RANDOMNESS IN THE HIGHER SETTING

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Abstract. We study the strengths of various notions of higher randomness: (i) strong Π^1_1-ML-randomness is separated from Π^1_1-ML-randomness; (ii) the hyperdegrees of Π^1_1-random reals are closed downwards (except for the trivial degree); (iii) the reals z in $NCR_{\Pi^1_1}$ are precisely those satisfying $z \in L_{\omega_1^z}$, and (iv) lowness for Δ^1_1-randomness is strictly weaker than that for Π^1_1-randomness.

1. Introduction

Randomness in the higher setting refers to the study of algorithmic randomness properties of reals from the point of view of effective descriptive set theory. Until recently, the study of algorithmic randomness has been focused on reals in the arithmetical hierarchy. The only exception was a paper by Martin-Löf [12], in which he showed the intersection of a sequence of Δ^1_1-sets of reals to be Σ^1_1 (Sacks [17] introduced the notion of Π^1_1 and Δ^1_1-randomness in two exercises). The first systematic study of higher randomness appeared in Hjorth and Nies [9] where the notion of Π^1_1-Martin-Löf randomness was defined and the key properties investigated. The paper also studied the stronger notion of Π^1_1-randomness and showed the existence of a universal test for Π^1_1-random reals. In Chong, Nies and Yu [2] the authors examined the relative strengths of Π^1_1-Martin-Löf randomness, Π^1_1 and Δ^1_1-randomness, as well as their associated notions of lowness.

Effective descriptive set theory offers a natural and different platform for the study of algorithmic randomness. Since the Gandy-Spector Theorem injects a new perspective to Π^1_1-sets of natural numbers, viewing them as Σ_1-definable subsets of $L_{\omega_1^{\mathrm{CK}}}$ and therefore recursively enumerable (r.e.) in the larger universe, the tools of hyperarithmetic theory are readily available for the investigation of random reals in the generalized setting. Just as arithmetical randomness has drawn new insights into the structure of Turing degrees below $\mathbf{0^{(n)}}$ (for $n < \omega$), the study of higher randomness properties has enhanced our understanding of hyperdegrees and Π^1_1-sets of reals, a point which we hope results presented in this paper will convey.

We consider several basic notions of randomness (see the next section for the definitions). In [2] it was shown that Π^1_1-Martin-Löf randomness, Π^1_1 and Δ^1_1-randomness are equivalent for reals x if and only if $\omega_1^x = \omega_1^{\mathrm{CK}}$. In [13], Nies introduced another notion called *strong* Π^1_1-Martin-Löf randomness which is an analog of weak 2-randomness

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in the literature. We prove (Theorem 3.5) that every hyperdegree greater than or equal to the hyperdegree of Kleene's $\mathcal{O}$ contains a real that is Π^1_1 but not strongly Π^1_1-Martin-Löf random, thus separating these two notions of randomness. In Theorem 4.4, we show that every nontrivial hyperdegree below the hyperdegree of a Π^1_1-random real contains a Π^1_1-random real. Such a downward closure property is not shared by weaker notions such as Π^1_1-Martin-Löf randomness. In fact, every nontrivial real below a Π^1_1-random is Π^1_1-random relative to a measure (Corollary 4.5), so that such reals are still essentially random. This result is strengthened in Theorem 5.1: We characterize the class $NCR_{\Pi^1_1}$ of reals x which are not Π^1_1-random relative to any representation of a continuous measure to be precisely those which satisfy $x \in L_{\omega_1^x}$. Our final result (Theorem 6.5) separates the notion of low for Δ^1_1-randomness from that of low for Π^1_1-randomness. To obtain this, we prove a general theorem about hyperdegrees (Theorem 6.3) which states that any two uncountable Σ^1_1-set of reals generate the cone of hyperdegrees with base the hyperdegree of Kleene's $\mathcal{O}$. The latter has its root in a result of Martin [11] that every uncountable Δ^1_1-set of reals contains a member of each hyperdegrees greater than or equal to the degree of $\mathcal{O}$. The paper concludes with a list of questions.

2. Preliminaries

We assume that the reader is familiar with hyperarithmetic theory and randomness theory. For a general reference, refer to [5], [13], [17] or [3]. The notations adopted are standard. Reals are denoted $x, y, z, \ldots$ A tree T is a subset of $2^{<\omega}$ or $\omega^{<\omega}$. $[T]$ denotes the set of infinite paths on T. By abuse of notation, we also write $x \in T$ (or $x \in U$) if the context is clear. We use $k \gg n$ to express the fact that the number k is "much bigger than" n. If λ is a measure on the Cantor space 2^ω, and $\sigma \in 2^{<\omega}$, denote $\lambda(\sigma)$ to be the measure of λ on the basic open set $\{x \mid \sigma \prec x\}$. We also let $[\sigma]$ denote the set of binary strings extending σ.

Definition 2.1. *Given a measure λ on 2^ω, a real $\hat{\lambda}$* represents *λ if for any $\sigma \in 2^{<\omega}$ and rational numbers p, q, $\langle \sigma, p, q\rangle \in \hat{\lambda} \Leftrightarrow p < \lambda(\sigma) < q$.*

Given a representation $\hat{\lambda}$ of a measure λ, one may define the notion of a $\hat{\lambda}$-Martin-Löf test as usual. More details can be found in [14].

Definition 2.2. *(i) A* Π^1_1-ML-test *is a sequence $\{U_m\}_{m\in\omega}$ of uniformly Π^1_1-open sets such that $\forall m(\mu(U_m) < 2^{-m})$. We say that x is* Π^1_1-ML random *if $x \notin \bigcap_m U_m$ for every such collection $\{U_m\}$, i.e. if x passes every Π^1_1-ML test.*

(ii) ([13, Problem 9.2.17]) *$\{U_m\}$ is a* Π^1_1-generalized ML test *if in (i) we have $\lim_m \mu(U_m) = 0$ instead. We say that x is* strongly Π^1_1-ML-random *if x passes every generalized Π^1_1-ML-test.*

Definition 2.2 (ii) is a generalization of the notion of weakly 2-randomness for reals, when Π^1_1 is replaced by r.e. (over L_ω). One may refine Definition 2.2 (i) as follows. A Δ^1_1-ML-test is obtained when Π^1_1 in the definition is replaced by Δ^1_1. Indeed, if $\{U_n\}_{n\in\omega}$ is a Δ^1_1-ML-test, then there is a recursive ordinal α such that $\{U_n\}_{n\in\omega}$ is uniformly $\emptyset^{(\alpha)}$-r.e. We call such a test a $\emptyset^{(\alpha)}$-ML-test. A real x is Δ^1_1-ML-random if it passes every Δ^1_1-ML-test. If x is not Δ^1_1-ML-random, then there is an $\alpha < \omega_1^{\mathrm{CK}}$ and an $\emptyset^{(\alpha)}$-ML-test in which x fails. This fact will be used in Section 4.

Definition 2.3. (Hjorth and Nies in [9]) *A real x is* Π^1_1-random *if it does not belong to any null Π^1_1-set of reals.*

Clearly $\bigcap_{m\in\omega} U_m$ is Π^1_1 for any uniformly Π^1_1-collection of Π^1_1-open sets, so that Π^1_1-randomness implies strong Π^1_1-ML-randomness. We say that a real x is Δ^1_1-dominated if every function hyperarithmetic in x is dominated by a hyperarithmetic function. As usual, ω_1^x is the least ordinal which is not an x-recursive ordinal, and Church-Kleene ω_1 is $\omega_1^{\emptyset}$ which is always denoted ω_1^{CK}. By a result in [2], we have the following proposition.

Proposition 2.4 (Chong, Nies and Yu)**.** *If $\omega_1^x = \omega_1^{\mathrm{CK}}$, then x is Π^1_1-ML-random if and only if it is Π^1_1-random. Moreover, each Π^1_1-random real is Δ^1_1-dominated.*

The Gandy Basis Theorem plays an important role in our present study:

Theorem 2.5 (Gandy [7])**.** *If $A \subseteq 2^\omega$ is a nonempty Σ^1_1-set, then there is an $x \in A$ such that $\omega_1^x = \omega_1^{\mathrm{CK}}$.*

Let L_α be the Gödel constructibility hierarchy at level α. The following is a set-theoretic characterization of Π^1_1-sets.

Theorem 2.6 (Spector[19] , Gandy [8])**.** *A set $A \subseteq 2^\omega$ is Π^1_1 if and only if there is a Σ_1-formula φ such that $x \in A \Leftrightarrow L_{\omega_1^x}[x] \models \varphi$.*

We use $\leq_h$ to denote hyperarithmetic reduction. $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$ is the structure for the ramified analytical hierarchy relative to x. For more details concerning the ramified analytical hierarchy, see [17].

If T is a tree that is $\Pi_1(L_{\omega_1^{\mathrm{CK}}})$-definable, then there is an effective enumeration over $L_{\omega_1^{\mathrm{CK}}}$ of the nodes not in T. For any $\gamma < \omega_1^{\mathrm{CK}}$, let $T[\gamma]$ be the Δ_1-tree which is an approximation of T at stage γ. Then $T = \bigcap_{\gamma<\omega_1^{\mathrm{CK}}} T[\gamma]$.

3. Strong Π^1_1-ML-randomness

In Nies [13], Problem 9.2.17 asks

Question 3.1. *Is strong Π^1_1-ML-randomness equivalent to Π^1_1-ML-randomness?*

The question was motivated by the following consideration. In the standard argument separating weak 2-randomness from ML-randomness, one exploits the fact that the rate of convergence of $\mu(U_n)$ (the measure of U_n) to 0 can be coded by the "size of the space" available to U_n, where $\{U_n\}_{n\in\omega}$ is a test designed to exhibit an ML-random real that is not weakly 2-random. Such an approach is no longer possible in the present setting, since U_n is now enumerated in ω_1^{CK}, instead of ω, -many stages. The following result leads to a negative solution.

Theorem 3.2. [1] *If x is the leftmost path of a Σ^1_1-closed set of reals, then x is not strongly Π^1_1-ML-random.*

[1] Bienvenu, Greenberg and Monin [1] have a shorter proof of this theorem.

The proof is measure-theoretic. More than separating the two notions of randomness, a measure-theoretic proof extracts useful information about the distribution of strong Π^1_1-ML random reals in the hyperdegrees. We first give a criterion for a uniformly Π^1_1-sequence of open sets to be a generalized Π^1_1-Martin-Löf test. This lemma will also be applied to show Theorem 3.5.

Lemma 3.3. *Suppose that $\{U_n\}_{n\in\omega}$ is a uniformly Π^1_1-sequence of open sets. If there is a $\Sigma_1(L_{\omega_1^{\mathrm{CK}}})$ enumeration $\{\hat{U}_{n,\gamma}\}_{n<\omega,\gamma<\omega_1^{\mathrm{CK}}}$ of the sequence with two numbers k and $m\geq 1$ such that for every n, $U_n=\bigcup_{\gamma<\omega_1^{\mathrm{CK}}}\hat{U}_{n,\gamma}$ and for every $\gamma<\omega_1^{\mathrm{CK}}$:*

(a) *$\hat{U}_{n+1,\gamma}\subseteq\hat{U}_{n,\gamma}$ and each string in $\hat{U}_n$ has length at least $2^{k\cdot n}$,*
(b) *$\forall\sigma\in 2^{k\cdot n-m}(\mu(\hat{U}_{n,\gamma}\cap[\sigma])<2^{-1+m-k\cdot n})$, and*
(c) *For $\gamma<\omega_1^{\mathrm{CK}}$ and any real z, if $z\in\hat{U}_{n,<\gamma}\setminus\hat{U}_{n,\gamma}$, where $\hat{U}_{n,<\gamma}=\bigcup_{\beta<\gamma}\hat{U}_{n,\beta}$, then $z\notin\hat{U}_{n,\beta}$ for any $\beta\geq\gamma$.*

Then $\{U_n\}_{n\in\omega}$ is a generalized Π^1_1-ML-test.

Proof. Note that by (c) the enumeration $\{\hat{U}_{n,\gamma}\}$ of U_n is not cumulative. Assume $\mu(\bigcap_{n\in\omega}U_n)>0$ for a contradiction. We will exhibit an infinite descending sequence of ordinals $\{\gamma_n\}_{n<\omega}$. First of all, the assumption implies that there is a σ_0 such that

$$\mu(\bigcap_{n\in\omega}U_n\cap[\sigma_0])>2^{-|\sigma_0|}\cdot(1-2^{-3}).$$

Moreover, we may assume that k divides $|\sigma_0|+m$. Let $n_0=\frac{|\sigma_0|+m}{k}$. Then there is a least $\gamma_0<\omega_1^{\mathrm{CK}}$ such that

$$\mu(\hat{U}_{n_0,\leq\gamma_0}\cap[\sigma_0])>\frac{7}{8}\cdot 2^{-|\sigma_0|}.$$

By (b),

$$\mu((\hat{U}_{n_0,<\gamma_0}\setminus\hat{U}_{n,\gamma_0})\cap[\sigma_0])>2^{-|\sigma_0|}\cdot(1-2^{-1}-2^{-3})\geq\frac{3}{8}\cdot 2^{-|\sigma_0|}.$$

By (a) and (c),

$$\mu(\bigcap_{n>n_0}\hat{U}_{n,<\gamma_0}\cap[\sigma_0])>(\frac{7}{8}-\frac{5}{8})\cdot 2^{-|\sigma_0|}=\frac{1}{4}\cdot 2^{-|\sigma_0|}.$$

Hence there is a $\sigma_1\succ\sigma_0$ such that

$$\mu(\bigcap_{n>n_0}\hat{U}_{n,<\gamma_0}\cap[\sigma_1])>\frac{7}{8}\cdot 2^{-|\sigma_0|}.$$

We may assume that k divides $|\sigma_1|+m$ and $|\sigma_1|\gg|\sigma_0|$. Let $n_1=\frac{|\sigma_1|+m}{k}\gg n_0$. Then there is a least $\gamma_1<\gamma_0$ such that

$$\mu(\hat{U}_{n_1,\leq\gamma_1}\cap[\sigma_1])>\frac{7}{8}\cdot 2^{-|\sigma_1|}.$$

Repeating the argument, we obtain an infinite descending sequence $\gamma_0>\gamma_1>\cdots$, which is not possible. ∎

Proof. (of Theorem 3.2).

Let $T \subseteq 2^{<\omega}$ be a Σ^1_1-tree. For any $n < \omega$ and $\gamma < \omega_1^{\mathrm{CK}}$, let

$$\hat{U}_{n,\gamma} = \{\sigma \mid \exists z (z \text{ is the leftmost path in } T[\gamma] \wedge \sigma \upharpoonright n+1 = z \upharpoonright n+1)\}.$$

Define

$$\hat{U}_{n,<\gamma} = \bigcup_{\beta<\gamma} \hat{U}_{n,\beta}$$

and

$$U_n = \bigcup_{\gamma<\omega_1^{\mathrm{CK}}} \hat{U}_{n,\gamma}.$$

The following facts are immediate.

(1) For any n and $\gamma < \omega_1^{\mathrm{CK}}$, $\hat{U}_{n+1,\gamma} \subseteq \hat{U}_{n,\gamma}$ and every string in $\hat{U}_n$ has length at least 2^n;

(2) $\forall \sigma \in 2^{n-1}(\mu(\hat{U}_{n,\gamma} \cap [\sigma]) < 2^{-n})$;

(3) For any n, $\gamma < \omega_1^{\mathrm{CK}}$ and real z, if $z \in \hat{U}_{n,<\gamma} \setminus \hat{U}_{n,\gamma}$, then $z \notin \hat{U}_{n,\beta}$ for any $\beta \geq \gamma$.

Clearly $\{U_n\}_{n\in\omega}$ is uniformly Π^1_1. By (1)—(3) and setting $k = m = 1$ in Lemma 3.3, i $\{U_n\}_{n\in\omega}$ is a generalized Π^1_1-ML-test. Obviously $x \in \bigcap_{n\in\omega} U_n$. We conclude that x is not strongly Π^1_1-ML-random. □

Corollary 3.4. *Π^1_1-ML-randomness is strictly weaker than strong Π^1_1-ML-randomness.*

Proof. By a result in [9], there is a Σ^1_1-tree T such that $[T]$ is uncountable and consists entirely of Π^1_1-ML-random reals. According to Theorem 3.2, the leftmost path in T is not strongly Π^1_1-ML-random. □

We give another application of Lemma 3.3. The theorem may be proved by combining results in [1] and [9]. We give a direct proof here.

Theorem 3.5. *For any real $x \geq_h \mathcal{O}$, there is a Π^1_1-ML-random $y \equiv_h x$ which is not strongly Π^1_1-ML-random.*

Proof. Given a tree T, let $\mathcal{T}(T)$ be the smallest subtree of T such that

- $\emptyset \in \mathcal{T}(T)$, and
- For $\sigma \in \mathcal{T}(T)$, let $V_\sigma = \{\nu \mid \nu \succ \sigma \wedge |\nu| = |\sigma| + 2 \wedge [\nu] \cap T \text{ is infinite}\}$. If τ is the leftmost or rightmost string in V_σ, then $\tau \in \mathcal{T}(T)$.

Now let $T \subseteq 2^{<\omega}$ be a Σ^1_1-tree of positive measure so that $[T]$ consists entirely of Π^1_1-ML-random reals. Note that T has no isolated infinite paths.

For any $\gamma < \omega_1^{\mathrm{CK}}$, let

$$\hat{U}_{n,\gamma} = \bigcup_{\sigma\in\mathcal{T}(T[\gamma])\wedge|\sigma|=2n} ([\sigma] \cap \mathcal{T}(T[\gamma]).$$

Define

$$\hat{U}_{n,<\gamma} = \bigcup_{\beta<\gamma} \hat{U}_{n,\beta}$$

and

$$U_n = \bigcup_{\gamma<\omega_1^{\mathrm{CK}}} \hat{U}_{n,\gamma}.$$

The following facts are immediate.

(1) For any n and $\gamma < \omega_1^{\mathrm{CK}}$, $\hat{U}_{n+1,\gamma} \subseteq \hat{U}_{n,\gamma}$ and every string in $\hat{U}_{n,\gamma}$ has length at least 2^{2n};
(2) $\forall \sigma \in 2^{2n-2}(\mu(\hat{U}_{n,\gamma} \cap [\sigma]) < 2^{-2n-1})$;
(3) For any n, $\gamma < \omega_1^{\mathrm{CK}}$ and real z, if $z \in \hat{U}_{n,<\gamma} \setminus \hat{U}_{n,\gamma}$, then $z \notin \hat{U}_{n,\beta}$ for any $\beta \geq \gamma$.

By (1)—(3) and Lemma 3.3 by setting $k = m = 2$, $\{U_n\}_{n<\omega}$ is a generalized Π^1_1-ML-test. It is obvious that $\bigcap_{n\in\omega} U_n$ contains a perfect subset of $[T]$. Furthermore, $\mathcal{O}$ hyperarithmetically computes a perfect tree S with $[S] \subseteq \bigcap_{n\in\omega} U_n$ so that no path in S is strongly Π^1_1-ML-random. Hence no path in S is Π^1_1-random and by Proposition 2.4, any $y \in [S]$ satisfies $\omega_1^y > \omega_1^{\mathrm{CK}}$ and so $\mathcal{O} \leq_h y$. Such a y exists in every hyperdegree above the degree of $\mathcal{O}$. Theorem 3.5 is proved. □

4. Hyperdegrees of Π^1_1-random reals

While the hyperdegrees of Δ^1_1-random reals cover the cone of hyperdegrees above the hyperjump, it is not difficult to see that the situation is quite different outside this cone:

Proposition 4.1. *If x is Δ^1_1-random and $\omega_1^x = \omega_1^{\mathrm{CK}}$, then there is a real $y \geq_h x$ with $\omega_1^y = \omega_1^{\mathrm{CK}}$ whose hyperdegree contains no Δ^1_1-random real.*

Proof. Suppose that x is Δ^1_1-random and $\omega_1^x = \omega_1^{\mathrm{CK}}$. Let

$$H(x) = \{y \mid y \geq_T x \wedge \exists f \leq_T y \forall g \leq_h x (g \text{ is dominated by } f)\}.$$

Then $H(x)$ is $\Sigma^1_1(x)$. Since $\mathcal{O}^x \in H(x)$, $H(x)$ is not empty. Relativizing Gandy's Basis Theorem 2.5 to x, there is a real $y \in H(x)$ with $\omega_1^y = \omega_1^x = \omega_1^{\mathrm{CK}}$. Thus y is not Δ^1_1-dominated and so by Proposition 2.4, no real $z \equiv_h y$ is Δ^1_1-random. □

By contrast, the hyperdegrees of Π^1_1-random reals are downward closed.

Lemma 4.2. [2] *If x is Π^1_1-random and $y \leq_h x$, then there is a recursive ordinal γ such that $y \leq_T x \oplus \emptyset^{(\gamma)}$.*

Proof. Suppose that x is Π^1_1-random and $y \leq_h x$. Then $\omega_1^x = \omega_1^{\mathrm{CK}}$ and there is a formula $\varphi(\dot{x}, n)$ with rank $\alpha_0 < \omega_1^{\omega_1^{\mathrm{CK}}}$ such that

$$n \in y \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi(x, n).$$

Recall that for a ranked sentence ψ, the relation "$\mu(\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi) > 0$" is Π^1_1 (Theorem 1.3.IV of [17]). Hence by the admissibility of ω_1^{CK}, there is a recursive ordinal $\beta > \alpha_0$ such that

$$A_{\alpha_0} = \{\ulcorner\psi\urcorner \mid \psi \text{ has rank at most } \alpha_0 \wedge \mu(\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi\}) > 0\}$$

[2] The lemma was also proved by Bienvenu, Greenberg and Monin [1] independently.

is recursive in $\emptyset^{(\beta)}$. Then there is a recursive $\alpha_1 \geq \beta$ such that for any natural number i and formula ψ of rank at most β, there is a formula ψ' of rank at most α_1 such that $\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi'\}$ is a $\Pi^0_1(\emptyset^{(\alpha_1)})$-subset of $\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi\}$ and the difference in measure between these two sets is less than 2^{-i}.

Repeating this, we obtain a Δ_1-definable ω-sequence of ordinals $\alpha_0 < \alpha_1 < \cdots$ in $L_{\omega_1^{\mathrm{CK}}}$ whose supremum $\gamma = \bigcup_{i<\omega} \alpha_i$ satisfies the following two properties: for any $\beta < \gamma$,

(i) The set

$$A_\beta = \{\ulcorner \varphi \urcorner \mid \varphi \text{ has rank at most } \beta \wedge \mu(\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \varphi\}) > 0\}$$

is recursive in $\emptyset^{(\gamma)}$; and

(ii) For any natural number i and formula ψ with rank at most β, there is a formula ψ' of rank less than γ such that for some $\beta' < \gamma$, $\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi'\}$ is a $\Pi^0_1(\emptyset^{(\beta')})$-subset of $\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi\}$ and the difference in measure between these two sets is less than 2^{-i}.

Note that by Π^1_1-randomness, for any ranked formula ψ, if $x \in P_\psi = \{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi\}$, then P_ψ has positive measure.

By Proposition 2.4, x is Δ^1_1-dominated and so there is a hyperarithmetic function $f : \omega \to \omega$ such that for any $n \in \mathcal{O}$ with $|n| < \gamma$ and any e for which $\Phi_e^{H_n}$ computes a tree $T_{e,n}$, if $x \notin [T_{e,n}]$, then $x \upharpoonright f(\langle e, n\rangle) \notin T_{e,n}$. This allows us to implement the following construction.

Recursively in $x \oplus \emptyset^{(\gamma)} \oplus f$, first find a ψ_0 with rank less than γ such that $P_0 = \{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi_0\}$ contains x, has positive measure, and is a closed subset of either $\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \varphi(z, 0)\}$ or $\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \neg\varphi(z, 0)\}$. Since x is Π^1_1-random, by (ii), such a ψ_0 exists. Note that $x \oplus \emptyset^{(\gamma)} \oplus f$ is able to decide if $x \in P_0$. In general, for any n recursively in $x \oplus \emptyset^{(\gamma)} \oplus f$ choose the formula ψ_{n+1} with rank less than γ such that $P_{n+1} = \{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi_{n+1}\}$ contains x, has positive measure, and is a closed subset of either $P_n \cap \{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \varphi(z, n)\}$ or $P_n \cap \{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \neg\varphi(z, n)\}$. Since x is Π^1_1-random, by (ii) there is such a ψ_{n+1}.

Thus $y \leq_T x \oplus \emptyset^{(\gamma)} \oplus f$. Without loss of generality, we may assume that $f \leq_T \emptyset^{(\gamma)}$. Then $y \leq_T x \oplus \emptyset^{(\gamma)}$. □

Corollary 4.3. *For any Π^1_1-random x and $y \leq_h x$, there is a recursive ordinal α, a function $f \leq_T \emptyset^{(\alpha)}$ and an oracle function Φ such that for every n, $y(n) = \Phi^{x \oplus \emptyset^{(\alpha)} \upharpoonright f(n)}(n)[f(n)]$.*

Proof. Suppose that x is Π^1_1-random and $y \leq_h x$. By Lemma 4.2, there is a recursive ordinal γ and an oracle function Φ such that for every n, $y(n) = \Phi^{x \oplus \emptyset^{(\gamma)}}(n)$. Let $g <_h x$ such that for every n, $y(n) = \Phi^{x \oplus \emptyset^{(\gamma)} \upharpoonright g(n)}(n)[g(n)]$. Since x is Δ^1_1-dominated, there is a hyperarithmetic h such that for all n, $h(n) > g(n)$. Hence there is a recursive ordinal $\alpha \geq \gamma$ such that h is many-one reducible to $\emptyset^{(\alpha)}$. Then it is not difficult to define an $f \leq_T \emptyset^{(\alpha)}$ and an oracle function Ψ such that for every n, $y(n) = \Psi^{x \oplus \emptyset^{(\alpha)} \upharpoonright f(n)}(n)[f(n)]$. □

Theorem 4.4. *If x is Π^1_1-random and $\emptyset <_h y \leq_h x$, then there is a Π^1_1-random $z \equiv_h y$.*

Proof. Suppose that x is Π^1_1-random and $y \leq_h x$ is not hyperarithmetic. Then there is a recursive ordinal α, a nondecreasing function $f \leq_T \emptyset^{(\alpha)}$ and an oracle function Ψ such that $\lim_{n\to\infty} f(n) = \infty$ and for every n,

$$y(n) = \Psi^{x\oplus\emptyset^{(\alpha)}\upharpoonright f(n)}(n)[f(n)].$$

We use a technique which is essentially due to Demuth [4]. For any $u, \tau \in 2^{<\omega}$, let

$$C(u,\tau) = \{\sigma \mid \sigma \in 2^{f(|u|)} \wedge \Psi^{\sigma\oplus\emptyset^{(\alpha)}\upharpoonright f(|u|)}[f(|u|)] \upharpoonright |u| = \tau\}.$$

For strings τ and u, let $\tau <_\ell u$ mean "τ is to the left of u". Define $\emptyset^{(\alpha)}$-recursive functions:

$$l(u) = \sum_{\tau\in 2^{|u|}\wedge\tau<_\ell u} (\sum_{\sigma\in C(u,\tau)} 2^{-|\sigma|})$$

and

$$r(u) = l(u) + \sum_{\sigma\in C(u,u)} 2^{-|\sigma|}.$$

One may view $\sum_{\sigma\in C(u,\tau)} 2^{-|\sigma|}$ as a "measure" of τ, see Demuth [4]. For each n, let

$$l_n = l(y\upharpoonright n), \text{ and } r_n = r(y\upharpoonright n).$$

Then $l_n \leq l_{n+1} \leq r_{n+1} \leq r_n$ for every n.

Since y is not hyperarithmetic, it is not difficult to see that $\lim_{n\to\infty} r_n = 0$. Hence there is a unique real

$$z = \bigcap_{n\in\omega}(l_n, r_n).$$

Obviously $z \leq_T y \oplus \emptyset^{(\alpha)}$.

For any n, $\emptyset^{(\alpha)}$-recursively find a string u such that z lies in the interval $(l(u), r(u))$ and $|l(u) - r(u)| < 2^{-f(n)-n-2}$. Then $u \upharpoonright n = y \upharpoonright n$. So $y \leq_T z \oplus \emptyset^{(\alpha)}$. And thus $z \equiv_h y$. We claim that z is Δ^1_1-random.

Suppose otherwise. Then there is a recursive ordinal $\beta < \omega_1^{\mathrm{CK}}$ and a $\emptyset^{(\beta)}$-ML-test $\{V_n\}_{n\in\omega}$ such that $z \in \bigcap_{n\in\omega} V_n$. Let

$$\begin{aligned}\hat{V}_n = \{u \mid \exists\nu\exists k(&\nu \text{ is the } k\text{-th string in } V_n \wedge \\ &\exists p, q \in \mathbb{Q}(l(u) \leq p < q \leq r(u) \wedge [p,q] \subseteq [\nu] \wedge q - p > r(u) - 2^{-n-k-2})\}.\end{aligned}$$

Since $z \in V_n$, we have $y \in \hat{V}_n$ for every n. Note that $\{\hat{V}_n\}_{n\in\omega}$ is $\emptyset^{(\beta+1+\alpha)}$-r.e. Let

$$U_n = \{\sigma \mid \exists\tau \in \hat{V}_n(|\sigma| = f(|\tau|) \wedge \Phi^{\sigma\oplus\emptyset^{(\alpha)}\upharpoonright f(|\tau|)}[f(|\tau|)] \upharpoonright |\tau| = |\tau|)\}.$$

Then $\{U_n\}_{n\in\omega}$ is $\emptyset^{(\beta+1+\alpha)}$-r.e and $x \in \bigcap_{n\in\omega} U_n$. Note that for every n,

$$\mu(U_n) \leq \mu(V_n) + \sum_{k\in\omega} 2^{-n-k-2+1} < 2^{-n} + 2^{-n} = 2^{-n+1}.$$

Then $\{U_{n+1}\}_{n\in\omega}$ is a $\emptyset^{(\beta+1+\alpha)}$-ML-test. So x is not a Δ^1_1-random, a contradiction. □

The following is an immediate consequence of the proof of Theorem 4.4:

Corollary 4.5. *For any Π^1_1-random x, if $\emptyset <_h y \leq_h x$, then y is Π^1_1-random relative to some measure λ.*

We will prove a stronger version of this result in Theorem 5.1.

5. On $NCR_{\Pi^1_1}$

This section is inspired by the work of Reimann and Slaman in [14] and [15], where they investigated reals not Martin-Löf random relative to any continuous measure. They prove that NCR_1, the collection of such reals, is countable. In fact their proof shows that for any recursive ordinal α, the collection NCR_α of reals not $\emptyset^{(\alpha)}$-ML-random relative to any continuous measure is countable. Hence a natural question to ask is how far the countability property extends. We set an upper limit for this by proving Theorem 5.1.

Given a representation $\hat{\lambda}$ of a measure λ over 2^ω, define a real x to be Π^1_1*-random relative to* $\hat{\lambda}$ if it does not belong to a $\hat{\lambda}$-null set which is $\Pi^1_1(\hat{\lambda})$. Define

$$NCR_{\Pi^1_1} = \{x \mid x \text{ is not } \Pi^1_1\text{-random relative to any representation } \hat{\lambda} \text{ of a continuous measure}\}.$$

Let $\mathcal{C} = \{x \in 2^\omega \mid x \in L_{\omega_1^x}\}$. It is known that $\mathcal{C}$ is the largest Π^1_1-thin set.

Theorem 5.1. $NCR_{\Pi^1_1} = \mathcal{C}$.

We decompose the proof of Theorem 5.1 into a sequence of lemmas.

Lemma 5.2. *$NCR_{\Pi^1_1}$ does not contain a perfect subset.*

Proof. The proof is essentially due to Reimann and Slaman [14]. Suppose that there is a perfect tree $T \subseteq 2^{<\omega}$ such that every member of $[T]$ is $NCR_{\Pi^1_1}$. Define a measure λ as follows:

$$\lambda(\emptyset) = 1, \text{ and}$$
$$\lambda([\sigma^\frown i]) = \begin{cases} \lambda([\sigma]) & \text{If } \sigma^\frown(1-i) \notin T; \\ \frac{1}{2}\lambda([\sigma]) & \text{Otherwise.} \end{cases}$$

Then λ is a continuous measure so that $\lambda([T]) = 1$. Thus $[T]$ must contain a Π^1_1-random relative to any representation $\hat{\lambda}$ of λ. □

Lemma 5.3. *$NCR_{\Pi^1_1}$ is a thin Π^1_1-set, and hence $NCR_{\Pi^1_1} \subseteq \mathcal{C}$.*

Proof. By Lemma 5.2, $NCR_{\Pi^1_1}$ does not contain a perfect subset.

Relative to any representation $\hat{\lambda}$ of a continuous measure λ, we may perform the same proofs as in [16] so that all the results remain valid upon replacing Lesbegue measure μ by $\hat{\lambda}$. Then the set $\{z \mid \omega_1^{z\oplus\hat{\lambda}} > \omega_1^{\hat{\lambda}}\}$ is $\Pi^1_1(\hat{\lambda})$ and $\hat{\lambda}$-null. Hence as in [2], there is a Π^1_1 set $\mathcal{Q} \subseteq (2^\omega)^2$ such that for each real $\hat{\lambda}$ representing a continuous measure, the set $\mathcal{Q}_{\hat{\lambda}} = \{y \mid (\hat{\lambda}, y) \in \mathcal{Q}\}$ is the largest $\Pi^1_1(\hat{\lambda})$ $\hat{\lambda}$-null set. Then, as in Reimann and Slaman [15],

$$z \in NCR_{\Pi^1_1} \Leftrightarrow \forall\hat{\lambda}(\hat{\lambda} \text{ represents a continuous measure } \rightarrow z \in \mathcal{Q}_{\hat{\lambda}}).$$

Thus $NCR_{\Pi^1_1}$ is Π^1_1. □

Lemma 5.4. *If $x \in L_{\omega_1^x}$ and $z \not\geq_h x$, then $z \oplus x \geq_h \mathcal{O}^z$.*

Proof. Suppose that $x \in L_{\omega_1^x}$ and $z \not\geq_h x$. Then $\omega_1^z < \omega_1^x$. So $\omega_1^{x\oplus z} > \omega_1^z$. Thus $z \oplus x \geq_h \mathcal{O}^z$. □

Lemma 5.5. *If $x \in \mathcal{C}$, then $x \in NCR_{\Pi^1_1}$.*

Proof. Let λ be a continuous measure with representation $\hat{\lambda}$. If $x \leq_h \hat{\lambda}$, then x obviously is not Π^1_1-random relative to $\hat{\lambda}$. By Lemma 5.4, $x \oplus \hat{\lambda} \geq_h \mathcal{O}^{\hat{\lambda}}$. But $\{z \mid z \oplus \hat{\lambda} \geq \mathcal{O}^{\hat{\lambda}}\}$ is a $\Pi^1_1(\hat{\lambda})$ $\hat{\lambda}$-null set. This implies that x is not Π^1_1-random relative to $\hat{\lambda}$. □

6. Separating lowness for higher randomness notions

In [2], Chong, Nies and Yu investigated lowness properties for Δ^1_1 and Π^1_1-randomness. It is unknown whether there is a nonhyperarithmetic real low for Π^1_1-random. However, there is a characterization of reals which are low for Π^1_1-randomness.

Proposition 6.1 (Harrington, Nies and Slaman [2])**.** *Being low for Π^1_1-randomness is equivalent to being low for Δ^1_1-randomness and not cuppable above $\mathcal{O}$ by a Π^1_1-random.*

We may apply Proposition 6.1 to separate lowness for Δ^1_1-randomness from lowness for Π^1_1-randomness. Recall that given a class of sets of reals $\mathbf{\Gamma}$, a real x is $\mathbf{\Gamma}$-Kurtz random if it does not belong to any $\mathbf{\Gamma}$-closed null set.

In [10], Kjos-Hanssen, Nies, Stephan and Yu investigated lowness for Δ^1_1-Kurtz randomness and lowness for Π^1_1-Kurtz randomness. They proved that lowness for Π^1_1-Kurtz randomness implies lowness for Δ^1_1-randomness. We show that the implication cannot be reversed.

In [20], Yu gave a new proof of the following theorem.

Theorem 6.2 (Martin [11] and Friedman)**.** *Every Σ^1_1-tree T with uncountably many infinite paths has a member of each hyperdegree $\geq_h \mathcal{O}$ as a path.*

We apply the technique introduced in [20] to prove the following result.

Theorem 6.3. *Let A_0 and A_1 be uncountable Σ^1_1-sets of reals. For any $z \geq_h \mathcal{O}$, there are reals $x_0 \in A_0$ and $x_1 \in A_1$ such that $x_0 \oplus x_1 \equiv_h z$.*

Proof. Fix a real $z \geq_h \mathcal{O}$ and two uncountable Σ^1_1-sets A_0 and A_1. Then there are two recursive trees $T_0, T_1 \subseteq 2^{<\omega} \times \omega^{<\omega}$ such that for $i \leq 1$, $A_i = \{x \mid \exists f \forall n (x \upharpoonright n, f \upharpoonright n) \in T_i\}$. We may assume that neither A_0 nor A_1 contains a hyperarithmetic real. Let $T_2 \subseteq \omega^{<\omega}$ be recursive so that $[T_2]$ is uncountable and does not contain a hyperarithmetic infinite path. Let $f_{\mathcal{O}}$ be the leftmost path in T_2. Then $f_{\mathcal{O}} \equiv_h \mathcal{O}$.

For any $i \leq 1$ and $(\sigma, \tau) \in T_i$, define

$$T_i(\sigma,\tau) = \{(\sigma',\tau') \in T_i \mid (\sigma',\tau') \succeq (\sigma,\tau) \vee (\sigma',\tau') \prec (\sigma,\tau)\}.$$

We say that a string $\sigma^* \in 2^{<\omega}$ is *splitting over* (σ, τ) for a tree $T \subseteq 2^{<\omega} \times \omega^{<\omega}$ if $\sigma^* \succeq \sigma$ and for any $j \leq 1$, $T_{\sigma^{*\frown} j}(\sigma,\tau) = \{(\sigma',\tau') \mid \sigma' \succeq \sigma^{*\frown} j \wedge \tau' \succeq \tau \wedge (\sigma',\tau') \in T\}$

contains an infinite path. Node that σ^* does not lie on T but some pair (σ^*, τ') does and we call (σ^*, τ') a *splitting node on* T,

For each $i \leq 1$, we construct a sequence $(\sigma_{i,0}, \tau_{i,0}) \prec (\sigma_{i,1}, \tau_{i,1}) \prec \cdots$ in T_i and let $x_i = \bigcup_j \sigma_{i,j}$. The idea is to apply a "mutual coding" technique so that x_0 codes the witness function (in the Σ^1_1-definition) for x_1 and x_1 codes the witness function for x_0. For our purpose, we also assign x_0 the additional responsibility of coding z as well as $f_{\mathcal{O}}$. More precisely, for each $s \in \omega$ we use $\sigma_{0,j}$ to code $z(s)$, $f_{\mathcal{O}}(s)$ and $\tau_{1,s-1}$, and use $\sigma_{1,s}$ to code $\tau_{0,s}$.

At stage 0, let $(\sigma_{i,0}, \tau_{i,0}) = (\emptyset, \emptyset)$ for $i \leq 1$. Without loss of generality, assume that $(\emptyset, \emptyset)$ is a splitting node in both T_0 and T_1.

The construction at stage $s+1$ proceeds as follows:

Substage (i). First let σ^* be the shortest splitting string over $(\sigma_{9,s}, \tau_{0,s})$ for T_0. Thus $T_{0,\sigma^{*\frown} j}(\sigma_{0,s}, \tau_{0,s})$ contains an infinite path for $j \leq 1$. Let $\sigma^*_{0,s+1}$ be the leftmost splitting string over $(\sigma_{0,s}, \tau_{0,s})$ extending $\sigma^{*\frown} z(s)$ for T_0. Thus $z(s)$ is coded here. Next we code $\tau_{1,s}$. Let $n^0_{s+1} = |\tau_{1,s}| - |\tau_{1,s-1}|$. Inductively, for any $k \in [1, n^0_{s+1}]$, let $\sigma^k_{0,s+1}$ be the left-most splitting string over $(\sigma_{0,s}, \tau_{0,s})$ extending $(\sigma^{k-1}_{0,s+1})^\frown 1$ for T_0 so that there are $\tau_{0,s}(k + |\tau_{0,s-1}|)$-many splitting strings over $(\sigma_{0,s}, \tau_{0,s})$ for T_0 between $\sigma^{k-1}_{0,s+1}$ and $\sigma^k_{0,s+1}$. Let $\sigma^{n^0_{s+1}+1}_{0,s+1}$ be the leftmost splitting string extending $(\sigma^{n^0_{s+1}}_{0,s+1})^\frown 1$ over $(\sigma_{0,s}, \tau_{0,s})$ for T_0 so that there are $f_{\mathcal{O}}(s)$-many splitting strings for T_0 over $(\sigma_{0,s}, \tau_{0,s})$ between $\sigma^{n^0_{s+1}}_{0,s+1}$ and $\sigma^{n^0_{s+1}+1}_{0,s+1}$. Thus $f_{\mathcal{O}}(s)$ is coded here. For $j \leq 1$, let $\sigma^{n^0_{s+1}+1+j+1}_{0,s+1}$ be the next splitting string T_0 over $(\sigma_{0,s}, \tau_{0,s})$ extending $(\sigma^{n^0_{s+1}+1+j}_{0,s+1})^\frown 1$. This coding tells us that the action at this stage for the "σ_0 side" is completed. Define $\sigma_{0,s+1} = \sigma^{n^0_{s+1}+3}_{0,s+1}$. Let $\tau_{0,s+1} \in \omega^{|\sigma_{0,s+1}|}$ be the leftmost string such that the tree $T_{0,(\sigma_{0,s+1}, \tau_{0,s+1})}$ has an infinite path.

Substage (ii). Let $\sigma^0_{1,s+1} = \sigma_{1,s}$ and $n^1_{s+1} = |\tau_{0,s+1}| - |\tau_{0,s}|$. Inductively, for any $k \in [1, n^1_{s+1}]$, let $\sigma^k_{1,s+1}$ be the leftmost splitting string over $(\sigma_{1,s}, \tau_{1,s})$ extending $(\sigma^{k-1}_{1,s+1})^\frown 1$ for T_1 so that there are $\tau_{0,s+1}(k + |\tau_{0,s}|)$-many splitting strings over $(\sigma_{1,s}, \tau_{1,s})$ between $\sigma^{k-1}_{1,s+1}$ and $\sigma^k_{1,s+1}$. Hence $\tau_{0,s+1}$ is coded. For $j \leq 1$, let $\sigma^{n^1_{s+1}+j+1}_{1,s+1}$ be the next splitting string over $(\sigma_{1,s}, \tau_{1,s})$ for T_1 extending $(\sigma^{n^1_{s+1}+j}_{1,s+1})^\frown 1$. This coding tells us that the action of coding $\tau_{0,s+1}$ at this stage for the 'σ_1 side" is completed. Define $\sigma_{1,s+1} = \sigma^{n^1_{s+1}+2}_{1,s+1}$. Let $\tau_{1,s+1} \in \omega^{|\sigma_{1,s+1}|}$ be the leftmost finite string such that the tree $T_{1,(\sigma_{1,s+1}, \tau_{1,s+1})}$ has an infinite path. Thus we have coded $\tau_{0,s+1}$ into $\sigma_{1,s+1}$.

This completes the construction at stage $s+1$.

Let $x_i = \bigcup_{s<\omega} \sigma_{i,s}$ for $i \leq 1$. Obviously $z \geq_h x_0 \oplus x_1$.

Now we use x_0 and x_1 to decode the coding construction. The decoding method is a finite injury method which is quite similar to that used in the new proof of Theorem 6.2. We construct a sequence of ordinals $\{\alpha_s\}_{s<\omega}$ Δ_1-definable in $L_{\omega_1^{x_0 \oplus x_1}}[x_0 \oplus x_1]$ so that $\lim_{s \to \omega} \alpha_s = \omega_1^{\mathrm{CK}}$, and use it as a parameter to decode the reals z and f_0, thereby concluding that $x_0 \oplus x_1 \geq_h z$.

As in [20], we may fix a Σ_1-enumeration $\{T_i[\alpha]\}_{i\leq 2,\alpha<\omega_1^{\mathrm{CK}}}$ over $L_{\omega_1^{\mathrm{CK}}}$ such that for $i \leq 1$,

- $T_i[0] = T_i$
- $T_i[\alpha] \subseteq T_i[\beta]$ for $\omega_1^{\mathrm{CK}} > \alpha \geq \beta$
- $T_i[\omega_1^{\mathrm{CK}}] = \bigcap_{\alpha<\omega_1^{\mathrm{CK}}} T_i[\alpha]$
- $T_i[\omega_1^{\mathrm{CK}}]$ has no dead end nodes, and
- $A_i = \{x \mid \exists f \forall n (x \upharpoonright n, f \upharpoonright n) \in T_i[\omega_1^{\mathrm{CK}}]\}$.

Since $[T_i]$ does not contain a hyperarithmetic infinite path, we have $[T_i[\omega_1^{\mathrm{CK}}]] = [T_i]$ for $i \leq 1$. For each $\alpha < \omega_1^{\mathrm{CK}}$, one may define similarly the notion of a string σ' being splitting over (σ, τ) for $T[\alpha]$.

We make the following observations:

(a) On the tree T_2 of which $f_{\mathcal{O}}$ is its leftmost path, the enumeration of strings to the left of $f_{\mathcal{O}}$ is $\Sigma_1(L_{\omega_1^{\mathrm{CK}}})$. This implies that for each s, one may approximate $f_{\mathcal{O}} \upharpoonright s$ "from the left" (of $f_{\mathcal{O}}$) in a $\Sigma_1(L_{\omega_1^{\mathrm{CK}}}$ way on the tree T_2. Furthermore, at most only finitely many errors are made in the approximation, i.e. beginning with initial guess set as 0 for $f_{\mathcal{O}}(n)$, where $n \leq s$, and increasing by 1 each time a wrong guess is detected at stage α, one may let $f'_{\mathcal{O}}(\alpha, n)$ be the value of $f_{\mathcal{O}}(n)$ at stage α. Then $f'_{\mathcal{O}}$ is non-decreasing on each n and changes value only finitely many times.

(b) Using x_i as oracle set, and letting f_i be the leftmost witness path for x_i (so that for all n, $(x_i \upharpoonright n, f_i \upharpoonright n) \in T_i$, and no $\langle x_i, f\rangle$ has this property if f is "left of" f_i), the set of strings *to the left of* f_i is $\Sigma_1(L_{\omega_1^{\mathrm{CK}}}[x_i])$. Furthermore, there is a $\Sigma_1(L_{\omega_1^{\mathrm{CK}}}[x_i])$-approximation of f_i "from the left" such that for each s, there are at most finitely many wrong guesses of $f_i(n)$, $n \leq s$, to be made if the approximation proceeds "from the left" as in (a). Thus one may define a $\Sigma_1(L_{\omega_1^{\mathrm{CK}}}[x_i])$-function $f'_i : \omega_1^{\mathrm{CK}} \times \omega \to \omega$ so that f'_i is non-decreasing for each n and $\{\alpha | f'_i(\alpha+1, n) \neq f'_i(\alpha, n)\}$ is finite.

(c) $f_i = \bigcup_s \tau_{i,s}$.

Since for each $i \leq 1$ and $\alpha < \omega_1^{\mathrm{CK}}$, $\langle x_i, f_i\rangle$ is a path on $T_i[\alpha]$, one may use $x_0 \oplus x_1$ to approximate the values of $f_{\mathcal{O}}(n)$ and $f_i(n)$ by simulating the construction above on $T_i[\alpha]$. This is achieved by relativizing to $x_0 \oplus x_1$ the algorithm described in the construction of the sequences $\langle \sigma_{i,s}, \tau_{i,s}\rangle$, $i \leq 1$. Firstly, for $(\sigma, \tau) \in T_i[\alpha]$ such that $\sigma \prec x_i$, one may define the notion of $x_i \upharpoonright n$ being splitting over (σ, τ) for $T_i[\alpha]$ *after* α *steps of computation*. Next let $\sigma_{i,s}[\alpha[$ be the initial segments of x_i (in ascending order of length) so that $\sigma_{i,s+1}[\alpha]$ is splitting over $(\sigma_{i,s}[\alpha], \tau_{i,s}[\alpha])$ for some $\tau_{i,s}[\alpha]$ that is an approximation of $f_i \upharpoonright s$ at stage α.

The algorithm we adopt proceeds as follows: For $i = 0$, $\sigma_{0,0}[\alpha](|\sigma_{0,0}[\alpha]|)$ is a guess of $z(0)$. Then $\sigma_{0,1}[\alpha](|\sigma_{0,1}[\alpha]|) = 1$ to signify the end of coding $z(0)$. Let $s(1)$ be the least $s > 1$ such that $\sigma_{0,s}[\alpha](|\sigma_{0,s}[\alpha]|) = 1$. Then the cardinality of $\{2, \ldots, s(1) - 1\}$, i.e. $s(1) - 3$, is an approximation of $f_1(0)$ at stage α from the point of view of x_0 via the process of decoding. Let $s(2) > s(1)$ be the least s such that $\sigma_{0,s}[\alpha](|\sigma_{0,s}[\alpha]|) = 1$. Then $s(2) - s(1) - 1$ is an approximation of $f_{\mathcal{O}}(0)$ at stage α according to x_0. By induction, one computes approximate values of $f_1(n)$ and $f_{\mathcal{O}}(n)$ for each n at stage α relative to x_0. Similarly, using x_1 as oracle, one computes an approximation of $f_0(n)$

at stage α. We say that the approximation of $f_{\mathcal{O}}(n)$ (or f_1 as the case may be) is *correct* at stage α if it agrees with $f'_{\mathcal{O}}(\alpha, n)$ (respectively $f'_1(\alpha, n)$). Define correctness for f_0 similarly.

We now describe the construction of the sequence $\{\alpha_s\}$. Set $\alpha_0 = 0$. Suppose $\alpha_s < \omega_1^{\mathrm{CK}}$ is defined. Let γ_{s+1} be the least $\gamma > \alpha_s$ such that for some $n \leq s$, the approximation of one of $f_{\mathcal{O}}(n)$, $f_i(n)$ is incorrect. Let n_{s+1} be the least such n. If either $f_{\mathcal{O}}(n_{s+1})$ or $f_1(n_{s+1})$ is found to be incorrectly approximated, then let α_{s+1} be the least ordinal greater than γ_{s+1} such that $f_{\mathcal{O}}(n_{s+1})$ and $f_1(n_{s+1})$ are both correct. This ordinal exists by the definition of the trees $T_i[\alpha]$ and the functions $f'_{\mathcal{O}}$ and f'_1 defined earlier. If $f_{\mathcal{O}}(n_{s+1})$ and $f_1(n_{s+1})$ are correctly approximated at stage γ_{s+1}, then let α_{s+1} be the least ordinal greater than γ_s where $f_0(n_{s+1})$ is correctly approximated.

Now $\alpha_s < \omega_1^{\mathrm{CK}}$ for each $s < \omega$, and is $\Sigma_1(L_{\omega_1^{\mathrm{CK}}}[x_0 \oplus x_1])$. Thus $\alpha^* = \sup_s \alpha_s < \omega_1^{x_0 \oplus x_1}$. We make the following claim.

Claim. For each n, $x_0 \oplus x_1$ computes $f_{\mathcal{O}}(n)$ and $f_i(n)$ correctly.

Proof of Claim. It is sufficient to verify that for each n, $x_0 \oplus x_1$ computes the correct approximation of $f_{\mathcal{O}}(n)$ and $f_i(n)$ at all but finitely many s. First of all, as noted in (a) and (b), each n may be found to have been incorrectly approximated only finitely many times. Secondly, if for some least n, $f_i(n)$ is seen to be correctly approximated at all sufficiently large s, and yet not equal to its actual value, then it implies that $\langle x_i, f_i \rangle$ is not the leftmost path on T_i, which is a contradiction. The argument for $f_{\mathcal{O}}(n)$ is similar and is omitted.

Thus over $L_{\alpha^*}[x_0 \oplus x_1]$ one may decode the construction and correctly compute $f_{\mathcal{O}}$ and f_i. Hence $\mathcal{O} \equiv_h f_{\mathcal{O}} \leq_h x_0 \oplus x_1$. With this, we conclude that $z \in L_{\alpha^*+1}[x_0 \oplus x_1]$ so that $z \equiv_h x_0 \oplus x_1$.

□

Let $\mathcal{F}$ be the collection of all finite subsets of ω. A real x is Δ^1_1-*traceable* if for any function $f \leq_h x$, there is a Δ^1_1-function $g : \omega \to \mathcal{F}$ such that for every n, $|g(n)| = n$ and $f(n) \in g(n)$.

Lemma 6.4. *There is an uncountable Σ^1_1-set A in which every member is Δ^1_1-traceable.*

Proof. This is precisely what was proved in Theorem 4.7 of [18]. □

By [2] and [10], each Δ^1_1-traceable real is low for Δ^1_1-randomness and hence low for Δ^1_1-Kurtz randomness. By [9], the Π^1_1-random reals form a Σ^1_1-set. Then by Lemma 6.4 and Theorem 6.3, there is an x which is low for Δ^1_1-randomness and $x \oplus y \equiv_h \mathcal{O}$ for some Π^1_1-random y. So y is a $\Pi^1_1(x)$-singleton. We thus conclude:

Theorem 6.5. *Lowness for Δ^1_1-randomness does not imply lowness for Π^1_1-randomness. And lowness for Δ^1_1-Kurtz-randomness does not imply lowness for Π^1_1-Kurtz-randomness.*

Remark. Theorem 6.3 may be used to answer Question 58 in [6] and Question 3 in [18], whose solutions were announced by Friedman and Harrington but have remain unpublished.

We end this paper with two problems.

It is still unknown whether strong Π^1_1-ML-randomness coincides with Π^1_1-randomness. To separate these two notions, one way is to investigate the Borel ranks of different notions of randomness. Obviously the collection of Π^1_1-ML-random reals is $\mathbf{\Pi}^0_3$ and it can be shown that it is not $\mathbf{\Sigma}^0_3$ (see Part 2, [21]) . Moreover, it is not hard to see that the collection of Π^1_1-random reals is neither $\mathbf{\Sigma}^0_2$ nor $\mathbf{\Pi}^0_2$. Its exact Borel rank remains unknown. We have the following conjecture.

Conjecture 6.6. *The collection of Π^1_1-random reals is not $\mathbf{\Pi}^0_3$.*

Also the question whether lowness for Π^1_1-randomness coincides with hyperarithmeticity remains open. In view of Theorem 6.1, we have the following question.

Question 6.7. *Is it true that for any nonhyperarithmetic x and uncountable Σ^1_1-set $A \subseteq 2^\omega$, there is a $y \in A$ such that $x \oplus y \geq_h \mathcal{O}$?*

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