SOME NOTES ON RANKED STRUCTURES

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1. Inductive definitions and Δ^1_1-boundedness

Let Γ be a map from 2^ω to 2^ω. Γ is *monotonic* if $X \subseteq Y$ implies $\Gamma(X) \subseteq \Gamma(Y)$.
Γ is *progress* if $X \subseteq \Gamma(X)$ for all X.
If Γ is progress, we define Γ_α by recursion on α.

$$\Gamma_0 = \emptyset.\ \Gamma_{\alpha+1} = \Gamma(\Gamma_\alpha) \cup \Gamma_\alpha.$$
$$\Gamma_\beta = \bigcup_{\beta<\alpha} \Gamma_\alpha \ (\beta \text{ is a limit}).$$
$$\Gamma_\infty = \bigcup_\alpha \Gamma_\alpha.$$

We use $|\Gamma|$ to denote the least ordinal α so that $\Gamma(\Gamma_\alpha) = \Gamma_\alpha$.
Some facts about inductive definitions can be found in [4].
The following fact is obvious.

Fact 1.1. *If Γ is monotonic and is progress, then $\Gamma(\Gamma_{|\gamma|}) = \bigcap_{\Gamma(X)=X} X$.*

The following theorem is essentially due to Spector (For more details, see Section 8 III [4]).

Theorem 1.2 (Spector [5]). *If Γ progress, then*
1. *if Γ is monotonic and Π^1_1, then $|\Gamma| \leq \omega_1^{\mathrm{CK}}$ and Γ_∞ is Π^1_1.*
2. *if Γ is Π^0_1, then $|\Gamma| \leq \omega_1^{\mathrm{CK}}$ and Γ_∞ is Π^1_1.*

The following proposition can be found in [2] (Corollary 2.20 IV).

Proposition 1.3. *If Γ is progress and Γ_∞ is Δ^1_1, then*
1. *if Γ is monotonic and Π^1_1, then $|\Gamma| < \omega_1^{\mathrm{CK}}$.*
2. *if Γ is Π^0_1, then $|\Gamma| < \omega_1^{\mathrm{CK}}$.*

From (2) of Proposition 1.3, one can deduce all of the results related the height of ranked structures in [3] and [1]. We just give an example and leave the others as exercises.

A linear ordering $\mathcal{L} = (L, <_L)$ is *scattered* if the rantional linear ordering $\mathcal{Q} = (Q, <_Q)$ cannot be embedded into $\mathcal{L}$. For the scattered linear ordering $\mathcal{L} = (L, <_L)$, we can define a Hausdorff rank on $\mathcal{L}$.

$x \approx_0 y$ if $x = y$.

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$x \approx_\alpha y$ if there are finitely many elements $x_1, x_2, ..., x_n$ so that $\forall z(x <_L z <_L y \implies \exists i < n(z \approx_\beta x_i))$ in case $\alpha = \beta + 1$.

$x \approx_\alpha y$ if $x \approx_\beta y$ for some $\beta < \alpha$ in the case that α is a limit ordinal.

The Hausdorff rank of $\mathcal{L}$, $\mathrm{rk}_H(\mathcal{L})$, is the least α so that $x \approx_\alpha y$ for all $x, y \in L$. Note that a linear ordering $\mathcal{L}$ is scattered if and only if $\mathrm{rk}_H(\mathcal{L})$ exists.

Montalbán proved the following theorem.

Theorem 1.4 (Montalbán [3]). *If $\mathcal{L}$ is hyerarithmetic scattered linear ordering, then* $\mathrm{rk}_H(\mathcal{L}) < \omega_1^{\mathrm{CK}}$.

Proof. Without loss of generality, we assume $\mathcal{L} = (\omega, \leq_L)$ where $\leq_L$ is Δ^1_1 and scattered.

Define $n \in \Gamma(X)$ if and only if either $n \in X$ or there is a number $m < n$ for which there are only finitely many elements not in X and $\leq_L$-between m and n.

Intuitively, $\Gamma_{\alpha+1}$ picks out the least representative from each equivalence class $\approx_\alpha$.

Obviously Γ is Π^1_1, progress and monotonic. Moreover, $\omega - \Gamma_\infty$ contains only one member. By (1) of Proposition 1.3, $|\Gamma| < \omega_1^{\mathrm{CK}}$. So $\mathrm{rk}_H(\mathcal{L}) < \omega_1^{\mathrm{CK}}$. □

2. Σ^1_1-boundedness on linear ordering

Theorem 2.1. *If $\mathcal{L}$ is a Σ^1_1 scattered linear ordering, then* $\mathrm{rk}_H(\mathcal{L}) < \omega_1^{\mathrm{CK}}$.

Proof. The proof is based on Montalbán's proof of Theorem 1.4. Assume that $\mathcal{L} = (L, <_L)$ is Σ^1_1 and $\mathrm{rk}_H(\mathcal{L}) \geq \omega_1^{\mathrm{CK}}$. Take an arithmetical function f so that W_n codes a recursive non-empty linear ordering if and only if there is a number e so that $f(e) = n$ where $\{W_n\}_n$ is an effective enumeration of r.e. binary relations. Then the set $\mathcal{W} = \{e | W_{f(e)} \text{ is well ordering}\}$ is Π^1_1. Given a set $E \subseteq \omega \times \omega \times \omega$, we define an E-arithmetical relation $\leq_E$ as following:

$b \leq_E a$ iff there are numbers x, y so that $(b, x, y) \notin E$ and $(a, x, y) \in E$.

Define a set $\mathcal{E}$ so that $E \in \mathcal{E}$ if and only if $E \subseteq \omega \times \omega \times \omega$ and

(1) $\leq_E$ is a partial ordering.
(2) For all a, the set $E_a = \{(x, y) | (a, x, y) \in E\}$ is an equivalence binary relation.
(3) $\forall (a, x, y)((a, x, y) \in E \implies x \in L \wedge y \in L)$.
(4) $\forall e \exists f \in \omega^\omega (e \in \mathcal{W} \implies f$ embeds $W_{f(e)}$ into $\leq_E)$.
(5) $\forall a \forall b \forall c \forall (x, y) \exists z (a >_E b >_E c \wedge (a, x, y) \in E \wedge (b, x, y) \notin E \implies ((x <_L z <_L y \vee y <_L z <_L x) \wedge (c, z, x) \notin E \wedge (c, z, y) \notin E \wedge (a, z, x) \in E))$.

(1)-(5) are Σ^1_1. Thus $\mathcal{E}$ is a Σ^1_1 set. Note that $\mathcal{E}$ is non-empty since the set coding Hausdorff ranks of $\mathcal{L}$ is in $\mathcal{E}$.

By Gandy's base theorem, there is a set $E \in \mathcal{E}$ so that $\omega_1^E = \omega_1^{\mathrm{CK}}$.

By (4), each recursive well ordering can be embedded into $\leq_E$. But $\omega_1^E = \omega_1^{\mathrm{CK}}$, $\leq_E$ is not a well founded partial ordering. Hence there is a descending chain in $\leq_E$. In other words, there is a descending chain $\{a_i\}_i$ so that $a_{i+1} <_E a_i$ for all $i \in \omega$. The remaining thing is just simulating Montalbán's proof to construct a copy of $\mathcal{Q}$ in $\mathcal{L}$. Readers having difficulty to fill in the details can refer to [3]. □

Remarks

(1) In (1) of Theorem 1.2, Π^1_1-ness cannot be replaced with Σ^1_1-ness. In (2), Π^0_1-ness cannot be replaced with Π^0_2-ness.

(2) Theorem 2.1 can be generalized. For example, the same method can be used to show that any scattered Σ^1_1-upper-semilattice has a rank less or equal to ω_1^{CK}. But I have not found a sweeping theorem just like Theorem 1.2 to easily deduce all of the related results in the paper [1].

References

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