THIN MAXIMAL ANTICHAINS IN THE TURING DEGREES

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Abstract. We study existence problems of maximal antichains in the Turing degrees. In particular, we give a characterization of the existence of a thin Π^1_1 maximal antichains in the Turing degrees in terms of (relatively) constructible reals. A corollary of our main result gives a negative solution to a question of Jockusch under the assumption that every real is constructible.

1. Introduction

Let $\langle \mathfrak{D}, \leq \rangle$ denote the structure of Turing degrees. If $\mathbf{A} \subset \mathfrak{D}$, then it is an *antichain* if $\mathbf{x} \not\leq \mathbf{y}$ and $\mathbf{y} \not\leq \mathbf{x}$ for any distinct $\mathbf{x}$, $\mathbf{y} \in \mathbf{A}$. $\mathbf{A}$ is *maximal* if it is not properly contained in an antichain. By contrast, $\mathbf{A}$ is a *chain* if all of its elements are pairwise Turing comparable. $\mathbf{A}$ is a *maximal chain* if it is not properly contained in any chain. In [3] we studied the existence problem of maximal chains in $\mathfrak{D}$ under various set-theoretic assumptions. In this paper we turn our attention to existence problems of maximal antichains in $\mathfrak{D}$. Since there are $2^{\aleph_0}$ many minimal degrees, we have immediately the following proposition.

Proposition 1.1 (Folklore)**.** *(ZFC) Every maximal antichain has size* $2^{\aleph_0}$.

In parallel with Turing degrees, we say that $A \subset 2^\omega$ is an antichain if its elements are pairwise Turing incomparable. We define the related notions similarly. Our interest here are twofold: (i) In view of Proposition 1.1, is there an analytically definable (say $\mathbf{\Pi^1_1}$) maximal antichain? (ii) Does every maximal antichain $A \subset 2^\omega$ contain a perfect subset? Theorem 2.5 (ii) says that under ZFC, the existence of a thin $\mathbf{\Pi^1_1}$ maximal antichain of Turing degrees is equivalent to the assertion that $2^\omega = (2^\omega)^{L[x]}$ for some real x. Comparing the consistency strength of the existence of a thin $\mathbf{\Pi^1_1}$ maximal antichain in the Turing degrees with that of a $\mathbf{\Pi^1_1}$ maximal chain, where a large cardinal axiom is needed for it to be refuted (see [3])), one sees that the former is a much weaker statement.

In §3, we apply the results of §2 to study a measure-theoretic problem on the Turing degrees, and provide a negative answer to a question raised by Jockusch.

The following notations are adopted: x, y, z etc. denote elements of 2^ω, while the collection of paths of a perfect tree T is denoted by $[T]$.

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2. Thin maximal antichains

Firstly, it is a consequence of ZFC that there does exist a thin maximal antichain in the Turing degrees:

Proposition 2.1. *(ZFC)There exists a thin maximal antichain.*

Proof. Fix an enumeration $\{[T_\alpha]\}_{\alpha<2^{\aleph_0}}$ of perfect sets whose Turing degrees form an antichain, and fix an enumeration of all reals $\{x_\alpha\}_{\alpha<2^{\aleph_0}}$. We construct a thin set $A = \{z^0_\alpha | \alpha < 2^{\aleph_0}\} \cup \{z^1_\alpha | \alpha < 2^{\aleph_0}\}$ whose Turing degrees form a maximal antichain, by induction on $\alpha < 2^{\aleph_0}$, so that both $A - [T_\alpha]$ and $[T_\alpha] - A$ are nonempty. :

At step α, check whether $\{x_\alpha\} \cup \{z^i_\beta | \beta < \alpha \wedge i \leq 1\}$ is an antichain. If the answer is yes, then check whether the Turing degrees of $\{x_\alpha\} \cup \{z^i_\beta | \beta < \alpha \wedge i \leq 1\} \cup [T_\alpha]$ form an antichain. There are two cases to consider:

(i) If they form an antichain, select a real $y \equiv_{\mathrm{T}} x_\alpha$ but $y \neq x_\alpha$. Obviously $y \notin T_\alpha$. Define $z^0_\alpha = y$. Then select another real $y_0 \notin [T_\alpha]$ so that there is a real $y_1 \in [T_\alpha]$ with $y_0 \equiv_{\mathrm{T}} y_1$. Define $z^1_\alpha = y_0$.
(ii) Otherwise, define $z^0_\alpha = z^1_\alpha = x_\alpha$.

If the Turing degrees of $\{x_\alpha\} \cup \{z^i_\beta | \beta < \alpha \wedge i \leq 1\}$ do not form an antichain, check whether $\{z^i_\beta | \beta < \alpha \wedge i \leq 1\} \cup [T_\alpha]$ is an antichain.

(iii) If the answer is yes, select a real $x \in [T_\alpha] - \{z^i_\beta | \beta < \alpha \wedge i \leq 1\}$. Then select a real $y \equiv_{\mathrm{T}} x_\alpha$ but $y \neq x_\alpha$. Define $z^0_\alpha = z^1_\alpha = y$.
(iv) Otherwise, define $z^0_\alpha = z^1_\alpha$ to be any real forming an antichain with $\{z^i_\beta | \beta < \alpha \wedge i \leq 1\}$.

The set $A = \{z^i_\alpha | \alpha < 2^{\aleph_0} \wedge i \leq 1\}$ is an antichain by construction. We claim that it is maximal. Otherwise, there is a real x_α whose Turing degree is incomparable with those of all the reals in A. Let α_0 be the least ordinal α for which x_α has this property. Then according to (i) and (ii) at step α_0, either x_{α_0} or some real y of the same degree is chosen to be $z^i_{\alpha_0}$ for some (or all) $i \leq 1$, which is a contradiction. Furthermore, for each α, both $A - [T_\alpha]$ and $[T_\alpha] - A$ are nonempty since $A \cup [T_\alpha]$ is not an antichain. Thus A is a maximal antichain that is thin. □

How complicated must a thin maximal antichain be? Since every maximal antichain of reals has size $2^{\aleph_0}$, it cannot be $\mathbf{\Sigma}^1_1$ (else it would contain a perfect subset). We show it is consistent with ZF that there exists a Π^1_1 thin maximal antichain. The idea of the proof is similar to that used in constructing a Π^1_1 maximal chain presented in [3]. But the technique required to derive the result is quite different.

Lemma 2.2. *(ZF) Let $X \cup \{x_0\}$ be a countable antichain in the Turing degrees. Let x_1 be a real. Then there is a z such that*

(1) $z'' \geq_{\mathrm{T}} x_1$;
(2) $\{z\} \cup X$ *is an antichain;*
(3) $z \geq_{\mathrm{T}} x_0$.

Proof. Let $X = \{y_i\}_{i\in\omega}$. We construct a real z so that the following requirements are satisfied:

$$N_{e,i} : \Phi_e^{z\oplus x_0} \text{ is total } \implies \Phi_e^{z\oplus x_0} \neq y_i.$$

Then $\{z \oplus x_0\} \cup X$ is an antichain. We also need to make $(z \oplus x_0)'' \geq_{\mathrm{T}} x_1$.
We construct a sequence of finite strings $\sigma_0 \prec \sigma_1 \prec ...$ so that $z = \bigcup_n \sigma_n$.

Construction:
At step 0, define $\sigma_0 = \emptyset$.
At step $n + 1 = \langle e, i\rangle$.

Substep 1: (Satisfying $N_{e,i}$). Consider the following statement:

$$(\exists \tau \succeq \sigma_n)(\forall \tau_0 \succeq \tau)(\forall \tau_1 \succeq \tau)(\forall m)(\Phi^{\tau_0\oplus x_0}(m) \downarrow \wedge \Phi^{\tau_1\oplus x_0}(m) \downarrow \implies \Phi^{\tau_0\oplus x_0}(m) = \Phi^{\tau_1\oplus x_0}(m)).$$

If the statement is true, then find the least τ (in a recursive well ordering of strings) and define $\sigma^0_{n+1} = \tau$. Then for every real $z \succ \sigma^0_{n+1}$, $\Phi_e^{z\oplus x_0}$ is total implies $\Phi_e^{z\oplus x_0} \leq_T x_0$. Thus $\Phi_e^{z\oplus x_0} \neq y_i$ since $X \cup \{x_0\}$ is an antichain. If the statement is not true, find the least $\tau_0 \succeq \sigma_n$ for which there exists (a least) $\tau_1 \succeq \sigma_n$ such that $\Phi^{\tau_0\oplus x_0}(m) \downarrow \neq \Phi^{\tau_1\oplus x_0}(m) \downarrow$ for some m. Define $\sigma^0_{n+1} = \tau_k$ for the $k < 2$ where $\Phi^{\tau_k\oplus x_0}(m) \neq y_i(m)$.

Substep 2: (Coding x_1). Define $\sigma_{n+1} = (\sigma^0_{n+1})^\frown(x_1(n))$.

Finally, define $z = \bigcup_n \sigma_n$. This finishes the construction.

Since $x_0 \not\leq_{\mathrm{T}} y_i$ for all i, $z\oplus x_0 \not\leq_{\mathrm{T}} y_i$ for all i. By the construction above, $z\oplus x_0 \not\geq_{\mathrm{T}} y_i$ for all i, so $X \cup \{z \oplus x_0\}$ is an antichain.

To see that $(z\oplus x_0)'' \geq_{\mathrm{T}} x_1$, we look at the statement considered in Substep 1. The statement is decidable by x_0''. If the statement is true, then we can x_0''-recursively find the τ. Then $\tau = \sigma^0_{n+1}$. Otherwise, we can x_0''-recursively find both τ_0 and τ_1. Then we use z to decide which one is the σ^0_{n+1}. Thus $x_1(n) = 0$ if and only if $z(|\sigma^0_{n+1}|+1) = 0$. Moreover, $\sigma_{n+1} = z \upharpoonright (|\sigma^0_{n+1}| + 1)$. So the sequence $\{\sigma_n\}_n$ can be computed from $z \oplus x_0''$. Hence $(z \oplus x_0)'' \geq_{\mathrm{T}} z \oplus x_0'' \geq_{\mathrm{T}} x_1$. □

Corollary 2.3. *(ZF+DC) Let $X \cup \{x_0\}$ be a countable antichain in the Turing degrees. Then there is a real x_1 so that for all real $y \geq_{\mathrm{T}} x_1$ there is a real z such that*

(1) $z'' \equiv_{\mathrm{T}} y$;
(2) $\{z\} \cup X$ *is an antichain;*
(3) $z \geq_{\mathrm{T}} x_0$.

Proof. Fix an enumeration $\{y_i\}_{i\in\omega}$ of X. Then the set

$$B = \{y|(\exists z)(z'' = y \wedge X \cup \{z\} \text{ is an antichain} \wedge z \geq_{\mathrm{T}} x_0)\}$$

is a Borel set. Moreover, by Lemma 2.2, for each real x, there is a real $y \in B$ so that $y \geq_{\mathrm{T}} x$. By Borel determinacy [8], there exists a real x_1 so that for all $x \geq_{\mathrm{T}} x_1$, there is a real $y \in B$ so that $y \equiv_{\mathrm{T}} x$. □

The proof of the following theorem depends heavily on the results of Boolos and Putnam [2]. Call a set $E \subseteq \omega\times\omega$ an arithmetical copy of a structure $(S, \in)$ if there is a 1-1 function $f : S \to \omega$ so that for all $x, y \in S$, $x \in y$ if and only if $(f(x), f(y)) \in E$. In ([2]) it is proved that if $(L_{\alpha+1} \setminus L_\alpha) \cap 2^\omega \neq \emptyset$ then there is an arithmetical copy $E_\alpha \in L_{\alpha+1}$ of $(L_\alpha, \in)$ so that any $x \in (L_{\alpha+1} \setminus L_\alpha) \cap 2^\omega$ is arithmetical in E_α (i.e. E_α is a master code for α in the sense of Jensen [6]). Moreover, each $z \in L_\alpha \cap 2^\omega$ is one-one reducible to E_α. Hence E_α may be viewed as a real. Note that for each constructibly countable β, there is an $\alpha > \beta$ such that $(L_{\alpha+1} \setminus L_\alpha) \cap 2^\omega \neq \emptyset$. For a given ordinal α and $X \subseteq \alpha\times\omega$, we denote by $X[\beta]$ the real $\{n \in \omega | (\beta, n) \in X\}$. We may regard X as a sequence of reals of length α.

Lemma 2.4. *Assume $V = L$. There exists a Π^1_1 thin maximal antichain in the Turing degrees.*

Proof. A set A of reals is Π^1_1 if and only if there is a Σ_0-formula φ such that

$$y \in A \Leftrightarrow (\exists x \in L_{\omega_1^y}[y])(L_{\omega_1^y}[y] \models \varphi(x, y)),$$

where ω_1^y is the least ordinal $\alpha > \omega$ such that $L_\alpha[y]$ is admissible (see [1] and [10]).

Our proof combines Corollary 2.3 and the argument in [3] which is based on [5].

Assuming $V = L$, we define a function F on $\omega_1 \times \bigcup_{\alpha<\omega_1} \mathcal{P}(\alpha\times\omega)$ as follows:

For each $\alpha < \omega_1$ and antichain $X \subseteq \alpha\times\omega$ with $\alpha < \omega_1$, we define $F(\alpha, X)$ to be the real z such that there exists a lexicographically least triple (β, E, e_0) (where the ordering on the second coordinate is $<_L$) satisfying the following properties:

1. There is a 1-1 function $h \in L_\beta$ which maps ω onto α, a real $x_0 \in L_\beta$ so that $\{x_0\} \cup \{X[h(n)]|n \in \omega\}$ is an antichain and $(L_{\beta+1} \setminus L_\beta) \cap 2^\omega \neq \emptyset$;
2. $E \in L_{\beta+1}$ is an arithmetical copy of $(L_\beta, \in)$ as described above,
3. $z \geq_{\mathrm{T}} x_0$ and $\{z\} \cup \{X[h(n)]|n \in \omega\}$ is an antichain. Furthermore,
4. $z'' \equiv_{\mathrm{T}} E$ and
5. $z = \Phi^E_{e_0}$.

We show that $F(\alpha, X)$ is defined if X is an antichain.

Fix (α, X) where X is an antichain. Since $V = L$, there is a $\gamma > \alpha$ such that there is a real $x_0 \in L_\gamma$ with $\{x_0\} \cup X$ forming an antichain. Choose a real x_1 for $X \cup \{x_0\}$ as guaranteed by Corollary 2.3. Since $V = L$, there is a $\beta > \gamma$ so that $x_1 \in L_\beta$, $(L_{\beta+1} \setminus L_\beta) \cap 2^\omega \neq \emptyset$ and there is a function h_α mapping ω onto α. By the discursion above, there is an arithmetical copy $E \subseteq \omega\times\omega$ in $L_{\beta+1}$ so that $E \geq_T x_1$. By Corollary 2.3, there is a real $z \geq_{\mathrm{T}} x_0$ so that $z'' \equiv_{\mathrm{T}} E$ and $\{z\}\cup\{X[h_\alpha(n)]|n \in \omega\}$ is an antichain. Obviously, $L_{\beta+1} \in L_{\omega_1^z}[z]$. By the absoluteness of $<_L$, it is easy to see that F is a well-defined function.

Moreover, one can verify using the absoluteness of $<_L$ that there is a Σ_0 formula $\varphi(\alpha, X, z, y)$ such that $F(\alpha, X) = z$ if and only if $L_{\omega_1^{(X,z)}}[X, z] \models (\exists y)\varphi(\alpha, X, z, y)$, with a function $h \in L_{\omega_1^{(X,z)}}[X, z]$ mapping ω onto α.

Thus we can perform transfinite induction on α to construct a maximal antichain of Turing degrees. But care has to be exercised here since in general sets constructed this way are Σ_1 over L_{ω_1}, i.e. Σ^1_2 and not necessarily Π^1_1.

Define $G(\alpha) = z$ if and only if $\alpha < \omega_1^z$ and there is a function $f : \alpha + 1 \to 2^\omega$ with $f \in L_{\omega_1^z}[z]$ so that for all $\beta \leq \alpha$, $f(\beta) = F(\beta, \{(\gamma, n)|n \in f(\gamma) \wedge \gamma < \beta\})$ and $f(\alpha) = z$. Since $L_{\omega_1^z}[z]$ is admissible, $\{f(\gamma)|\gamma \leq \alpha\} \in L_{\omega_1^z}[z]$. So $G(\alpha) = z$ if and only if there is a function $f : \alpha + 1 \to 2^\omega$ with $f \in L_{\omega_1^z}[z]$ such that

$$L_{\omega_1^z}[z] \models ((\forall \beta \leq \alpha)(\exists y)\varphi(\beta, \{(\gamma, n)|n \in f(\gamma) \wedge \gamma < \beta\}, y, f(\beta))) \wedge f(\alpha) = z.$$

Since $L_{\omega_1^z}[z]$ is admissible, G is Σ_1-definable. In other words, $G(\alpha) = z$ if and only if there is a function $f : \alpha + 1 \to 2^\omega$ with $f \in L_{\omega_1^z}[z]$ such that

$$L_{\omega_1^z}[z] \models ((\exists s)(\forall \beta \leq \alpha)(\exists y \in s)\varphi(\beta, \{(\gamma, n)|n \in f(\gamma) \wedge \gamma < \beta\}, y, f(\beta))) \wedge f(\alpha) = z.$$

Define the range of G to be T. Then $z \in T$ if and only if there exists an ordinal $\alpha < \omega_1^y$ and a function $f : \alpha + 1 \to 2^\omega$ with $f \in L_{\omega_1^z}[z]$ such that

$$L_{\omega_1^z}[z] \models ((\exists s)(\forall \beta \leq \alpha)(\exists y \in s)\varphi(\beta, \{(\gamma, n)|n \in f(\gamma) \wedge \gamma < \beta\}, y, f(\beta))) \wedge f(\alpha) = z.$$

So T is Π_1^1.

All that remains is to show that G is a well-defined total function on ω_1. This can be done using the same argument as that for showing the recursion theorem over admissible structures (see Barwise [1]). The only difficult part is to argue, as was done earlier, that the function f defined above exists. We leave this to the reader.

We show that G is a maximal antichain. Suppose not, then there is a $<_L$-least real $x_0 \notin G$ so that $\{x_0\} \cup G$ is an antichain. Since $V = L$, $x_0 \in L_\gamma$ for some $\gamma < \omega_1$. Then, by the construction above, there must be some real $y \in G$ so that $y \geq_{\mathrm{T}} x_0$, a contradiction.

To see that G is thin, it suffices to show that $z \in L_{\omega_1^z}$ if z is in the range of F. By (2), $E \in L_{\beta+1}$ and $\beta + 1 < \omega_1^E$. So $E \in L_{\omega_1^E}$. By (4), $\omega_1^E = \omega_1^z$ and $z \in L_{\beta+2} \subseteq L_{\omega_1^E}$. So $z \in L_{\omega_1^z}$. By a result of Mansfield-Solovay [7], G is a thin set. $\square$

Theorem 2.5. *(ZFC)*

(i) *There is a thin Π_1^1 maximal antichain of Turing degrees if and only if $(2^\omega)^L = 2^\omega$.*

(ii) *There is a thin $\mathbf{\Pi}_1^1$ maximal antichain of Turing degrees if and only if $(2^\omega)^{L[x]} = 2^\omega$ for some real x.*

Proof. (i) Suppose A is a thin Π_1^1 maximal antichain. Then, by Solovay's result [11], $A \subset L$. Now let x be a real. By a theorem of Cooper [4], there is a real y of minimal degree such that $x \leq_T y'$. Since A is a maximal antichain, there is a real $z \in A$ with $z \geq_{\mathrm{T}} y$. So $x \leq_{\mathrm{T}} z'$. Hence $x \in L$.

Conversely, suppose $(2^\omega)^L = 2^\omega$. Fix a Π_1^1 set G as in Lemma 2.4. Since the statement "G is an antichain in the Turing degrees" is Π_2^1 and

$$L \models \text{"}G \text{ is an antichain in the Turing degrees"},$$

G is an antichain in the Turing degrees by absoluteness. Fix a real x. Since $(2^\omega)^L = 2^\omega$, $x \in L$. The statement $T(x)$:"there exists $y \in G$ so that y is

Turing comparable with x" is $\Sigma^1_2(x)$ and $L \models T(x)$. It follows that $T(x)$ is true. Thus G is a maximal antichain.

(ii) Relativize the proof of (i).

□

It follows that to construct a model in which there is no thin $\mathbf{\Pi}^1_1$ maximal antichain of Turing degrees, one just needs to refute CH in the model. It is natural to ask whether there is a model of $ZFC+CH$ with no thin $\mathbf{\Pi}^1_1$ maximal antichain of Turing degrees. The answer is yes: Apply iterated Cohen forcing with finite support of length $(\omega_1)^L$, i.e. conditions of the form $((<\omega, 2)_\alpha, <: \alpha < \omega_1)$ over L to obtain a generic set G. Notice that this notion of forcing satisfies the (set-theoretic) countable chain condition (c.c.c), and so preserves all cardinals. Now $L[G] \models ZFC + CH$. If there is a real $x \in L[G]$ so that $(2^\omega)^{L[x]} = 2^\omega$, then $x \in L[G_\alpha]$ for some $\alpha < \omega_1$ where G_α is the generic set obtained from iterated forcing up to α. Then for any real $y \in L[G] - L[G_\alpha]$, y is not constructible in x. It follows from Theorem 2.5 that there is no thin $\mathbf{\Pi}^1_1$ maximal antichain of Turing degrees in $L[G]$.

3. Applications to the measure theory of Turing degrees

In [12], Yu investigated measure theoretic aspects of the Turing degrees. In this section, we continue the investigation by applying the results in the previous section to study some problems in this area.

Given a set A of reals, we define $\mathcal{U}(A) = \{y | \exists x (x \in A \wedge x \leq_{\mathrm{T}} y)\}$. We have the following proposition.

Proposition 3.1. *If A is a Π^1_1 thin set, then $\mu(\mathcal{U}(A)) = 0$.*

Proof. Fix a ZFC model $\mathfrak{M}$. If A is Π^1_1, then $\mathcal{U}(A)$ is Π^1_1 and so measurable. By a result of Sacks [9], the set $C = \{n \in \omega | \mu(\mathcal{U}(A)) > 2^{-n}\}$ is Π^1_1. Since A is a thin Π^1_1 set, $A \subset L$. Extend $\mathfrak{M}$ to a generic $\mathfrak{N}$ by any notion of forcing that collapses $(\omega_1)^L$ to ω. Then A is still thin by absoluteness since the statement "A is a thin set" is Π^1_2. In the generic extension $\mathfrak{N}$, A is countable since A is a subset of constructible reals. So $\mathcal{U}(A)$ is a null set in $\mathfrak{N}$. I.e. "$\forall n (n \notin C)$" is true in $\mathfrak{N}$. Since the statement "$\forall n (n \notin C)$" is Σ^1_1, it is true in $\mathfrak{M}$. Thus $\mu(\mathcal{U}(A)) = 0$ in $\mathfrak{M}$. □

Together with Theorem 2.5, we have the following corollary.

Corollary 3.2. *Assume $(2^\omega)^L = 2^\omega$. There is a maximal antichain A in the Turing degrees such that $\mu(A) = \mu(\mathcal{U}(A)) = 0$.*

We say that a set $X \subset 2^\omega$ is a *quasi-antichain* in the Turing degrees if there is an antichain $\mathbf{X} \subset \mathfrak{D}$ so that $X = \{x | x \text{ is of } \mathbf{x} \wedge \mathbf{x} \in \mathbf{X}\}$.

Yu [12] showed that there is a nonmeasurable quasi-antichain in the Turing degrees and no quasi-antichain has positive measure. In response to Yu's results, Jockusch [12] asked the following question:

Question 3.3 (Jockusch)**.** *Is every maximal quasi-antichain in the Turing degrees nonmeasurable?*

We answer this question in the negative under the assumption that every real is constructible:

Corollary 3.4. *Assume $(2^\omega)^L = 2^\omega$. There is a null maximal quasi-antichain in the Turing degrees.*

Proof. By Corollary 3.2, there is a maximal antichain A so that $\mu(\mathcal{U}(A)) = 0$. Then $B = \{y|\exists x(x \in A \wedge x \equiv_{\mathrm{T}} y)\} \subseteq \mathcal{U}(A)$ is a null maximal quasi-antichain. □

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