HIGHER RANDOMNESS NOTIONS AND THEIR LOWNESS PROPERTIES

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Abstract. We study randomness notions given by higher recursion theory, establishing the relationships Π^1_1-randomness $\subset$ Π^1_1-Martin-Löf randomness $\subset$ Δ^1_1-randomness $=$ Δ^1_1-Martin-Löf randomness. We characterize the set of reals that are low for Δ^1_1 randomness as precisely those that are Δ^1_1 -traceable. We prove that there is a perfect set of such reals.

1. Introduction

The study of algorithmic randomness has been focused almost exclusively in recent years on the arithmetical hierarchy, and with considerable success. In particular, n-randomness and weak n-randomness were investigated for $n < \omega$. (Recall here that a real in 2^ω is n-random if it is not in the intersection of any nested uniformly Σ^0_n sequence $(V_n)_{n\in\omega}$ of sets of reals so that $\mu(V_n) \leq 2^{-n}$. A real is weakly n-random if it is not in any Π^0_n null set of reals.) Nevertheless, the conceptualization of algorithmic randomness may be approached from a different plane. If one accepts the view that a real is random if it does not satisfy any "reasonable" set of properties of measure zero, then it makes sense to study randomness relative to a naturally defined notion, and investigate the mathematical properties of reals that are random in the given context. There are two ways of doing this: The first is to study algorithmic n-randomness by varying the notion of the underlying measure (recent work of Reimann and Slaman (unpublished) points to a significant link between being n-random and the measure that determines randomness), while the second is to retain the classical notion of Lebesgue measure and raise the logical complexity of the sets of reals being considered in the investigation of randomness. In this paper we adopt the second approach and consider randomness within the realm of second order arithmetic. In the spirit of higher recursion theory, we call this the theory of higher randomness.

From the point of view of higher recursion theory, a natural extension of the notion of recursive enumerability for subsets of ω in second order arithmetic is Π^1_1 definability. An extensive theory has been developed by Kleene, Spector, Gandy, Sacks and others (cf. Sacks [16] for a thorough treatment of the subject). Martin-Löf [10] was the first to study randomness in the setting of higher recursion theory, when he showed that the intersection of a sequence of hyperarithmetical sets of reals of measure one forms

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a nonempty Σ^1_1 set. For almost 40 years this remained the only contribution to the subject of higher randomness, with the marginal exception of Sacks [16] (Chapter IV, Exercise 2.5). He defined what we call in this paper Π^1_1 and Δ^1_1 random reals, namely those reals avoiding Π^1_1 and Δ^1_1 null sets, respectively. The recent work of Hjorth and Nies [5] may be regarded as the first systematic study of randomness via effective descriptive set theory. In this paper we follow this direction, by examining various notions we consider to be central to any reasonable theory of randomness. We study them in the setting of higher recursion theory. The motivation is to understand how the choice of a mathematical definability setting determines the key properties of random reals within the structure. We first investigate the analogs of various naturally defined, competing and inequivalent notions of randomness in first order theory. We show that under some circumstances, their analogs are equivalent in second order arithmetic. For instance, a real x is Δ^1_1 random if and only if it is Δ^1_1 random in the sense of Martin-Löf tests. In the case when $\omega_1^x = \omega_1^{\mathrm{CK}}$, the equivalence extends to x being Π^1_1 random and being Π^1_1 random in the sense of Martin-Löf (Theorem 3.3 and Corollary 3.5; see §3 for the definitions). In general, however, the last two notions do not coincide (Theorem 3.12). In §4 we study an analog of the notion of a real of hyperimmune-free degree, being Δ^1_1 dominated. We show that the set of Δ^1_1-dominated reals has measure 1, and that every Π^1_1-random real is Δ^1_1-dominated (Theorem 4.2 and Corollary 4.3). In §5 we study the class of Δ^1_1 and Π^1_1 traceable sets as analogs of recursive and r.e. traceable reals. It turns out that these two classes are identical, of size the continuum (Theorem 5.4), and properly contained in the class of Δ^1_1 dominated reals. This is used to study the class of low for Δ^1_1 random reals where it is proved in §6 Theorem 6.2 that a real is low for Δ^1_1 random if and only if it is Δ^1_1 traceable. We end the paper with further comments on higher randomness, one result on low for Π^1_1-randomness, and some open problems.

2. Preliminaries

We assume that the reader is familiar with elements of higher recursion theory such as that presented in Sacks [16]. Fix a standard Π^0_2 set $H \subseteq \omega \times 2^\omega \times 2^\omega$ so that for all x and $n \in \mathcal{O}$, there is a unique real y satisfying $H(n, x, y)$. Moreover, if $\omega_1^x = \omega_1^{\mathrm{CK}}$, then each real $z \leq_h x$ is Turing reducible to some y so that $H(n, x, y)$ holds for some $n \in \mathcal{O}$. Roughly speaking, y is the $|n|$-th Turing jump of x. These y's are called H^x sets and denoted by H^x_n's.

We use the Cantor pairing function, the bijection $p : \omega^2 \to \omega$ given by $p(n, s) = \frac{(n+s)^2+3n+s}{2}$, and write $\langle n, s\rangle = p(n, s)$.

The following results will be used in later sections.

Theorem 2.1 (Gandy). *If $A \subset 2^\omega$ is a nonempty Σ^1_1 set, then there is a real $x \in A$ so that $\mathcal{O}^x \leq_h \mathcal{O}$.*

Theorem 2.2 (Spector [17] and Gandy [4]). *$A \subset 2^\omega$ is Π^1_1 if and only if there is an arithmetical predicate $P(x, y)$ such that*

$$y \in A \leftrightarrow \exists x \leq_h y P(x, y).$$

Theorem 2.3 (Sacks[15]). *If x is non-hyperarithmetical, then $\mu(\{y|y \geq_h x\}) = 0$.*

Theorem 2.4 (Sacks [16]). *The set $\{x|x \geq_h \mathcal{O}\}$ is Π^1_1. Moreover, $x \geq_h \mathcal{O}$ if and only if $\omega_1^x > \omega_1^{\mathrm{CK}}$.*

A consequence of the last two theorems above is that the set $\{x|\omega_1^x > \omega_1^{\mathrm{CK}}\}$ is a Π^1_1 null set.

The ramified analytical hierarchy was introduced by Kleene, and applied by Feferman [2] and Cohen [1] to study forcing, a tool that turns out to be powerful in the investigation of higher randomness theory. We recall some basic facts here following Sacks [16] whose notations we mostly follow, as given below:

The ramified analytic hierarchy language $\mathfrak{L}(\omega_1^{\mathrm{CK}}, \dot{x})$ contains the following symbols:

(1) Number variables: $j, k, m, n, ...$;
(2) Numerals: 0,1,2,...;
(3) Constant: $\dot{x}$;
(4) Ranked set variables: $x^\alpha, y^\alpha, ...$ where $\alpha < \omega_1^{\mathrm{CK}}$;
(5) Unranked set variables: $x, y, ...$;
(6) Others symbols include: $+$, $\cdot$ (times), $'$ (successor) and $\in$.

Formulas are built in the usual way. A formula φ is *ranked* if all of its set variables are ranked. Due to its complexity, the language is not codable in a recursive set but rather in the countable admissible set $L_{\omega_1^{\mathrm{CK}}}$.

To code the language in a uniform way, we fix a Π^1_1 path $\mathcal{O}_1$ through $\mathcal{O}$ (by [3] such a path exists). Then a ranked set variable x^α is coded by the number $(2, n)$ where $n \in \mathcal{O}_1$ and $|n| = \alpha$. Other symbols and formulas are coded recursively. With such a coding, the set of Gödel number of formulas is Π^1_1. Moreover, the set of Gödel numbers of ranked formulas of rank less than α is r.e. uniformly in the unique notation for α in $\mathcal{O}_1$. Hence there is a recursive function f so that $W_{f(n)}$ is the set of Gödel numbers of the ranked formula of rank less than $|n|$ when $n \in \mathcal{O}_1$ ($\{W_e\}_e$ is, as usual, an effective enumeration of r.e. sets).

One now defines a structure $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$, where x is a real, analogous to the way Gödel's L is defined, by induction on the recursive ordinals. Only at successor stages are new sets defined in the structure. The reals constructed at a successor stage are arithmetically definable by the reals constructed at earlier stages. The details may be found in [16]. We define $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi$ for a formula φ of $\mathfrak{L}(\omega_1^{\mathrm{CK}}, \dot{x})$ by allowing the unranked set variables to range over $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$, while the symbol x^α will be interpreted as the reals built before stage α. In fact, the domain of $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$ is the set $\{y|y \leq_h x\}$ if and only if $\omega_1^x = \omega_1^{\mathrm{CK}}$ (see [16]).

A sentence φ of $\mathfrak{L}(\omega_1^{\mathrm{CK}}, \dot{x})$ is said to be Σ^1_1 if it is ranked, or of the form $\exists x_1, ..., \exists x_n \psi$ for some formula ψ with no unranked set variables bounded by a quantifier.

We have the following result which is a model-theoretic version of the Gandy-Spector Theorem.

Theorem 2.5 (Sacks [16]). *The set $\{(n_\varphi, x)|\varphi \in \Sigma^1_1 \wedge \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi\}$ is Π^1_1, where n_φ is the Gödel number of φ. Moreover, for each Π^1_1 set $A \subseteq 2^\omega$, there is a formula $\varphi \in \Sigma^1_1$ so that*

(1) $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi \implies x \in A$;
(2) *if* $\omega_1^x = \omega_1^{\mathrm{CK}}$, *then* $x \in A \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi$.

Note that if φ is ranked, then both the sets $\{x|\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi\}$ (the Gödel number of φ is omitted) and $\{x|\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \neg\varphi\}$ are Π^1_1 and so Δ^1_1. Moreover, if $A \subseteq 2^\omega$ is Δ^1_1, then there is a ranked formula φ so that $x \in A \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi$ (see Sacks [16]).

Theorem 2.6 (Sacks [15])**.** *The set* $\{(n_\varphi, p)|\mu(\{x|\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi\}) > p \wedge \varphi \in \Sigma^1_1 \wedge$ p *is a rational number*$\}$ *is* Π^1_1 *where* n_φ *is the Gödel number of* φ.

Theorem 2.7 (Sacks [15])**.** *There is a recursive function* $f : \omega \times \mathbb{Q} \to \omega$ *so that for all* n *which is Gödel number of a ranked formula*

(1) $f(n,p)$ *is Gödel number of a ranked formula;*
(2) *The set* $\{x|\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_{f(n,p)}\} \supseteq \{x|\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_n\}$ *is open;*
(3) $\mu(\{x|\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_{f(n,p)}\} - \{x|\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_n\}) < p$.

Theorem 2.8 (Sacks [15] and Tanaka [19])**.** *If* A *is a* Π^1_1 *set of positive measure, then* A *contains a hyperarithmetical real.*

3. Defining higher randomness notions

A sequence of open sets $\{U_n\}_{n\in\omega}$ is a Martin-Löf test (ML-test) if $\mu(U_n) \leq 2^{-n}$ for all n. Given a class of sets of reals $\mathbf{\Gamma}$ (e.g. Π^1_1 or Δ^1_1), $\{U_n\}_{n\in\omega}$ is a $\mathbf{\Gamma}$-ML test if $\{(n,\sigma)|\sigma \in 2^{<\omega} \wedge [\sigma] \in U_n\} \in \mathbf{\Gamma}$.

Definition 3.1. *Given a class of sets of reals* $\mathbf{\Gamma}$,

(1) *A real* x *is* $\mathbf{\Gamma}$ *random if no* $\mathbf{\Gamma}$ *null set contains* x.
(2) *A real* x *is* $\mathbf{\Gamma}$*-ML-random if* $x \notin \bigcap_{n\in\omega} U_n$ *for any* Γ *ML-test* $\{U_n\}_n$.

In this paper, we focus on Δ^1_1-ML, Δ^1_1, Π^1_1-ML and Π^1_1-randomness. First we show that Δ^1_1-randomness and Δ^1_1 ML randomness coincide. For this we need a lemma which will be used later. In effect it says that at the hyperarithmetical level, recursive randomness and Schnorr randomness are the same.

Lemma 3.2. *Let* $\mathcal{A}$ *be a* Δ^1_1 *null set. Then* $\mathcal{A} \subseteq \bigcap U_n$ *for some* Δ^1_1*-ML test* $\{U_n\}_{n\in\omega}$ *such that, in addition,* $\mu(U_n) = 2^{-n}$ *for each* n.

Proof. If $\mathcal{A}$ is a Δ^1_1-null set, then by Theorem 2.7 there is a recursive sequence of Δ^1_1 open sets U_n for which $\mu(U_n) < 2^{-n}$ and $\mathcal{A} \subseteq U_n$ for all n. So $\{U_n\}_{n\in\omega}$ is a Δ^1_1-ML-test.

It now suffices to show that the Δ^1_1-ML test $\{U_n\}_{n\in\omega}$ can be improved to a Δ^1_1-ML test $\{\hat{U}_n\}_{n\in\omega}$ such that $U_n \subseteq \hat{U}_n$ and $\mu(\hat{U}_n) = 2^{-n}$ for each n. For this, it clearly suffices to show that for each Δ^1_1 open set $S \subseteq 2^\omega$ and each rational $q \geq \mu(S)$ one can effectively obtain a Δ^1_1 open set $\hat{S}$ such that $S \subseteq \hat{S} \subseteq 2^\omega$ and $\mu(\hat{S}) = q$. Recall the isometry F between the conull subset of Cantor space 2^ω consisting of the coinfinite sets and the interval $I = [0,1)_{\mathbb{R}}$: for a coinfinite set $z \subseteq \omega$, let

$$F(z) = \sum_{i\in z} 2^{-i-1}.$$

Note that under F, the product measure μ turns into Lebesgue measure, and the lexicographical ordering $<_L$ becomes the usual ordering of real numbers. The function $f : I \to I$ given by $f(x) = \mu([0, x) \cup F(S))$ is continuous, non-decreasing and $f(0) \leq q$ while $f(x) \geq x$ for each $x \in I$. Thus there is a least r such that $f(r) = q$. Since $f \in \Delta^1_1$ and the left cut of r is $\{s \in \mathbb{Q} | f(s) < q\}$, the real number r is Δ^1_1, so $F(z) = r$ for some hyperarithmetical coinfinite $z \subseteq \omega$. Now the open set $\hat{S} = \{y | y <_L z\} \cup S$ is as desired. $\square$

Theorem 3.3. *The following are equivalent for a real x.*

(i) *x is Δ^1_1-random*
(ii) *x is Δ^1_1-ML random*

Proof. (i) $\Rightarrow$ (ii): If $\{\hat{U}_n\}_{n\in\omega}$ is a Δ^1_1-ML-test, then $V = \bigcap_{n\in\omega} \hat{U}_n$ is a Δ^1_1-null set. So $x \notin V$.

(ii) $\Rightarrow$ (i): This is an immediate consequence of the previous lemma.

$\square$

Hjorth and Nies [5] gave a direct proof of the following result, which may also be obtained as a special case of the more general result [7, Theorem 1A-2]. We give yet another proof via the ramified analytical hierarchy, in order to extract more information about the set.

Theorem 3.4 (Kechris [7]; Hjorth and Nies [5])**.** *There is a largest null Π^1_1-set.*

Proof. Define

$$\mathcal{P} = \{(n, x) | n \text{ is the Gödel number of a ranked formula} \\ \wedge\, \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_n(\dot{x}) \;\wedge \mu(\{x | \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \neg\varphi_n(\dot{x})\}) \geq 1)\}$$

and

$$\mathcal{Q}_n = \{x | (n, x) \in \mathcal{P}\}.$$

Define

$$\mathcal{Q} = \bigcup_{n\in\omega} \mathcal{Q}_n \cup \{x | \omega_1^x > \omega_1^{\mathrm{CK}}\}.$$

By Theorem 2.6, the sequence $\{\mathcal{Q}_n\}_{n\in\omega}$ is a Π^1_1-sequence of Δ^1_1 sets. $\mathcal{Q}$ is Π^1_1. Moreover, $\mu(\mathcal{Q}_n) = 0$ for all $n \in \omega$. Since $\mu(\{x | \omega_1^{\mathrm{CK}} = \omega_1^x\}) = 1$, $\mu(\mathcal{Q}) = 0$.

If A is a Π^1_1 null set, then, by Theorem 2.5, there is a ranked formula $\varphi \in \Sigma^1_1$ so that if $\omega_1^x = \omega_1^{\mathrm{CK}}$, then $x \in A \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \exists y \varphi(\dot{x}, y)$. So if $\omega_1^x = \omega_1^{\mathrm{CK}}$, then $x \in A \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \exists y^\alpha \varphi(\dot{x}, y^\alpha)$ for some $\alpha < \omega_1^{\mathrm{CK}}$. Since the set $\{x | \omega_1^x > \omega_1^{\mathrm{CK}}\}$ is null, it is easy to see that $A \subseteq \mathcal{Q}$.

Thus $\mathcal{Q}$ is a largest null Π^1_1 set. $\square$

Corollary 3.5. *Suppose $\omega_1^x = \omega_1^{\mathrm{CK}}$. Then x is Δ^1_1-random if and only if x is Π^1_1-ML-random, and this is equivalent to x being Π^1_1-random.*

Proof. Clearly Π^1_1-randomness implies Π^1_1-ML-randomness. By Theorem 3.3, it suffices to show that if x is Δ^1_1-random and $\omega_1^x = \omega_1^{\mathrm{CK}}$, then x is Π^1_1-random. Assume $\omega_1^x = \omega_1^{\mathrm{CK}}$. If x is Δ^1_1-random, then $x \notin \mathcal{Q}_n$ for all n. So $x \notin \mathcal{Q}$. Hence x is Π^1_1-random. □

In contrast to Theorem 3.4, we have the following.

Proposition 3.6. *There is no largest null Σ^1_1-set.*

Proof. Suppose A is the largest null Σ^1_1-set. Then by the Tanaka-Sacks Theorem 2.8, there is a Δ^1_1 real $x \notin A$. $X = \{x\}$ is Δ^1_1 and $X \cap A = \emptyset$, a contradiction. □

By Theorem 2.4 and the proof of Theorem 3.4, we have the following result.

Proposition 3.7 (Hjorth and Nies [5]). *If x is Π^1_1-random, then $\omega_1^x = \omega_1^{\mathrm{CK}}$.*

Together with Corollary 3.5, the Π^1_1 random reals are precisely the Δ^1_1 random reals x that also satisfy $\omega_1^x = \omega_1^{\mathrm{CK}}$.

By the Gandy Basis Theorem 2.1, there is a Π^1_1-random real x with $\mathcal{O}^x \leq_h \mathcal{O}$.

Theorem 3.8 (Hjorth and Nies [5]). *Given any real x, there is a Π^1_1-ML-random real $y \geq_h x$.*

Combining Theorem 3.8 and Proposition 3.7, we have the following consequence.

Corollary 3.9 (Hjorth and Nies [5]). *There is a Π^1_1-ML-random real that is not Π^1_1-random.*

We now separate Δ^1_1 randomness from Π^1_1-ML randomness, which is needed for the proof of Theorem 3.12 below. If one views the randomness notions as operators mapping oracles to classes, the separation can be obtained as a consequence of Theorem 5.4, Theorem 6.2, and the result of Hjorth and Nies [5] that every low for Π^1_1-ML random real is hyperarithmetical. We now obtain the separation for the plain randomness notions. Recall that in [5] a Π^1_1-version of prefix free Kolmogorov complexity was introduced, denoted by K. It was shown that a Theorem analogous to Schnorr's Theorem holds, namely: z is Π^1_1-ML random if and only if there is a $b \in \omega$ such that for each n, $K(z \upharpoonright n) \geq n - b$. So the following result implies the separation:

Theorem 3.10. *Let h be a nondecreasing Δ^1_1 function such that* $\lim_n h(n) = \infty$. *Then there is a Δ^1_1 random real z such that $\forall^\infty n\, K(z \upharpoonright n \mid n) \leq h(n)$.*

Here, $K(\sigma \mid n)$ is the complexity of σ given n. A number n is encoded in some effective way by a string (say the binary expansion). Then $K(\sigma) \leq K(\sigma \mid n) + 2\log n$ (up to constants), so if we let $h(n) = \log n$ then we obtain $K(z \upharpoonright n) \leq 3\log n$.

First we need some preliminaries. A function $f : 2^{<\omega} \to \mathbb{R}^+ \cup \{0\}$ is hyperarithmetical if there is a hyperarithmetical approximation function $g : 2^{<\omega} \times \omega \to \mathbb{Q}^+ \cup \{0\}$ such that for each σ and n, we have $|f(\sigma) - g(\sigma, n)| \leq 2^{-n}$. A hyperarithmetical martingale is a hyperarithmetical function $M : 2^{<\omega} \to \mathbb{R}^+ \cup \{0\}$ that satisfies for every $\sigma \in 2^{<\omega}$ the martingale equality $M(\sigma 0) + M(\sigma 1) = 2M(\sigma)$. For a martingale

M and a real z, let $M(z) = \sup_n M(z \upharpoonright n)$. We say that the martingale M succeeds on z if the capital it reaches along z is unbounded, that is, $M(z) = \infty$. Let $S(M) = \{z | M \text{ succeeds on } z\}$.

Of course, $S(M)$ is a Δ^1_1 null set for any hyperarithmetical martingale M. Here is the converse. This equivalence is an effectivization of Ville's theorem.

Lemma 3.11. *Let $\mathcal{A}$ be a Δ^1_1 null set. Then there is a hyperarithmetical martingale $M_{\mathcal{A}}$ such that $\mathcal{A} \subseteq S(M)$.*

Proof. By Lemma 3.2 there is a Δ^1_1 ML test $\{U_n\}_{n\in\omega}$ such that $\mu(U_n) = 2^{-n}$ and $\mathcal{A} \subseteq U_n$ for all n. Let $M_n(\sigma) = \mu(U_n \cap [\sigma])2^{|\sigma|}$. Then M_n is a hyperarithmetical martingale, uniformly in n, and $M_n(z) = 1$ if $z \in U_n$. Moreover, the start capital $M_n(\emptyset)$ is 2^{-n}. Now let $M(\sigma) = \sum_n M_n(\sigma)$, then M is as required. $\square$

The proof of Theorem 3.10 is a straightforward adaptation of the proof of a corresponding theorem for the case of recursively random reals, see for instance [13, Thm 5.33]. We build a real z of slowly growing initial segment complexity (in the sense above) on which no $\mathbb{Q}$-valued hyperarithmetical martingale L succeeds. The martingale $M_{\mathcal{A}}$ is not necessarily $\mathbb{Q}$-valued, but by adaptation of a standard argument due to Schnorr (see [13, Prop. 5.24]), for each hyperarithmetical martingale M there is a $\mathbb{Q}$-valued hyperarithmetical martingale $\hat{M}$ such that $\hat{M}(\sigma) \geq M(\sigma)$ for each σ.

In the following theorem we summarize the implications between the various randomness notions.

Theorem 3.12. $\Delta^1_1(\mathcal{O})$*-randomness* $\Rightarrow$ Π^1_1*-randomness* $\Rightarrow$ Π^1_1*-ML randomness* $\Rightarrow$ Δ^1_1*-randomness* $\Leftrightarrow$ Δ^1_1*-ML randomness, and none of the implications may be reversed.*

Proof. $\Delta^1_1(\mathcal{O})$-randomness $\Rightarrow$ Π^1_1-randomness: Fix an $\mathcal{O}$-recursive well ordering $<_R$ on ω of order type ω_1^{CK}. Then

$$\omega_1^x > \omega_1^{\mathrm{CK}} \iff$$
$$\exists S \subseteq \omega\times\omega \exists f \in \omega^\omega (S \leq_T x \wedge \forall n \exists m (f(m) = n) \wedge \forall n \forall m (S(n,m) \iff f(n) <_R f(m))).$$

So the set $\{x | \omega_1^x > \omega_1^{\mathrm{CK}}\}$ is $\Sigma^1_1(\mathcal{O})$. By Theorem 2.4, $\{x | \omega_1^x > \omega_1^{\mathrm{CK}}\}$ is $\Delta^1_1(\mathcal{O})$. Note that the sequence $\{\mathcal{Q}_n\}_{n\in\omega}$ is a Π^1_1-sequence, and so is an $\mathcal{O}$-recursive sequence of $\Delta^1_1(\mathcal{O})$ sets. So $\mathcal{Q} = \bigcup_{n\in\omega} \mathcal{Q}_n \cup \{x | \omega_1^x > \omega_1^{\mathrm{CK}}\}$ is a null $\Delta^1_1(\mathcal{O})$ set. Hence $\Delta^1_1(\mathcal{O})$-randomness $\Rightarrow$ Π^1_1-randomness. By the Gandy Basis Theorem 2.1, there is a Π^1_1 random real $x \leq_h \mathcal{O}$. Now x cannot be $\Delta^1_1(\mathcal{O})$-random. Thus the implication cannot be reversed.

Π^1_1-randomness $\Rightarrow$ Π^1_1-ML randomness: It is clear that Π^1_1-randomness $\subseteq$ Π^1_1-ML randomness. By Theorem 3.8, there exists a Π^1_1-ML random real $x \geq_h \mathcal{O}$. x cannot be Π^1_1-random.

Obviously Π^1_1-ML randomness $\Rightarrow$ Δ^1_1-randomness. It follows from the Theorem 3.10 that the implication cannot be reversed.

Finally, Δ^1_1-randomness $\Leftrightarrow$ Δ^1_1-ML-randomness is a consequence of Theorem 3.3. $\square$

The reader may wonder why we do not study Σ^1_1-randomness. In fact this is done implicitly—the following proposition says that Σ^1_1-randomness coincides with Δ^1_1-randomness.

Proposition 3.13. *If A is Π^1_1 and $\mu(A) = 1$, then there is a Δ^1_1 conull set $B \subseteq A$.*

Proof. Suppose A is a Π^1_1-set for which $\mu(A) = 1$. Then, by Theorem 2.5, there is a ranked formula $\varphi(\dot{x}, y)$ so that for all $n \in \mathcal{O}_1$, $A_n \subseteq A$, where $A_n = \{x | \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \exists y^{|n|}\varphi(\dot{x}, y^{|n|})\})$. Since the set $\{x|\omega_1^x > \omega_1^{\mathrm{CK}}\}$ is null and $A_n \subseteq A_m$ for all $n <_o m$ in $\mathcal{O}_1$, by Theorem 2.5, $\mu(A) = \mu(\bigcup_{n\in\mathcal{O}_1} A_n)$. Define $R(k, n)$ if and only if $\mu(A_n) > 1-2^{-k}$. By Theorem 2.6, R is a Π^1_1 relation. By the Π^1_1 Uniformitarian Theorem (see [12]), there is a Π^1_1 function $f : \omega \to \omega$ uniformizing R. Since $\mu(A) = 1$, f is a total function. So f is Δ^1_1. Hence the range S of f is Δ^1_1. Then there must a recursive ordinal α so that $|n| < \alpha$ for all $n \in S$ (otherwise, $\mathcal{O}_1$ would be Δ^1_1). Fix the notation $n \in \mathcal{O}_1$ so that $|n| = \alpha$. Define $B = A_n$. Then $\mu(B) = 1$ and $B \subseteq A$. □

4. Δ^1_1-dominated reals

A real x is hyperimmune-free if every function Turing reducible to x is dominated by a recursive function. The following may be viewed as an analog in the setting of effective descriptive set theory:

Definition 4.1. *A real x is Δ^1_1-dominated if for all functions $f : \omega \to \omega$ with $f \leq_h x$, there is a hyperarithmetic function g so that $g(n) > f(n)$ for all n (written as $g > f$).*

Theorem 4.2. $\mu(\{x|x$ *is Δ^1_1-dominated*$\}) = 1$.

Proof. We prove that for any rational number p, the measure of

$$\{x|x \text{ is } \Delta^1_1\text{-dominated}\}$$

is not less than p. We apply a fusion argument to achieve this.

Firstly we show that for any number e, rational r, notation $n \in \mathcal{O}$ and Δ^1_1 set A for which $p + r < \mu(A)$, there is a hyperarithmetic function f so that

$$\mu(\{x|x \in A \wedge \Phi_e^{H_n^x} \text{ is total } \implies \Phi_e^{H_n^x} < f\}) > p + \frac{r}{2}.$$

Since the set $\{(x, i, m)|\Phi_e^{H_n^x}(i)\downarrow \implies \Phi_e^{H_n^x}(i) < m\}$ is Δ^1_1, there is a ranked formula $\varphi(\dot{x}, i, m)$ so that $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi(\dot{x}, i, m)$ if and only if $\Phi_e^{H_n^x}(i) < m$. Since A is Δ^1_1, by Theorem 2.6, the set

$$C = \{(i, m, k)| \wedge \mu(\{x|x \in A \wedge (\Phi_e^{H_n^x}(i)\downarrow \implies \Phi_e^{H_n^x}(i) < m)\}) > \mu(A) - \frac{r}{2^{k+2}}\}$$

is Δ^1_1. Note that for each k, there is a number m so that $(k, m, k) \in C$. So there is a Δ^1_1 total function f so that for all k, $(k, f(k), k) \in C$. Define

$$B_k = \{x|x \in A \wedge (\Phi_e^{H_n^x}(k)\downarrow \implies \Phi_e^{H_n^x}(k) < f(k))\}.$$

Then the set $\{(k,x)|x \in B_k\}$ is Δ^1_1. Moreover, for every k, $B_k \subseteq A$ and $\mu(B_k) > \mu(A) - \frac{r}{2^{k+2}}$. So the set $B = \bigcap_k B_k$ is Δ^1_1 and

$$\mu(B) \geq \mu(A) - \sum_{k\geq 0} \mu(A - B_k) \geq p + r - \sum_{k\geq 0} \frac{r}{2^{k+2}} = p + \frac{r}{2}.$$

Moreover, for every $x \in B$, if $\Phi_e^{H^x_n}$ is total, then $\Phi_e^{H^x_n} < f$. Thus we may construct an ω-sequence of Δ^1_1 sets $\{B^{\langle e,n\rangle}\}_{e\in\omega \wedge n\in\mathcal{O}}$ so that for all $e \in \omega$ and $n \in \mathcal{O}$,

(1) If $\langle e,n\rangle > \langle e',n'\rangle$, then $B^{\langle e,n\rangle} \subseteq B^{\langle e',n'\rangle}$;
(2) $\mu(B^{\langle e,n\rangle}) > p$.

Define $D = \bigcap_{e\in\omega\wedge n\in\mathcal{O}} B^{\langle e,n\rangle}$. Then $D \subseteq \{x|x \text{ is } \Delta^1_1\text{-dominated}\}$ and $\mu(D) \geq p$. Moreover, each real in D is Δ^1_1-dominated. □

Corollary 4.3. *Each Π^1_1-random real is Δ^1_1-dominated.*

Proof. By the proof of Theorem 4.2, for each $e \in \omega$ and $n \in \mathcal{O}$, the set $A_{e,n} = \{x|\exists f \in \Delta^1_1(\Phi_e^{H^x_n} \text{ is total } \implies \Phi_e^{H^x_n} < f)\}$ has measure 1. Note that $A_{e,n}$ is Π^1_1. So, by Proposition 3.13, if x is Δ^1_1-random, then $x \notin A_{e,n}$. Now if x is Π^1_1-random, then, by Proposition 3.7, $\omega_1^{\mathrm{CK}} = \omega_1^x$. So if $g \leq_h x$, then $g = \Phi_e^{H^x_n}$ for some e, n. Thus each Π^1_1-random is Δ^1_1-dominated. □

Note that Π^1_1-randomness cannot be improved to Δ^1_1-randomness in Corollary 4.3 since there exists a Δ^1_1 random real $x \geq_h \mathcal{O}$ (see [5]) but $\{x|x \text{ is } \Delta^1_1\text{-dominated }\} \subset \{x|\omega_1^x = \omega_1^{\mathrm{CK}}\}$ by the following proposition.

Proposition 4.4. $\{x|x$ *is* Δ^1_1*-dominated* $\} \subset \{x|\omega_1^x = \omega_1^{\mathrm{CK}}\}$.

Proof. If $\omega_1^x > \omega_1^{\mathrm{CK}}$, then $x \geq_h \mathcal{O}$. Since there is an $\mathcal{O}$-arithmetical enumeration of Δ^1_1 functions $\{f_n\}_{n\in\omega}$, there is a $\Delta^1_1(x)$ enumeration. Define $g(n) = f_n(n) + 1$. Then $g \leq_h x$. So x is not Δ^1_1-dominated. Thus $\{x|x \text{ is } \Delta^1_1\text{-dominated }\} \subseteq \{x|\omega_1^x = \omega_1^{\mathrm{CK}}\}$.

To see that the relation is proper, we apply Cohen forcing developed in [16]. The forcing conditions are elements of $2^{<\omega}$. A real is said to be generic if each Σ^1_1-sentence or its negation is forced by a finite initial segment of x. So generic reals form a comeager set. Feferman (see [2] or [16]) proved that $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$ satisfies Δ^1_1-comprehension for any generic real x. So $\omega_1^x = \omega_1^{\mathrm{CK}}$ (see [16]). We claim that no generic real can be Δ^1_1-dominated.

Given a real x, define $g_x(n) = m_n$ if m_n is the n-th bit of x so that $x(m_n) = 1$. So there is a recursive oracle function $\Phi^x = g_x$ for all x. Hence there is a ranked (and so Σ^1_1) formula φ defining g_x, i.e. $g_x(n) = m \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi(\dot{x}, n, m)$. For any Δ^1_1 function f, there is a ranked formula ψ_f defining f, i.e. $f(n) = m \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \psi_f(n,m)$. So if $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \forall n(f(n) > g_{\dot{x}}(n))$, then there is a finite string $p \prec x$ so that $p \Vdash \forall n(f(n) > g_{\dot{x}}(n))$. This is impossible since one can easily find a condition q stronger then p so that $q \Vdash \exists n(f(n) < g_{\dot{x}}(n))$.

Thus $\{x|x \text{ is } \Delta^1_1\text{-dominated }\} \subset \{x|\omega_1^x = \omega_1^{\mathrm{CK}}\}$. □

One might conjecture that Δ^1_1-dominated reals form a basis for Σ^1_1 sets. This is however false.

Proposition 4.5. *There is a nonempty Σ^1_1 $A \subseteq 2^\omega$ which does not contain any Δ^1_1-dominated real.*

Proof. As in the proof of Proposition 4.4, there is a recursive oracle function $x \mapsto \Phi^x$ so that the set $A = \{x | \forall f \in \Delta^1_1(f \not\geq \Phi^x)\}$ is non-empty. By Theorem 2.2 (the Spector-Gandy Theorem), A is a nonempty Σ^1_1 set. □

5. Δ^1_1-TRACEABLE REALS

Next we consider the notions analogous to being r.e. traceable and recursively traceable in first order randomness theory, both of which are studied in [20, 9] (see for instance [9, Section 2.2] for the formal definition). The corresponding notions are called Π^1_1 traceability and Δ^1_1 traceability respectively. We shall show that they are in fact equivalent.

Definition 5.1. (i) *Let $h : \omega \to \omega$ be a nondecreasing unbounded function that is hyperarithmetical. A Π^1_1- trace/Δ^1_1-trace with bound h is a uniformly Π^1_1/uniformly Δ^1_1 sequence $(T_e)_{e\in\omega}$ such that $|T_e| \leq h(e)$ for each e.*

(ii) *$A \subseteq \omega$ is Π^1_1-traceable/Δ^1_1-traceable if there is $h \in \Delta^1_1$ such that, for each $f \leq_h A$, there is a Π^1_1-trace/Δ^1_1 -trace with bound h such that, for each e, $f(e) \in T_e$.*

Note that, if $(T_e)_{e\in\omega}$ is a uniformly Δ^1_1 sequence of finite sets, then there is $g \in \Delta^1_1$ such that for each e, $D_{g(e)} = T_e$ (where D_n is the nth finite set according to some recursive ordering). Thus

$$g(e) = \mu n \, \forall u \, [u \in D_n \leftrightarrow u \in T_e].$$

In this formulation, the definition of Δ^1_1 traceability is very close to that of recursive traceability. It is not difficult to see that every Δ^1_1-traceable real is Δ^1_1-dominated.

Also notice that the choice of a bound as a witness for traceability is immaterial:

Proposition 5.2 (Terwijn and Zambella [20]). *Let A be a real that is Δ^1_1 traceable with bound h. Then for any monotone and unbounded Δ^1_1 function h', A is Δ^1_1 traceable with bound h'. The same holds for Π^1_1 traceability.*

The class of r.e. traceable sets is strictly larger than the class of recursively traceable sets, since the former contains nonrecursive r.e. sets [9]. In contrast, we have the following equivalence:

Proposition 5.3. *If x is Π^1_1-traceable, then x is Δ^1_1-traceable.*

Proof. We first claim that $\omega^x_1 = \omega^{\mathrm{CK}}_1$. Otherwise $x \geq_h \mathcal{O}$. So it is sufficient to show that $\mathcal{O}$ is not Π^1_1 traceable. Since each Π^1_1 set is many-one reducible to $\mathcal{O}$ [14, 5.4 I], there is a uniformly $\mathcal{O}$-recursive list $(T^e)_{e\in\omega}$ of all Π^1_1-traces for the bound $h(e) = e$. Define $f \leq_h \mathcal{O}$ by

$$f(e) = \mu n \, [n \notin T^e_e],$$

then f does not have a Π^1_1 trace.

To complete the proof, given $f \leq_h x$, there is a Π^1_1 trace $(T_e)_{e\in\omega}$ such that $f(e) \in T_e$ for each e. Then there is a recursive function $h : \omega^2 \to \omega$ so that $k \in T_e$ if and only if $h(k, e) \in \mathcal{O}$. Define a $\Pi^1_1(x)$-relation $R \subseteq \omega \times \mathcal{O}$ by

$$(e, n) \in R \Leftrightarrow h(e, f(e)) \in \mathcal{O}_n,$$

where $\mathcal{O}_n = \{m \in \mathcal{O} \| m| < |n|\}$, a Δ^1_1 set. Note that for each e, there is a notation $n \in \mathcal{O}$ so that $(e, n) \in R$. By the Kreisel Uniformization Theorem, there is a total $\Pi^1_1(x)$ (and so $\Delta^1_1(x)$) function g uniformizing R. Hence the range $S = \{n|\exists e[g(e) = n]\}$ of g is a $\Delta^1_1(x)$ set. Since $\omega^x_1 = \omega^{\mathrm{CK}}_1$, there exists a notation $n_0 \in \mathcal{O}$ so that $S \subseteq \mathcal{O}_{n_0}$ (otherwise the well-founded relation "$i <_o j$" would be $\Delta^1_1(x)$). Define a set $\hat{T}_e \subseteq T_e$ as follows:

$$k \in \hat{T}_e \Leftrightarrow h(k, e) \in \mathcal{O}_{n_0}.$$

By the definition of n_0, $f(e) \in \hat{T}_e$ for all $e \in \omega$. Note that the relation $n \in \hat{T}_e$ is Δ^1_1. Hence $(\hat{T}_e)_{e\in\omega}$ is a Δ^1_1-trace for f. So f is Δ^1_1-traceable. □

Theorem 5.4. *There are $2^{\aleph_0}$-many Δ^1_1-traceable reals.*

Proof. We apply Sacks forcing to show this (see [16]). The forcing conditions are perfect trees coded by Δ^1_1 reals. A real x is Sacks generic if for each Σ^1_1 sentence φ, there is a condition T so that $x \in T$ and $T \Vdash \varphi$ or $T \Vdash \neg\varphi$. Sacks proved that the set $\{(T, n_\varphi)|\varphi \in \Sigma^1_1 \wedge T \Vdash \varphi\}$ is Π^1_1. We claim that each Sacks generic real is Δ^1_1-traceable. Thus there are $2^{\aleph_0}$-many Δ^1_1-traceable reals.

Suppose x is a Sacks generic real. Since x has minimal hyperdegree (see [16]), $\omega^{\mathrm{CK}}_1 = \omega^x_1$. So if $f \leq_h x$, then there is a number e and a notation $n \in \mathcal{O}$ so that $\Phi_e^{H^x_n} = f$. Since the set $A = \{(y, n, i, j)|\Phi_e^{H^y_n}(i) = j\}$ is Δ^1_1, there exists a ranked formula defining A. Since $\Phi_e^{H^x_n}$ is total, by the definition of Sacks genericity, there is a condition $T \Vdash$ "$\Phi_e^{H^{\dot{x}}_n}$ is total". We show that for each condition $S \subseteq T$, there is a condition $Q \subseteq S$ so that $Q \Vdash$ "$\exists f(f \in \Delta^1_1 \wedge \forall i(\Phi_e^{H^{\dot{x}}_n}(i) \in D_{f(i)} \wedge |D_{f(i)}| \leq 2^{i+1}))$". Then, by the definition of forcing, there is a Δ^1_1 function f so that for all i, $\Phi_e^{H^x_n}(i) \in D_{f(i)} \wedge |D_{f(i)}| \leq 2^{i+1}$.

Since $T \Vdash$ "$\Phi_e^{H^{\dot{x}}_n}$ is total", $S \Vdash$ "$\Phi_e^{H^{\dot{x}}_n}$ is total".

Case (1). There is a condition $R \subseteq S$ so that for all i, j_0, j_1, for all conditions $P_0, P_1 \subseteq R$, $P_0 \Vdash \Phi_e^{H^{\dot{x}}_n}(i) = j_0$ and $P_1 \Vdash \Phi_e^{H^{\dot{x}}_n}(i) = j_1$ implies $j_0 = j_1$. Then we define $f(i) = j$ if and only if there exists a condition $P \subseteq R$ so that $P \Vdash \Phi_e^{H^{\dot{x}}_n}(i) = j$. Then f is a total Π^1_1-function and hence Δ^1_1. This implies that $R \Vdash f = \Phi_e^{H^{\dot{x}}_n}$.

Case (2). Otherwise. Define a relation $\mathcal{R}(P, \sigma, i, j_0, j_1, Q_0, Q_1)$ if and only if $i \geq |\sigma|$, $j_0 \neq j_1$, $Q_0 \cap Q_1 = \emptyset$ and $Q_k \subseteq P \wedge Q_k \Vdash \Phi_e^{H^{\dot{x}}_n}(i) = j_k$ for $k \leq 1$. Obviously $\mathcal{R}$ is a Π^1_1 relation. By Kreisel's Uniformization Theorem, there is a partial Π^1_1 function $F : 2^\omega \times 2^{<\omega} \to (\omega)^3 \times (2^\omega)^2$ so that $\mathcal{R}(P, \sigma, i, j_0, j_1, Q_0, Q_1)$ for some i, j_0, j_1, Q_0, Q_1 if and only if $\mathcal{R}(P, \sigma, F(P, \sigma))$. Without loss of generality, we assume that if $P \Vdash \Phi_e^{H^{\dot{x}}_n}(i) = j_i$ then for all $k \leq i$, $P \Vdash \Phi_e^{H^{\dot{x}}_n}(k) = j_k$ for some j_k. We do an induction on ω. During the construction, we will define a Π^1_1 sequence of conditions $\{P_\sigma\}_{\sigma\in 2^{<\omega}}$.

Step 0, define $P_\emptyset = S$.
Step n+1, for each $\sigma \in 2^n$, define $P_{\sigma^\frown 0} = Q_0$, $P_{\sigma^\frown 1} = Q_1$ if $F(P_\sigma, \sigma) = (i, j_0, j_1, Q_0, Q_1)$.

Define $G(\sigma) = P_\sigma$. Then G is a total Π^1_1 and so Δ^1_1 function.
Note that for each σ, $G(\sigma^\frown 0) \cap G(\sigma^\frown 1) = \emptyset$ and if $\sigma \preceq \tau$ then $G(\sigma) \supseteq G(\tau)$. Define

$$Q = \bigcap_n \bigcup_{\sigma \in 2^n} G(\sigma).$$

Then Q is a Δ^1_1 perfect set.

Define a function $g : \bigcup_{i \in \omega} i \times 2^{i+1} \to \omega$ so that $g(i, \sigma) = k$ if $\sigma \in 2^{i+1}$ and $G(\sigma) \Vdash \Phi_e^{H_n^{\dot{x}}}(i) = k$. Hence g is a total Π^1_1 and therefore Δ^1_1 function. Define $f(i) = j$ if j is the least number such that $D_j = \{g(i, \sigma) | \sigma \in 2^{i+1}\}$. Then f is a Δ^1_1 function and $|D_{f(i)}| \leq 2^{i+1}$ for all i. Since for all i, $Q \subseteq \bigcup_{\sigma \in 2^n} G(\sigma)$, it is easy to see that $Q \Vdash \Phi_e^{H_n^{\dot{x}}}(i) \in D_{f(i)}$.

So x is Δ^1_1-traceable. □

6. Lowness for Δ^1_1-randomness

Definition 6.1. *Given a relativizable class of reals $\mathcal{C}$ (for instance, $\mathcal{C}$ is the class of random reals), a real x is low for $\mathcal{C}$ if $\mathcal{C} = \mathcal{C}^x$.*

For a randomness notion $\mathcal{C}$, we have $\mathcal{C}^x \subseteq \mathcal{C}$, and usually one would expect $\mathcal{C}^x$ to be a proper subset of $\mathcal{C}$. Thus being low for $\mathcal{C}$ means to be computationally weak, in the sense that the extra computational power of x does not help to recognize more reals as nonrandom.

It is shown in [5] that x is low for Π^1_1-ML randomness if and only if x is hyperarithmetical. The main result of this section is that a real is low for Δ^1_1-randomness if and only if it is Δ^1_1 traceable. This corresponds to the main result in [9] that a real A is low for Schnorr randomness if and only if it is recursively traceable. That result was an extension of the theorem in [20] that A is low for Schnorr tests if and only if it is recursively traceable. The equivalence of (i) and (ii) in the theorem below reveals this parallel phenomenon in the realm of effective descriptive set theory.

For $D \subset 2^{<\omega}$ we let $[D]^{\preceq}$ denote the open set $\bigcup \{[\sigma] | \sigma \in D\}$. We often identify an open set with the corresponding set of strings closed under extension. We let S_e be the eth finite subset of $2^{<\omega}$ under a suitable effective enumeration. Thus S_e is a finite set of strings, and $[S_e]^{\preceq} = \bigcup_{\sigma \in S_e} [\sigma]$ is then the clopen set coded by $e \in \omega$.

Theorem 6.2. *The following are equivalent for a real x.*

(i) *x is Δ^1_1 traceable (or equivalently, Π^1_1 traceable).*
(ii) *Each $\Delta^1_1(x)$ null set is contained in a Δ^1_1 null set.*
(iii) *x is low for Δ^1_1 randomness.*
(iv) *Each Π^1_1-ML random set is $\Delta^1_1(x)$ random.*

Proof. (i) → (ii): Assume that x is Δ^1_1 traceable. Let $\mathcal{S}$ be a $\Delta^1_1(x)$ null set. By Lemma 3.2 relativized to x, $\mathcal{S} \subseteq \bigcap U_n$ for a $\Delta^1_1(x)$-ML test $\{U_n\}_{n \in \omega}$ such that $\mu(U_n) =$

2^{-n} for each n. There is a function $f \leq_h x$ such that $[S_{f(\langle n,s\rangle)}]^{\preceq} =: U_{n,s}$ satisfies $U_{n,s} \subseteq U_{n,s+1}$, $U_n = \bigcup_{s\in\omega} U_{n,s}$, and, moreover, $\mu(U_{n,s}) > 2^{-n}(1-2^{-s})$.

Let $T = (T_e)_{e\in\omega}$ be a Δ^1_1 trace of f. By Proposition 5.2, we may choose T such that in addition $|T_e| \leq e$ for each $e > 0$.

We now define a subtrace $\hat{T}$ of T, i.e., $\hat{T}_{\langle n,s\rangle} \subseteq T_{\langle n,s\rangle}$ for each n, s. The objective is to define open sets V_n via $\hat{T}$ (in a way to be specified) small enough to give us a Δ^1_1- null set $\mathcal{V} = \bigcap_n V_n$, yet large enough as to keep all "relevant" reals out of $T_{\langle n,s\rangle} - \hat{T}_{\langle n,s\rangle}$, so that $\bigcap_{n\in\omega} U_n \subseteq \mathcal{V}$.

Let $\hat{T}_{\langle n,s\rangle}$ be the set of $e \in T_{\langle n,s\rangle}$ such that $2^{-n}(1-2^{-s}) \leq \mu([S_e]^{\preceq}) \leq 2^{-n}$ and $[S_e]^{\preceq} \supseteq [S_d]^{\preceq}$ for some $d \in \hat{T}_{\langle n,s-1\rangle}$ (where $\hat{T}_{\langle n,-1\rangle} = \omega$). Note that $f(\langle n,s\rangle) \in \hat{T}_{\langle n,s\rangle}$. Let

$$V_n = \bigcup \left\{ [S_e]^{\preceq} | e \in \hat{T}_{\langle n,s\rangle},\ s\in\omega \right\}.$$

Then $\mu(V_n) \leq 2^{-n}|\hat{T}_{\langle n,0\rangle}| + \sum_{s\in\omega} 2^{-s}2^{-n}|\hat{T}_{\langle n,s\rangle}|$. Since $|\hat{T}_{\langle n,s\rangle}| \leq |T_{\langle n,s\rangle}| \leq \langle n,s\rangle$ for $\langle n,s\rangle \neq 0$, and $\langle n,s\rangle$ has only polynomial growth in n and s, it is clear that $\lim_n \sum_{s\in\omega} 2^{-s}2^{-n}|\hat{T}_{\langle n,s\rangle}| = 0$, and hence $\lim_n \mu(V_n) = 0$. Then $\mathcal{V} = \bigcap_n V_n$ is a Δ^1_1-null set and $\bigcap U_n \subseteq \mathcal{V}$.

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are immediate.

(iv) $\Rightarrow$ (i): In [9, Lemma 4.7], it is shown that, if each ML-random set is Schnorr random relative to x, then x is r.e. traceable. With merely notational changes, the proof works in the present situation. First some preliminaries. Recall that $K(\sigma)$ denotes the Π^1_1 version of prefix free Kolmogorov complexity. For $b \in \omega - \{0\}$, let $R_b = [\{\sigma \in 2^{<\omega} | \ K(\sigma) \leq |\sigma| - b\}]$. In [5, Theorem 3.9] it is shown that $(R_b)_{b\in\omega}$ is a universal test for Π^1_1-ML randomness. Thus, by our hypothesis in (iv), we have $\mathcal{C} \subseteq \bigcap_b R_b$ for each $\Delta^1_1(x)$ null set $\mathcal{C}$.

For $k, l \in \omega$ define the clopen set

$$B_{k,l} = \bigcup \left\{ [\tau 1^k] | \tau \in 2^{<\omega}, |\tau| = l \right\},$$

where 1^k is a string of 1's of length k. Note that $\mu(B_{k,l}) = 2^{-k}$ for all l.

Given $\sigma \in 2^{<\omega}$ and a measurable set $C \subseteq 2^\omega$, let $\mu_\sigma(C) = \frac{\mu(C\cap[\sigma])}{\mu[\sigma]}$. For an open set W let

$$W|\sigma = \bigcup \left\{ [\tau] | \tau \in 2^{<\omega}, [\sigma\tau] \subseteq W \right\}.$$

Now to find a trace for a given function $g \leq_h x$, define the $\Delta^1_1(x)$-ML test U^g by stipulating that

$$U^g_n = \bigcup_{k>n} B_{k,g(k)}.$$

Hence by assumption $\bigcap_n U^g_n \subseteq \bigcap_{b\in\omega} R_b$. Thus $V = R_3$ contains $\bigcap_n U^g_n$ and $\mu(V) < \frac{1}{4}$. We may assume throughout that $g(k) \geq k$ for every k because from a trace for $g(k)+k$ one can obtain a trace for g with the same bound. By [9, Lemma 4.4], there exist σ and n such that $\mu_\sigma(U^g_n - V) = 0$ and $\mu_\sigma(V) < 1/4$. As $U^g_0 \supseteq U^g_1 \supseteq \cdots$, we can choose σ and n with the additional property $n \geq |\sigma|$. Hence for each $k > n$, we have $g(k) \geq k > n \geq |\sigma|$ and hence $g(k) \geq |\sigma|$.

Let $\tilde{V} = V|\sigma$, let $\tilde{g}(k) = \max\{0, g(k) - |\sigma|\}$, and

$$T_k = \left\{ l | \mu(B_{k,l} - \tilde{V}) < 2^{-(l+3)} \right\}.$$

Note that for each $l \in \omega$, if $l \geq |\sigma|$ then $B_{k,l}|\sigma = B_{k,l-|\sigma|}$. So since $g(k) \geq |\sigma|$,

$$U_n^g|\sigma = \bigcup_{k>n} B_{k,g(k)}|\sigma = \bigcup_{k>n} B_{k,g(k)-|\sigma|} = U_n^{\tilde{g}},$$

and we obtain $\mu(U_n^{\tilde{g}} - \tilde{V}) = \mu_\sigma(U_n^g - V) = 0$. Hence $\tilde{g}(k) \in T_k$ for all $k > n$.

Since $\tilde{V}$ is a Π^1_1 open set, it is evident that T is a Π^1_1 set of integers. A trace for g is obtained as follows:

$$G_k = \begin{cases} \{l + |\sigma| | l \in T_k\} & \text{if } k > n; \\ \{g(k)\} & \text{if } k \leq n. \end{cases}$$

We now show that G is a trace for g, i.e. for all $k \in \omega$, $g(k) \in G_k$. If $k \leq n$ then this holds by definition of G_k. Thus assume $k > n$. Then $g(k) > k > n > |\sigma|$, so $\tilde{g}(k) = g(k) - |\sigma|$ so $g(k) = \tilde{g}(k) + |\sigma|$. As $k > n$, $\tilde{g}(k) \in T_k$ and hence $g(k) \in G_k$.

Clearly G is Π^1_1; so it remains to show that $|G_k|$ is hyperarithmetically bounded, independently of g. As $|G_k| = |T_k|$ for $k > n$ and $|G_k| = 1$ for $k \leq n$, this is a consequence of Lemma 4.8 of [9], reproduced below: □

Lemma 6.3 ([9]). *If $\tilde{V}$ is a measurable set with $\mu(\tilde{V}) < \frac{1}{4}$, and $T_k = \{l | \mu(B_{k,l} - \tilde{V}) < 2^{-(l+3)}\}$, then for $k \geq 1$, $|T_k| < 2^k k$.*

Corollary 6.4. *There exists a Δ^1_1-dominated real which is not Δ^1_1-traceable.*

Proof. By Theorem 4.2, Δ^1_1-dominated reals form a measure 1 set but, by Theorem 6.2, the set of Δ^1_1-traceable reals form a null set, being disjoint from the set of Δ^1_1-random reals. □

7. Concluding remarks

A real x is said to be Π^1_1-*random cuppable*, or *random cuppable* for short, if $x \oplus y \geq_h \mathcal{O}$ for all Π^1_1-random real y. It is known [5] that if x is low for Π^1_1-randomness then $\omega_1^{\mathrm{CK}} = \omega_1^x$. Harrington, Nies and Slaman have obtained a further result on low for Π^1_1 randomness. We include a proof of this result here.

Theorem 7.1 (with Harrington and Slaman). *A real x is low for Π^1_1-randomness if and only if x is low for Δ^1_1-randomness and not random cuppable.*

Proof. For the direction from left to right, suppose x is low for Π^1_1-randomness, that is, each Π^1_1 random real is $\Pi^1_1(x)$ random. Since $x \not\geq_h \mathcal{O}$, the $\Pi^1_1(x)$ set $\{y | y \oplus x \geq_h \mathcal{O}\}$ is null, by relativizing Theorems 2.3 and 2.4. Thus x is not random cuppable. To see that x is low for Δ^1_1-randomness, suppose for a contradiction that y is a Δ^1_1-random real that is not $\Delta^1_1(x)$-random. Thus there is a $\Delta^1_1(x)$-null set A containing y. By the main result in Martin-Löf [10], the null set $B = \bigcup\{C \subset 2^\omega | \mu(C) = 0 \wedge C \text{ is } \Delta^1_1\}$ is Π^1_1. Since $y \in A - B$, $A - B$ is a nonempty $\Sigma^1_1(x)$ set. By the Gandy Basis Theorem

2.1 relative to x, there is a real $z \in A - B$ so that $\omega_1^{z \oplus x} = \omega_1^x = \omega_1^{\mathrm{CK}}$. Then z is Δ^1_1-random but not $\Delta^1_1(x)$-random, so by Corollary 3.5 and its relativization to x, z is Π^1_1-random but not $\Pi^1_1(x)$-random, a contradiction.

For the other direction, suppose x is low for Δ^1_1-randomness and not random cuppable. Then $x \not\geq_h \mathcal{O}$. Suppose z is a Π^1_1 random real. By the proof of Theorem 3.4 relative to x, the largest $\Pi^1_1(x)$ null set $\mathcal{Q}(x)$ is a union of countably many $\Delta^1_1(x)$ null sets $\mathcal{Q}_n(x)$ and the $\Pi^1_1(x)$ null set $\{y | y \oplus x \geq_h \mathcal{O}\}$. Since x is low for Δ^1_1-randomness, $z \notin \bigcup_n \mathcal{Q}_n(x)$. Since x is non-Π^1_1-random cuppable, $z \oplus x \not\geq_h \mathcal{O}$. So z is $\Pi^1_1(x)$-random. $\square$

The following question remains open:

Question 7.2. *Is there a real x that is low for Π^1_1-randomness but not hyperarithmetical?*

Reimann and Slaman have shown that if x is not 1-random for any continuous measure, then x is hyperarithmetical. In an analogy, one can ask:

Question 7.3. *Is there a characterization of a real x that is not Π^1_1-ML or Δ^1_1 random for any continuous measure?*

One may also study higher genericity theory as has been done for classical genericity theory ([21] and [18]). The third author has proved that lowness for Π^1_1-genericity is the same as being hyperarithmetical and there exists a non-hyperarithmetical real that is low for Δ^1_1-genericity.

The results of the previous sections show that several of the key notions of randomness, demonstrably different in first order theory, coalesce into equivalent ones in effective descriptive set theory. Thus finer distinctions are revealed only at the arithmetic level. It is tempting to venture beyond Π^1_1 and Δ^1_1 and explore the landscape of definable randomness in the analytical hierarchy. However, this will lead us very quickly to statements undecidable in ZFC. Assuming *projective determinacy* (PD), Kechris [6] has proved several measure and category-theoretic results in the analytical hierarchy in parallel with results for the Π^1_1 case in [15].[1] We believe that most of the results proved in the previous sections remain valid upon replacing Π^1_1 with Π^1_{2n+1} or Σ^1_{2n} under PD. However it seems that PD is not a correct tool to use for analyzing the analytical sets since it provides limited recursion-theoretic information. For example, PD does not give a ramified analytical hierarchy with properties similar to what one has for Π^1_1 sets. Instead, some deep results in inner model theory are necessary for this. Inner model theory (say Q-theory [8]) has been applied by some to study descriptive set theory in order to obtain powerful characterizations of analytical sets (under large cardinal assumptions, see [8])[2]. The results are of recursion-theoretic interest, and this area is worth further investigation.

[1]Since one may apply PD to obtain some dynamic properties of Π^1_{2n+1} and Σ^1_{2n}-sets, such as scales (see [11]).

[2]We thank W. Hugh Woodin for pointing this out to us.

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