

# THE STRENGTH OF PROJECTIVE MARTIN CONJECTURE

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ABSTRACT. We show that Martin's conjecture on  $\Pi_1^1$ -functions uniformly  $\leq_T$ -order preserving on a cone implies  $\Pi_1^1$  Turing Determinacy over  $ZF + DC$ . In addition, it is also proved that for  $n \geq 0$ , this conjecture for uniformly degree invariant  $\Pi_{2n+1}^1$ -functions is equivalent over  $ZFC$  to  $\Sigma_{2n+2}^1$ -Axiom of Determinacy. As a corollary, the consistency of the conjecture for  $\Pi_1^1$ -uniformly degree invariant functions implies the consistency of the existence of a Woodin cardinal.

## 1. INTRODUCTION

A cone  $C$  of reals with base  $z$  is a set of the form  $\{x \mid x \geq_T z\}$  where  $\leq_T$  denotes Turing reducibility. A function  $F : 2^\omega \rightarrow 2^\omega$  is *degree invariant* on  $C$  if any two reals  $x, y \geq_T z$  of the same Turing degree satisfy  $F(x) \equiv_T F(y)$ . The degree invariance is *uniform* on  $C$  if there is a function  $t$  such that if  $x, y \geq_T z$ , then  $\Phi_i^x = y$  and  $\Phi_j^y = x$  implies  $\Phi_m^{F(x)} = F(y)$  and  $\Phi_n^{F(y)} = F(x)$ , where  $t(i, j) = (m, n)$ .  $F$  is *increasing* on  $C$  if  $F(x) \geq_T x$  for all  $x \geq z$ .  $F$  is *order preserving* on  $C$  if  $z \leq_T x \leq_T y$  implies  $F(z) \leq_T F(x) \leq_T F(y)$ . If this order preservation is witnessed by a function  $t : \omega \rightarrow \omega$ , i.e.,  $\Phi_e^x = y \geq_T z$  implies  $\Phi_{t(e)}^{F(x)} = F(y)$ , then it is *uniform* (note that a uniformly order preserving function is necessarily uniformly degree invariant). Finally, given functions  $F$  and  $G$  degree invariant on a cone, write  $F \geq_M G$  if  $F(x) \geq_T G(x)$  on a cone. Donald A. Martin conjectured that, under the assumption of  $ZF$  set theory plus the Axiom of Determinacy (AD) and Dependent Choice (DC),

- (1) Every degree invariant function that is not increasing on a cone is a constant on a cone, and
- (2)  $\leq_M$  prewellorders degree invariant functions which are increasing on a cone. Furthermore, if the  $\leq_M$ -rank of  $F$  is  $\alpha$ , then  $F'$  has  $\leq_M$ -rank  $\alpha + 1$ , where  $F'(x) = (F(x))'$ , the Turing jump of  $F(x)$ .

Slaman and Steel [6] proved (1) for functions which are uniformly degree invariant on a cone and (2) for Borel functions which are increasing and order preserving. In [7] Steel showed (2) for uniformly degree invariant functions and conjectured that every function degree invariant on a cone is uniformly degree invariant on a cone.

While Martin [4] has shown that Borel determinacy is a theorem of  $ZF + DC$  (hence conjectures (1) and (2) hold for  $\Delta_1^1$  functions that are uniformly degree invariant),

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it is known that AD in the analytical hierarchy beyond  $\Delta_1^1$  is a large cardinal axiom. An analysis of the proof in [7] shows that conjecture (2) for uniformly degree invariant  $\Pi_{2n+1}^1$  functions follows from  $\Delta_{2n+2}^1$  Determinacy. Thus a natural question for Martin's Conjectures (1) and (2) is their set-theoretic strength for uniformly degree invariant functions beyond  $\Delta_1^1$  in the analytical hierarchy. There is also a related question concerning the more restrictive uniformly order preserving functions, i.e. while (2) holds for such functions under AD according to Steel [7], the set-theoretic strength of (2) for these functions has not been considered.

A set of reals is degree invariant if it is closed under Turing equivalence. Martin [3] showed that under AD, every degree invariant set of reals either contains or is disjoint from a cone. By  $\Pi_{2n+1}^1$ -Turing Determinacy ( $\Pi_{2n+1}^1$ -TD) we mean the assertion that every  $\Pi_{2n+1}^1$  set of reals that is degree invariant either contains or is disjoint from a cone. We show in this paper that Conjecture (2) for uniformly order preserving  $\Pi_1^1$  functions implies the existence of  $0^\#$ . Relativizing the argument to arbitrary reals  $x$  leads to the conclusion that  $x^\#$  exists for every  $x$ , so that by Harrington [1] the truth of Conjecture (2) for  $\Pi_1^1$  uniformly order preserving functions implies  $\Pi_1^1$ -TD. We also show that in general, for  $n \geq 0$ , Conjecture (2) for  $\Pi_{2n+1}^1$ -uniformly degree invariant functions implies  $\Sigma_{2n+2}^1$ -TD, assuming  $\Pi_{2n+1}^1$ -uniformization when  $n \geq 1$ . In fact, by this, Steel [7] and an unpublished work of W. H. Woodin, one concludes that Martin's conjecture (2) for  $\Pi_{2n+1}^1$ -uniformly degree invariant functions is equivalent to  $\Sigma_{2n+2}^1$ -AD.

We recall some facts and notations (see Sacks [5] which is used as the standard reference in this paper). For each real  $x$ ,  $\omega_1^x$  denotes the least ordinal  $\alpha$  for which  $L_\alpha[x]$  is admissible. Kleene constructed a  $\Pi_1^1(x)$  complete set  $\mathcal{O}^x$  with a  $\Pi_1^1(x)$  well founded relation  $<_{\mathcal{O}^x}$  on  $\mathcal{O}^x$ .  $\mathcal{O}^x$  is the hyperjump of  $x$ . The height of the ordering  $<_{\mathcal{O}^x}$  on  $\mathcal{O}^x$  is exactly  $\omega_1^x$ . Furthermore, Kleene's construction of  $\mathcal{O}^x$  is uniform. In other words, the relation  $\{(x, \mathcal{O}^x) \mid x \in 2^\omega\}$  is  $\Pi_1^1$ . A fact that will be used implicitly is that given reals  $x$  and  $y$ ,  $x$  is hyperarithmetic in  $y$  (written  $x \leq_h y$ ) if and only if  $x$  is  $\Delta_1^1$  in  $y$ , and this is in turn equivalent to  $x \in L_{\omega_1^y}[y]$ . We work under ZF + DC. As we will only be concerned with Conjecture (2), it will be referred to as the  $\leq_M$  Conjecture from here on.

## 2. THE $\leq_M$ CONJECTURE FOR UNIFORMLY ORDER PRESERVING $\Pi_1^1$ FUNCTIONS

Let

$$\mathcal{F} = \{x \mid \forall \alpha < \omega_1^x \forall a \subseteq \alpha (a \in L_{\omega_1^x} \implies a \in L_{\alpha+3}[x])\}.$$

$\mathcal{F}$  is a degree invariant  $\Sigma_1^1$  set introduced by H. Friedman [2]. We give a simpler proof of the following result given as Lemma 7.17 in [2].

**Lemma 2.1.**  *$\mathcal{F}$  is cofinal in the Turing degrees.*

*Proof.* For any real  $z$ , let

$$\mathcal{F}(z) = \{x \oplus z \mid \forall \alpha < \omega_1^z \forall a \subseteq \alpha (a \in L_{\omega_1^z} \implies a \in L_{\alpha+3}[x \oplus z])\}$$

be a degree invariant  $\Sigma_1^1(z)$  set. Obviously  $\mathcal{F}(z)$  is not empty. By the Gandy Basis Theorem relativized to  $z$ , there is an  $x$  such that  $\omega_1^{x \oplus z} = \omega_1^z$  and  $x \oplus z \in \mathcal{F}(z)$ . Then  $x \oplus z \in \mathcal{F}$ .  $\square$

The proof of the following lemma follows from Lemmas 7.20–7.22 in [2].

**Lemma 2.2.** *If  $0^\sharp$  does not exist, then  $\bar{\mathcal{F}} = 2^\omega - \mathcal{F}$  is cofinal in the Turing degrees.*

For  $x$  a real and  $n \in \omega$ , let  $x^{[n]}$  be the real such that  $x^{[n]}(i) = x(\langle n, i \rangle)$ .

**Theorem 2.3.** *If the  $\leq_M$  Conjecture holds for  $\Pi_1^1$  uniformly order preserving functions, then  $0^\sharp$  exists.*

*Proof.* If  $0^\sharp$  does not exist, then by Lemmas 2.1 and 2.2, both  $\mathcal{F}$  and  $\bar{\mathcal{F}}$  are cofinal. For a contradiction, we will define a  $\Pi_1^1$ -function  $G$  that is uniformly order preserving such that  $\{x \mid G(x) = O^{O^x}\}$  and  $\{x \mid G(x) = O^{O^{O^x}}\}$  are both cofinal in the Turing degrees,

Let  $P(x, y)$  be an arithmetic predicate such that

$$x \in \bar{\mathcal{F}} \Leftrightarrow \forall y P(x, y).$$

**Claim 2.4.** *If  $x \leq_T y$  are such that  $x \in \mathcal{F}$  and  $y \in \bar{\mathcal{F}}$ , then  $O^x \leq_h y$ .*

As  $y \notin \mathcal{F}$ , there are  $\alpha < \omega_1^y$  and  $a \subseteq \alpha$  with  $a \in L_{\omega_1^y} \setminus L_{\alpha+3}[y]$ . So  $a \notin L_{\alpha+3}[x]$  and  $\omega_1^x \leq \alpha < \omega_1^y$ . As  $x \leq_T y$ ,  $O^x \leq_h y$ .

By Claim 2.4, if  $x \leq_T y$  are such that  $x \in \mathcal{F}$  and  $y \in \bar{\mathcal{F}}$ , then  $O^{O^x} \leq_T O^y$ .

Now define  $G(x) = y$  as follows:

- (1)  $y^{[0]} = \langle 0 \rangle \wedge O^{O^x} \wedge x \in \bar{\mathcal{F}}$  or  $y^{[0]} = \langle 1 \rangle \wedge O^{O^{O^x}} \wedge \exists v \leq_T y^{[0]} \neg P(x, v)$ . Thus  $y^{[0]}$  gives a  $\Pi_1^1$  differentiation between  $x \in \bar{\mathcal{F}}$  and  $x \in \mathcal{F}$ ;
- (2)  $y^{[1]} = O^x$ ,
- (3) If  $\Phi_e^x$  is partial then let  $y^{[e+2]} = \emptyset$ ,
- (4) If  $\Phi_e^x$  is total and equal to  $u$ , the following three cases differentiate in a  $\Pi_1^1$  way between  $u \in \mathcal{F}$  and  $x \in \bar{\mathcal{F}}$ ,  $u, x \in \mathcal{F}$ , and  $u \in \bar{\mathcal{F}}$  for all  $u \leq_T x$ :
  - (a)  $y^{[0]}(0) = 0 \wedge \exists v \leq_T y^{[1]} \neg P(u, v) \wedge y^{[e+2]} = \langle 1 \rangle \wedge O^{\Phi_i^{y^{[1]}}}$  where  $i$  is the least index so that  $O^{O^u} = O^{\Phi_i^{y^{[1]}}}$ , or
  - (b)  $y^{[0]}(0) = 1 \wedge \exists v \leq_T y^{[1]} \neg P(u, v) \wedge y^{[e+2]} = \langle 1 \rangle \wedge O^{O^{O^u}}$ , or
  - (c)  $\forall v \leq_T y^{[1]} P(u, v) \wedge y^{[e+2]} = \langle 0 \rangle \wedge O^{O^u}$ .

$G(x)$  is obviously  $\Pi_1^1$ .

**Claim 2.5.** *If  $x \in \mathcal{F}$  then  $G(x) \equiv_T O^{O^{O^x}}$ .*

Clearly  $x \in \mathcal{F}$  implies  $O^{O^{O^x}} \leq_T G(x)$  and  $G(x)^{[0]} \oplus G(x)^{[1]} \leq_T O^{O^{O^x}}$ .

Given  $e < \omega$ ,  $O^{O^{O^x}}$  can uniformly decide whether  $\Phi_e^x$  is total. Suppose that  $\Phi_e^x$  is total. To calculate  $G(x)^{[e+2]}(n)$ , one verifies clauses (4b), (4c) above. But the predicate  $\forall v \leq_T y^{[1]} P(u, v)$  is  $\Delta_1^1(O^x)$ , hence recursive in  $O^{O^{O^x}}$ . Once this predicate is decided,  $O^{O^{O^x}}$  may use recursive functions  $f$  and  $g$ , where  $u = \Phi_e^w \rightarrow O^{O^u} = \Phi_{f(e)}^{O^{O^w}}$  and  $u = \Phi_e^w \rightarrow O^{O^{O^u}} = \Phi_{g(e)}^{O^{O^{O^w}}}$ , to finish the calculation.

**Claim 2.6.** *If  $x \in \bar{\mathcal{F}}$  then  $G(x) \equiv_T O^{O^x}$ .*

This is similar to the above claim, except for the final step calculating  $G(x)^{[e+2]}(n)$ .

Now  $O^{O^x}$  is able to decide whether (4a) or (4c) holds, as in the above claim. If (4c) holds, the calculation is the same. If (4a) holds, then  $u \in \mathcal{F}$ . By Claim 2.4,  $O^u \leq_h x$  and thus  $O^{O^u} \leq_T O^x = G(x)^{[1]}$ . So  $i$  exists. Moreover, the search for  $i$  is a procedure uniformly  $\Pi_1^1(O^x)$ . Hence  $O^{O^x}$  uniformly computes  $G(x)^{[e+2]}(n)$ .

It follows from the above two claims that  $G$  is degree invariant. Moreover  $G$  preserves  $\leq_T$  by Claim 2.4.

To show that  $G$  is uniformly order preserving, let  $h$  be a recursive function such that  $\forall x, y, e (x = \Phi_e^y \rightarrow O^x = \Phi_{h(e)}^{O^y})$ . In addition, let  $s$  be recursive with

$$\forall x, y, e, i (x = \Phi_e^y \rightarrow \Phi_i^x = \Phi_{s(e,i)}^y).$$

Suppose that  $x = \Phi_e^y$ . Then

- (1)  $G(x)^{[0]} = G(y)^{[e+2]}$ ,
- (2)  $G(x)^{[1]} = \Phi_{h(e)}^{G(y)^{[1]}}$ , and
- (3)  $G(x)^{[i+2]} = G(y)^{[s(e,i)+2]}$ .

Hence  $G$  is as desired.  $\square$

The above proof easily relativizes to any real  $x$  to guarantee the existence of  $x^\#$ . Since Harrington [1] has shown that the existence of sharps implies  $\Pi_1^1$ -TD, we have

**Corollary 2.7.** *If the  $\leq_M$  Conjecture holds for  $\Pi_1^1$  functions which are uniformly order preserving, then  $\Pi_1^1$ -TD is true.*

### 3. THE $\leq_M$ CONJECTURE FOR $\Pi_{2n+1}^1$ FUNCTIONS AND $\Sigma_{2n+2}^1$ -TD

**Lemma 3.1.**  *$\Pi_{2n+1}^1$ -uniformization and  $\Delta_{2n+2}^1$ -TD imply  $\Sigma_{2n+2}^1$ -TD for  $n \in \omega$ .*

*Proof.* Let  $A \in \Sigma_{2n+2}^1$  be degree invariant and  $\leq_T$ -cofinal. Define

$$R(x, y) \Leftrightarrow x \leq_T y \wedge y \in A.$$

So  $R(x, y) \in \Sigma_{2n+2}^1$ . Note that  $\Pi_{2n+1}^1$ -uniformization implies  $\Sigma_{2n+2}^1$ -uniformization. Let  $F \in \Sigma_{2n+2}^1$  uniformize  $R$ .  $F$  is actually a  $\Delta_{2n+2}^1$  function.

Define

$$B = \{u \mid \exists x \leq_T u, y \equiv_T u (F(x) = y)\}.$$

Then  $B$  is  $\Delta_{2n+2}^1$ , degree invariant and  $\leq_T$ -cofinal. Moreover  $B \subseteq A$ . By  $\Delta_{2n+2}^1$ -TD,  $B$  contains a cone of Turing degrees. Hence so does  $A$ .  $\square$

**Corollary 3.2.**  *$\Delta_2^1$ -TD implies  $\Sigma_2^1$ -TD.*

*Proof.* As  $\Pi_1^1$ -uniformization is a theorem of ZFC, the corollary follows immediately from Theorem 3.1.  $\square$

We prove the next result for the lightface version. The proof for the boldface version follows with obvious changes.

**Theorem 3.3.** *Assume  $\Pi_{2n+1}^1$ -uniformization. If the  $\leq_M$  Conjecture holds for uniformly degree invariant  $\Pi_{2n+1}^1$  functions, then  $\Sigma_{2n+2}^1$ -TD holds.*

*Proof.* Let  $A \in \Delta_{2n+2}^1$ , and  $P, Q \in \Pi_{2n+1}^1$  are such that

$$x \in A \Leftrightarrow \exists y P(x, y) \Leftrightarrow \forall y \neg Q(x, y).$$

Let  $R(x, y) \Leftrightarrow P(x, y) \vee Q(x, y)$ . By  $\Pi_{2n+1}^1$ -uniformization, let  $F \in \Pi_{2n+1}^1$  uniformize  $R$ . Define  $J_0(x) = z$  if and only if  $z^{[0]} = F(x)$  and

$$\forall e ((\Phi_e^x \text{ is total} \rightarrow z^{[e+1]} = F(\Phi_e^x)) \wedge (\Phi_e^x \text{ is partial} \rightarrow z^{[e+1]} = \emptyset)).$$

Obviously  $J_0 \in \Pi_{2n+1}^1$  is total. Moreover  $J_0$  is uniformly order preserving. To see this, let  $f$  be a recursive function such that

$$\forall e, x_0, x_1 (x_0 = \Phi_e^{x_1} \rightarrow \forall i (\Phi_{f(e,i)}^{x_1} \simeq \Phi_i^{x_0})).$$

Suppose  $x_0 = \Phi_e^{x_1}$ . Then  $(J_0(x_0))^{[0]} = (J_0(x_1))^{[e]}$  and  $(J_0(x_0))^{[i+1]} = (J_0(x_1))^{[f(e,i)+1]}$ . Thus  $J_0(x_0)$  may be effectively computed from  $J_0(x_1)$ .

Let  $g$  be a recursive function such that  $x_0 = \Phi_e^{x_1} \rightarrow J_0(x_0) = \Phi_{g(e)}^{J_0(x_1)}$ .

Define  $J(x) = x \oplus z_0 \oplus z_1$  if and only if  $z_0 = J_0(x)$  and

$$(P(x, z_0^{[0]}) \wedge z_1 = \emptyset) \vee (Q(x, z_0^{[0]}) \wedge z_1 = \langle 1 \rangle \wedge (x \oplus z_0)').$$

Note that  $J \in \Pi_{2n+1}^1$ . We claim that  $J$  is uniformly degree invariant. To see this, let  $h$  be a recursive function such that  $x_0 = \Phi_e(x_1) \rightarrow (x_0 \oplus J_0(x_0))' = \Phi_{h(e)}^{(x_1 \oplus J_0(x_1))'}$ . For each  $e$ , let  $t(e)$  be the index of the procedure  $\Psi$  defined by:

- (1)  $(\Psi^z)^{[0]} = \Phi_e^{z^{[0]}}$  and  $(\Psi^z)^{[1]} = \Phi_{g(e)}^{z^{[1]}}$ , and
- (2) if  $z^{[2]}(0) = 0$  then  $(\Psi^z)^{[2]} = \emptyset$ . Otherwise  $(\Psi^z)^{[2]} = \langle 1 \rangle \wedge \Phi_{h(e)}^w$ , where  $w$  is such that  $\langle 1 \rangle \wedge w = z^{[2]}$ .

If the  $\leq_M$  Conjecture holds for uniformly degree invariant  $\Pi_{2n+1}^1$  functions, then eventually  $J$  is either  $x \mapsto x \oplus J_0(x)$  or  $x \mapsto x \oplus J_0(x) \oplus (x \oplus J_0(x))'$ . Hence  $A$  either contains or avoids a cone of Turing degrees.

Thus we have  $\Delta_{2n+2}^1$ -TD.  $\Sigma_{2n+2}^1$ -TD follows from Lemma 3.1.  $\square$

**Corollary 3.4.** *Let  $n \geq 0$ . The  $\leq_M$  Conjecture for  $\Pi_{2n+1}^1$  uniformly degree invariant functions is equivalent to  $\Sigma_{2n+2}^1$ -AD.*

*Proof.* An analysis of Theorem 1 in Steel [7] shows that  $\Sigma_{2n+2}^1$ -AD (in fact  $\Delta_{2n+2}^1$ -AD) implies the  $\leq_M$  Conjecture for  $\Pi_{2n+1}^1$  uniformly degree invariant functions. We show the converse by induction on  $n$ : First note that if  $n = 0$ , then  $\Pi_1^1$ -uniformization is the Kondo-Addison Theorem, so that by Theorem 3.3,  $\Sigma_2^1$ -TD holds. Now assume by induction that  $\Sigma_{2n}^1$ -TD is true. Woodin (unpublished) has shown that over ZFC, for  $k \geq 1$ ,  $\Sigma_{2k}^1$ -TD is equivalent to  $\Sigma_{2k}^1$ -AD, and Moschovakis [?] has shown (Chapter 6) that  $\Pi_{2k+1}^1$ -uniformization is a consequence of  $\Sigma_{2k}^1$ -AD. Thus  $\Pi_{2n+1}^1$ -uniformization holds and so Theorem 3.3 yields  $\Sigma_{2n+2}^1$ -TD, hence  $\Sigma_{2n+2}^1$ -AD.  $\square$

The following corollary gives the consistency strength of the  $\leq_M$  Conjecture.

**Corollary 3.5.** *If it is consistent that the  $\leq_M$  Conjecture holds for  $\Pi_1^1$ -uniformly degree invariant functions, then it is consistent that there is a Woodin cardinal.*

*Proof.* The hypothesis and Theorem 3.3 imply that  $\Pi_2^1$ -TD is consistent. Woodin has shown that  $\Pi_2^1$ -TD is equiconsistent with the existence of a Woodin cardinal.  $\square$

**Remark.** We do not know if the conclusion of Corollary 2.7 may be strengthened to  $\Delta_2^1$ -TD (hence  $\Delta_2^1$ -AD). If this is true, then by Steel [7] it will give a characterization of the  $\leq_M$  Conjecture for  $\Pi_1^1$  uniformly order preserving functions. In general, one would like to understand better the role of order preserving functions in the study of the  $\leq_M$  Conjecture. For example, it is not clear if Corollary 3.4 applies to functions which are order preserving.

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