 2.4. *There exist two infinite sequences of c.e. degrees $\langle \mathbf{d}_n : n < \omega \rangle$ and $\langle \mathbf{a}_{n,k} : n, k < \omega \rangle$ such that*

(i) $\mathbf{d}_m \cap \mathbf{d}_n = 0$ *whenever* $m \neq n$;
(ii) $\mathbf{a}_{n,k} \leq \mathbf{d}_n$ *for arbitrary n and k;*
(iii) *for any finite subsets $F \subset \omega, G \subset \omega \times \omega$ and $\langle n, k \rangle \in \omega \times \omega$, if $n \notin F$ and $\langle n, k \rangle \notin G$ then $\mathbf{a}_{n,k} \not\leq \bigcup_{m \in F} \mathbf{d}_m \cup \bigcup_{\langle i,j \rangle \in G} \mathbf{a}_{i,j}$.*

Proof. We simultaneously construct two infinite sequences of c.e. sets $\langle D_n : n < \omega \rangle$ and $\langle A_{n,k} : n, k < \omega \rangle$ such that $D_n = \bigoplus_{k < \omega} A_{n,k}$,

$$\mathcal{M}_{m,n,e} : \Phi_e(D_m) = \Phi_e(D_n) \text{ is total } \Rightarrow \Phi_e(D_m) \leq_{\mathrm{T}} \text{ for all } m \neq n \text{ and } e$$

and

$$\mathcal{P}_{F,G,n,k,e} : A_{n,k} \neq \Psi_e(D_F \oplus A_G)$$

for all finite $F \subset \omega, G \subset \omega \times \omega, n \notin F$ and $\langle n, k \rangle \notin G$ (where $D_F = \bigoplus_{m \in F} D_m$ and $A_G = \bigoplus_{\langle i,j \rangle \in G} A_{i,j}$).

The strategies to make $\mathcal{M}$s are typical minimal pairs strategies.

To make a single $\mathcal{P}$, we wait for agreements between both sides of the inequality to be established, then enumerate witness into the left hand side and properly restraint the right hand side.

It is straightforward to combine the strategies.

Let $\mathbf{d}_n = \mathbf{deg}(D_n)$ and $\mathbf{a}_{n,k} = \mathbf{deg}(A_{n,k})$. $\square$

Note that (ii) and (iii) above together imply that $\langle \mathbf{d}_n : n < \omega \rangle$ is an independent sequence, and (iii) in addition implies that for F an arbitrary finite subset of ω the class of degrees $\{\mathbf{a}_{n,k} : \langle n, k \rangle \in (\omega \backslash F) \times \omega\} \cup \{\mathbf{d}_n : n \in F\}$ is also independent.

Lemma 2.5. *The sequence $\langle \mathbf{d}_n : n < \omega \rangle$ meets Theorem 2.3.*

Proof. Following the analysis for Proposition 2.1, it suffices to consider the question: given an embedding $f : \langle \overrightarrow{x}, \leq_P \rangle \to \langle \{\mathbf{d}_n : n < \omega\}, \leq \rangle$, is there an extension $g : P \to \mathcal{R}$ of f?

Two necessary conditions of positive solutions follow immediately from the assumption of $\langle \mathbf{d}_n : n < \omega \rangle$:

$$\overrightarrow{x} \text{ is an antichain, and} \tag{2.1}$$

$$\overrightarrow{x} \text{ pairwise forming minimal pairs.} \tag{2.2}$$

It turns out that these conditions together are also sufficient. From now on they are assumed for a given P.

If f maps 0_P to some d_n then the extension always exists. Assume otherwise, first of all we expand P. For each $u \in P \backslash \mathcal{A}(P)$ we add an atom z_u such that $z_u \leq_P u$ and

$$\text{if } u \not\leq_P v \in P \text{ then } z_u \not\leq_P v. \tag{2.3}$$

Denote the resulting partial order by P'. It suffices to extend f to some $g : P' \to \mathcal{R}$.

0_P. Let $g(0_P) = 0$.

Atoms. For $z \in \mathcal{A}(P') \backslash \overrightarrow{x}$, if it is bounded by some x, let $g(z) = \mathbf{a}_{n,k}$ where $\mathbf{d}_n = f(x)$ and k is arbitrary; otherwise let $g(z) = \mathbf{a}_{n,k}$ where $\mathbf{d}_n \notin ran(f)$ and k is arbitrary. Moreover we make $g \restriction \mathcal{A}(P')$ injective.

Non-atoms. For $w \in P' \backslash (\mathcal{A}(P') \cup \{0_P\})$, let $g(w)$ be the joint of

$$\bigcup \{\mathbf{d}_n : f^{-1}(\mathbf{d}_n) \leq_P w\}$$

and

$$\bigcup \{g(z) : z \in \mathcal{A}(w) \text{ and there is no } x \text{ such that } z \leq_P x \leq_P w\}.$$

g is obviously well defined. To verify that g extends f, if $w \in \overrightarrow{x}$ in the last case of the definition of g, then

$$\bigcup \{\mathbf{a}_n : f^{-1}(\mathbf{a}_n) \leq_P w\} = f(w)$$

and

$$\{g(z) : z \in \mathcal{A}(w) \text{ and there is no } x \text{ such that } z \leq_P x \leq_P w\} = \emptyset.$$

To verify that g is an embedding, the definition of non-atoms case guarantees that $g(u) \leq g(v)$ if $u \leq_P v$. Furthermore, if $u \not\leq_P v$, then by (2.3), $z_u \not\leq_P v$ and $z_u \not\leq_P x$ for $x \leq_P v$. Hence by Lemma 2.4(iii), $g(z_u) \not\leq g(v)$ and thus $g(u) \not\leq g(v)$.

Now we have proved that (2.1) and (2.2) are sufficient for existence of extensions. The lemma follows easily from the above this characterization. □

Note that Theorem 2.3 is not an implication of Theorem 2.2. We could map 1_P to arbitrary c.e. degree, e.g. $\mathbf{0}'$, to accommodate extensions ruled out by Theorem 2.2(i), because the original embedding f says nothing about 1_P.

3. Dividing Σ_1 and Σ_2 Formulas

The success of finding 1-indiscernibles may root in the weaknesses of Σ_1 formulas. For examples, we have the facts below.

Proposition 3.1. None *of* $\mathbf{0}, \mathbf{0}'$, *the join and meet operations is* Σ_1 *definable.*

Proof. Immediately. □

These facts also illustrate differences between the power of Σ_1 and Σ_2 formulas since the notions mentioned are all Σ_2 definable. Moreover, we prove a more significant difference between these two classes of formulas.

Theorem 3.2. *Every nonprincipal ideal is a* Σ_1 *elementary substructure of* $\mathcal{R}$.

Proof. Fix $\mathbf{I}$ a nonprincipal ideal of $\mathcal{R}$. Following the analysis in the previous section, it suffices to prove that for any finite partial order P and an embedding $f : P \to \mathcal{R}$, there exists an embedding $g : P \to \mathbf{I}$ with g^{-1} extending $f^{-1} \upharpoonright \mathbf{I}$.

Let $x_1, \ldots, x_n$ be an enumeration of $P^- = f^{-1}(\mathbf{I})$, $\mathbf{a}_i = f(x_i)$ for $i(0 < i \leq n)$ and $\mathbf{a}_0 = \mathbf{0}$. To extend f, define $g(x_i) = f(x_i)$ for $i \leq n$.

Since $\mathbf{I}$ is nonprincipal we can find some $\mathbf{c} \in \mathbf{I}$ such that $\mathbf{c} > \bigcup_{i\leq n} \mathbf{a}_i$. We need a technical lemma.

Lemma 3.3. *There exist an independent sequence of degrees in* $[\mathbf{0}, \mathbf{c}]$, *say* $\langle \mathbf{b}_{i,k} : i \leq n, k \in \omega\rangle$, *such that* $\mathbf{a}_i \leq \mathbf{b}_{i,k}$, *and for any finite* $H \subset \{0, 1, \ldots, n\} \times \omega$,

$$\mathbf{a}_i \not\leq \bigcup_{j\in H_0} \mathbf{a}_j \Rightarrow \mathbf{a}_i \not\leq \bigcup_{\langle j,k\rangle\in H} \mathbf{b}_{j,k} \tag{3.1}$$

where $H_0 = \{j : \exists k(\langle j, k\rangle \in H)\}$.

Proof. This lemma is essentially a generalization of Sacks Density Theorem.

Assume $A_i \subseteq \omega^{[2i]}$ is a c.e. representative of $\mathbf{a}_i$ for $i \leq n$ and C is a c.e. representative of $\mathbf{c}$. Let $A = \bigcup_{i\leq n} A_i$. We construct pairwise disjoin c.e. sets $\langle B_{i,k} : i \leq n, k \in \omega\rangle$ such that $B_{i,k} \leq_{\mathrm{T}} C$. Let $B = \bigcup_{i\leq n,k\in\omega} B_{i,k}$, we make $A \cap B = \emptyset$.

To make the sequence independent, for $m \in \omega$, $\langle i, k\rangle \in (n + 1) \times \omega$ and finite $H \subset (n + 1) \times \omega$, we make

$$\mathcal{P}_e : B_{i,k} = \Phi_m(\bigcup_{j\in H_0} A_j \cup \bigcup_{\langle j,l\rangle\in H} B_{j,l}) \Rightarrow C \leq_{\mathrm{T}} A$$

where e is an index of the requirement under some effective encoding of all possible combinations.

To make (3.1), for i and H satisfying the left hand side of (3.1) and m, we make

$$\mathcal{N}_e : A_i = \Psi_m(\bigcup_{j\in H_0} A_j \cup \bigcup_{\langle j,l\rangle\in H} B_{j,l}) \Rightarrow A_i \leq_{\mathrm{T}} \bigcup_{j\in H_0} A_j$$

where e is again an index under analogues settings. Note that there are only finitely many such pairs (i, H_0).

We employee the trick of *true stages computation* (see Chapter VIII of Soare [4]) where true stages are uniformly defined as true stages of the effective enumeration of $A \cup B$ arisen in the construction. But for convenience we do not take the trouble of putting hats on symbols.

The strategy for a single $\mathcal{N}_e$ is to preserve agreements between A_i and Ψ_m by imposing restraints on B. Strategies serving less prior requirements are required to respect the restraints. By the assumption of (i, H_0), the restraints imposed for $\mathcal{N}_e$ will drop at true stages.

The strategy for a single $\mathcal{P}_e$ is Sacks Coding with a slight modification. To be precise, at stage $s+1$, we enumerate $\langle x, t, 2e+1\rangle$ in $B_{i,k}$ iff $x \in C_{s+1}$, $\langle x, t, 2e+1\rangle$ is not restrained from entering B by prior requirements and $(\forall v)(t \leq v \leq s \rightarrow \langle x, t, e\rangle < l^{\Phi}(e, v))$ where l^{Φ} is the length of agreements defined as usual.

In addition, we impose a restraint on B to protect the computation $\Phi_m \upharpoonright l^{\Phi}(e, s)$. This is the only modification comparing to Sacks Coding strategy. Since $C \not\leq_{\mathrm{T}} A$, this restraint imposed for $\mathcal{P}_e$ will drop at true stages.

The verifications go essentially in the same way as those for Sacks Density Theorem.

Finally let $\mathbf{b}_{i,k} = \mathbf{deg}(A_i \cup B_{i,k})$. $\square$

Let us return to the proof of Theorem 3.2. Fix $\langle b_{i,k} : i \leq n, k \in \omega\rangle$ as in the above lemma. Let $y_1, \ldots, y_m$ enumerate $P\backslash P^-$, define

$$g(y_k) = \bigcup\{b_{i,l} : x_i <_P y_k \text{ and } y_l \leq_P y_k\}.$$

Immediately, if $u \leq_P y_k$ then $g(u) \leq g(y_k)$.

Assume $u \not\leq_P y_k$. If u is some x_i, then since f is an embedding, $a_i \not\leq \bigcup_{j\in H_0} a_j$ where $H_0 = \{j \leq n : x_j < y_k\}$. Hence (3.1) and the definition together imply that $g(u) \not\leq g(y_k)$.

If u is some y_l, then the independence of the sequence $\langle b_{i,k} : i \leq n, k \in \omega\rangle$ implies that again $g(u) \not\leq g(y_k)$.

Hence g is the desired embedding. $\square$

Although no principal ideal could be a Σ_1 elementary substructure since the predicate *x is not maximal* is Σ_1 definable in $\mathcal{R}$, the following follows immediately from the proof above.

Proposition 3.4. *For $\mathbf{c}$ c.e., $\varphi \in \Sigma_1$ and $\overrightarrow{\mathbf{a}} \in [\mathbf{0}, \mathbf{c}]$ such that $\cup\overrightarrow{\mathbf{a}} < \mathbf{c}$,*

$$\mathcal{R} \models \varphi[\overrightarrow{\mathbf{a}}] \Leftrightarrow [\mathbf{0}, \mathbf{c}] \models \varphi[\overrightarrow{\mathbf{a}}].$$

On the contrary, Σ_2 elementary substructures always contain $\mathbf{0}'$ since the fact *there exists a greatest element* is Σ_2 in $\mathcal{R}$ (plus the remark before the proposition). Hence *no* proper ideal could be a Σ_2 elementary substructure.

These differences may suggest that the following is far more complicate even for $n = 2$.

Question 3.5. *Do there exist infinite classes of n(-order)-indiscernibles in $\mathcal{R}$ for $n > 1$?*

4. Remarks and Questions

It is tempting to expect that $\mathcal{R}$ could be generated (i.e., a closure under join and meet) by some infinite class of 1-indiscernibles. However, this is inaccessible.

Proposition 4.1. *There is no infinite class of 1-indiscernibles generating all c.e. degrees.*

Proof. Suppose on the contrary there is an infinite class of 1-indiscernibles, say $\mathbf{I}$, generating all c.e. degrees. Then $\mathbf{0}$ is generated by some finite tuple $\overrightarrow{\mathbf{a}}$ of $\mathbf{I}$. It is obvious that $\mathbf{0} = \bigcap \overrightarrow{\mathbf{a}}$.

Let n be the least length of such tuples. The predicate *the meet of* $\mathbf{x_1}, \ldots, \mathbf{x_n}$ *is* $\mathbf{0}$ is Π_1 as it is the negation of the Σ_1 formula below

$$(\exists u, v)(u < v < x_1, \ldots, x_n).$$

By 1-indiscernibility, the meets of n-tuples of $\mathbf{I}$ are always $\mathbf{0}$. While by the choice of n, no element of $\mathbf{I}$ could be non-cappable. Since the cappables form an ideal of $\mathcal{R}$ (by Ambos-Spies et al. [1]), $\mathbf{I}$ could not generate any non-cappable degree. □

Note that nor could we be satisfied even if only expecting $\mathbf{I}$ generating $\mathbf{R} - \{\mathbf{0}\}$. To see this, take a minimal pair. If $\mathbf{I}$ could generate both elements of the pair, then it could also generate $\mathbf{0}$.

But an analogue of Proposition 4.1 below is open.

Question 4.2. *Does there exist a class of 1-indiscernibles, say* $\mathbf{I}$*, such that all c.e. degrees are definable with parameters from* $\mathbf{I}$*?*

Could we success if less ambitious?

Question 4.3. *Is there any class of 1-indiscernibles generating all cappables?*

Question 4.4. *Are there finitely many classes of 1-indiscernibles altogether generating all c.e. degrees?*

Of course we would like to impose restraints on relations of those classes in Question 4.4. For example, let $\mathbf{I}$ and $\mathbf{J}$ be an infinite class of 1-indiscernibles and a linear order without ends respectively. We call $(\mathbf{I}, \mathbf{J})$ *cross 1-indiscernibles* if for $\langle \overrightarrow{\mathbf{a}}, \overrightarrow{\mathbf{b}} \rangle, \langle \overrightarrow{\mathbf{c}}, \overrightarrow{\mathbf{d}} \rangle \in \mathbf{I}^m \times \mathbf{J}^n$, $\langle \overrightarrow{\mathbf{a}}, \overrightarrow{\mathbf{b}} \rangle$ and $\langle \overrightarrow{\mathbf{c}}, \overrightarrow{\mathbf{d}} \rangle$ realize the same Σ_1 type given that $(\overrightarrow{\mathbf{a}}, \overrightarrow{\mathbf{b}})$ and $(\overrightarrow{\mathbf{c}}, \overrightarrow{\mathbf{d}})$ with partial orders inherited from $\mathcal{R}$ are isomorphic.

Question 4.5. *Could $\mathcal{R}$ be generated by cross 1-indiscernibles?*

On the other hand it may be interesting to investigate into 1-indiscernibles themselves.

Question 4.6. *Characterize 1-indiscernibles.*

Lempp suggested a relating and more basic question.

Question 4.7 (Lempp)**.** *When do two finite tuple of c.e. degrees with same length realize the same Σ_1 type?*

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