MAXIMAL CHAINS IN THE TURING DEGREES

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Abstract. We study the problem of existence of maximal chains in the Turing degrees. We show that:

1. $ZF + DC+$"There exists no maximal chain in the Turing degrees" is equiconsistent with $ZFC+$"There exists an inaccessible cardinal";
2. For all $a \in 2^\omega$, $(\omega_1)^{L[a]} = \omega_1$ if and only if there exists a $\Pi^1_1[a]$ maximal chain in the Turing degrees.

As a corollary, $ZFC+$"There exists an inaccessible cardinal" is equiconsistent with $ZFC+$ "There is no (bold face) $\mathbf{\Pi}^1_1$ maximal chain of Turing degrees".

§1. Introduction. A chain in the Turing degrees is a set of reals in which any two distinct elements are Turing comparable but not equivalent. A maximal chain is a chain which cannot be properly extended. An antichain of Turing degrees, by contrast, is a set of reals in which any two distinct elements are Turing incomparable. A maximal antichain is an antichain that cannot be properly extended. In this paper, we study maximal chains in the Turing degrees. This is a classical topic in recursion theory which may be traced back to Sacks [13], in which he proved the existence of a minimal upper bound for any countable set of Turing degrees. As a consequence, assuming the Axiom of Choice AC, there is a maximal chain of order type ω_1. Abraham and Shore [1] even constructed an initial segment of the Turing degrees of order type ω_1. All of these results depend heavily on AC. We are interested in the following questions:

1. Is AC necessary to show that there exists a maximal chain? Can one construct a maximal chain without AC?
2. Is there a definable, say $\mathbf{\Pi}^1_1$, maximal chain?

For (1), we will prove in Section 2 that over ZF plus the Axiom of Dependent Choice DC, "there is no maximal chain in the Turing degrees" is equiconsistent with ZFC + "there exists an inaccessible cardinal". This shows that the existence of maximal chains is "decided" by one's belief, over $ZF + DC$, in

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the existence of an inaccessible cardinal. Thus if one believes that there is no large cardinal in L, then one would deduce the existence of a maximal chain from $ZF + DC$. On the other hand, if there is no maximal chain then ω_1 is inaccessible in L (in fact in $L[a]$ for any real a). For (2), the situation is more complicated. It is not difficult to show, under $ZF+DC$, that there exists no $\underset{\sim}{\mathbf{\Sigma}}^1_1$ maximal chain in the Turing degrees. This is the best result within $ZF + DC$. By Martin and Solovay's result [9], it is consistent with ZFC that *every* maximal chain is $\underset{\sim}{\mathbf{\Pi}}^1_1$. Is it consistent with ZFC that there exists a Π^1_1 maximal chain? We show (Theorem 3.5) that there is a Π^1_1 maximal chain under the assumption of $(\omega_1)^L = \omega_1$.

We organize the paper as follows: In Section 2, we study the relation between existence of maximal chains and large cardinals. In Section 3, we consider the problem of the existence of definable maximal chains.

Notations. A tree T is a subset of $2^{<\omega}$ which is downward closed. Given a tree T, a finite string σ is said to be a splitting node on T if both $\sigma^\frown 0$ and $\sigma^\frown 1$ are in T. The string σ is said to be an n-th splitting node on T if $\sigma \in T$ is a splitting node and there are $n-1$-many splitting nodes that are initial segments of σ. Let $T^\sigma = \{\tau \in T | \tau \succeq \sigma \vee \tau \preceq \sigma\}$ and $T \upharpoonright n = \{\sigma \in T | |\sigma| \leq n\}$. We use $[T]$ to denote the set of reals $\{z | (\forall n)(z \upharpoonright n \in T)\}$. A perfect tree T is a tree in which for each σ, there is a $\tau \in T$ so that $\tau \succeq \sigma$ and τ is a splitting node. Define the n-th level of T to be

$$Lev_n(T) = \{\sigma \notin Lev_{n-1}(T) | (\exists \tau)(\tau \in T \wedge \sigma \preceq \tau \wedge \tau \text{ is an n-th slpitting node on } T)\}.$$

For notations and definitions not given here, see [4], [8], [6],[12] and [14].

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§2. Chains and large cardinals. We use I to denote the statement "*there exists an inaccessible cardinal*" and NMT to denote "*there exists no maximal chain in the Turing degrees*". In this section, we prove that $ZF + DC + NMT$ is equiconsistent with $ZFC + I$.

A set A of reals is said to be an antichain if for all $x, y \in A$, $x \neq y$ implies $x|_T y$. The following result is essentially due to Sacks [13] which will play a critical role in this and later sections.

LEMMA 2.1. *($ZF + DC$) For each perfect tree T, there is a perfect tree $S \subseteq T$ so that $[S]$ is an antichain.*

PROPOSITION 2.2. *Assume $Con(ZFC + I)$. Then $Con(ZF + DC + NMT)$ holds.*

PROOF. By Solovay's theorem [17], assuming $Con(ZFC + I)$, $ZF + DC$ is consistent with the statement: "*Every uncountable set contains a perfect subset*".

Now every maximal chain is uncountable and, by Lemma 2.1, does not contain a perfect set. Hence $ZF + DC$ is consistent with the statement "There is no maximal chain in the Turing degrees". $\dashv$

PROPOSITION 2.3. *If $Con(ZF + DC + NMT)$, then $Con(ZFC + I)$.*

PROOF. By Solovay's result [16], it suffices to prove that

$$ZF + DC + NMT \vdash (\forall x \in 2^\omega)(\omega_1^{L[x]} < \omega_1).$$

To show this, assume that there is a real x so that $\omega_1^{L[x]} = \omega_1$. Then since $2^\omega \cap L[x]$ has a well-ordering in $L[x]$ of order type $\omega_1^{L[x]}$, it is straightforward to obtain a maximal chain $A \subseteq L[x] \cap 2^\omega$ in $L[x]$ of size $(\aleph_1)^{L[x]}$. Since $\omega_1^{L[x]} = \omega_1$, $|A| = \aleph_1$. Let $A = \{z_\alpha\}_{\alpha<\omega_1}$.

If A is not a maximal chain in V (the real world), then take a witness $z \notin A$ which is comparable with all of the reals in A. So there is an $\alpha < \omega_1$ so that $z \leq_{\mathrm{T}} z_\alpha$. Then $z \in L[x]$ since $A \subseteq L[x]$. So A is not a maximal chain in $L[x]$, a contradiction. $\dashv$

Note that the proofs above also show that $ZFC + I$ is equiconsistent with $ZF + DC+$"Every chain of the Turing degrees is countable".

§3. The existence of definable maximal chains.

In this section, we study the existence of definable maximal chains of Turing degrees.

PROPOSITION 3.1. *($ZF + DC$) There is no $\utilde{\mathbf{\Sigma}}^1_1$ maximal chain in the Turing degrees.*

PROOF. If A is a maximal $\utilde{\mathbf{\Sigma}}^1_1$ chain in the Turing degrees, then A must have a perfect subset since A is uncountable. By Lemma 2.1, A will then contain a pair of T-incomparable reals, which contradicts the fact that A is chain. $\dashv$

Assuming $MA + \neg CH + (\omega_1)^L = \omega_1$, by Martin-Solovay's result that each set of reals with size at most $\aleph_1$ is $\utilde{\mathbf{\Pi}}^1_1$ [9], one sees that each maximal chain in the Turing degrees is a $\utilde{\mathbf{\Pi}}^1_1$-set. The question then is whether there is a Π^1_1 maximal chain. We give a positive answer assuming $(\omega_1)^L = \omega_1$.

Recall that a real x is a minimal cover of a countable set A of reals if (i) x is an upper bound of every real in A, and (ii) no $y <_{\mathrm{T}} x$ is an upper bound of A. Sacks [13] showed that every countable collection of Turing degrees has a minimal cover. The next lemma implies that a minimal cover exists with arbitrarily high double jump.

LEMMA 3.2. *(ZF) Assume A is a countable set of reals and x is a real. There is a minimal cover z of A so that $z'' \geq_{\mathrm{T}} x$.*

Proof. We fix an effective enumeration of partial recursive oracle functions $\{\Phi_e\}_{e<\omega}$.

Let $\{p_n\}_{n<\omega}$ be a recursive 1-1 enumeration of prime numbers. Allowing ambiguity, we will also use p_σ to denote the prime number which codes the string σ. Given a recursive oracle function Φ, we use $\Phi^y = T$ to express the following:

1. Φ^y is total;
2. $(\forall n)(\Phi^y(n) = \prod_{\sigma \in Lev_n(T)} p_\sigma)$.

Define $L_n(T)$ to be the leftmost $n+1$-th splitting node and $R_n(T)$ to be the rightmost $n+1$-th splitting node.

For each finite string σ^*, real x and perfect tree $T \subseteq 2^{<\omega}$ with $\sigma^* \in T$, define $\mathcal{T}(\sigma^*, x, T)$ to be a perfect tree so that $\mathcal{T}(\sigma^*, x, T)$ is the intersection of a sequence of perfect trees T_n given as follows:

$T_0 = T^{\sigma^*}$.

$T_{n+1} = \{\tau | \exists \sigma \in Lev_n(T_n)(\tau \preceq \sigma)\} \cup S_{n+1} \subseteq T_n$ where S_{n+1} is defined as:

Case(1) : $x(n) = 0$. $\sigma \in S_{n+1}$ if and only if there exists $\tau \in Lev_{n+1}(T_n)$ so that $\sigma \succeq \tau$ and $\tau = L_1(T_n^\nu)$ for some ν which is an n-th splitting node of T_n.

Case(2) :$x(n) = 1$. $\sigma \in S_{n+1}$ if and only if there exists $\tau \in Lev_{n+1}(T_n)$ so that $\sigma \succeq \tau$ and $\tau = R_1(T_n^\nu)$ for some ν which is an n-th splitting node of T_n.

Define
$$\mathcal{T}(\sigma^*, x, T) = \bigcap_n T_n.$$

In other words, $\mathcal{T}(\sigma^*, x, T)$ is a subtree which, roughly speaking, codes $x(n)$ at a $2n$-th splitting node of T^{σ^*}. Note that $\mathcal{T}(\sigma^*, x, T) \oplus T \geq_{\mathrm{T}} x$. Moreover, suppose for some recursive oracle function Φ, we have $\Phi^y = T$ for all $y \in [T]$. Then there is a recursive oracle function Ψ such that $\Psi^y = x$ for all $y \in \mathcal{T}(\sigma^*, x, T)$. Furthermore, given an index of the oracle function Φ, an index of the oracle function Ψ may be effectively obtained from σ^*. In other words, there is a recursive function f such that $\Phi^y_{f(\sigma^*)} = x$ for all $y \in [\mathcal{T}(\sigma^*, x, T)]$. Since $\Phi^y = T$ for all $y \in [\mathcal{T}(\sigma^*, x, T)] \subseteq [T]$ and $x \oplus T \geq_{\mathrm{T}} \mathcal{T}(\sigma^*, x, T)$, there is a recursive function g so that $\Phi^y_{g(\sigma^*)} = \mathcal{T}(\sigma^*, x, T)$ for all $y \in [\mathcal{T}(\sigma^*, x, T)]$.

We give a sketch of the idea behind the construction of z. To obtain a minimal cover of a countable set $A = \{x_i\}_{i<\omega}$, one makes appropriate modifications of the construction of a minimal degree (see [8]). To make the minimal degree relatively high (i.e. to make it compute a given x through jumps), one needs to code the indices of the perfect trees in the course of the construction ([15] is a good source where this idea is made precise). Were the construction uniform, one could use the Recursion Theorem to code the index of the next perfect tree being defined during the step by step construction. This technique could be applied to code x and x_i into z for each $i < \omega$. However, to achieve minimality, the construction is non-uniform (one needs to decide whether the next tree will be an "e-splitting tree" or a "full tree", which in general is a "double jump" question). Although it is highly non-uniform, the construction does become uniform once it is decided which situation one is in (see Substep 3 of the construction below). This is the reason for using z'' to "get up" to x.

We now turn to the construction.

Fix a real x and an enumeration $\{x_i\}_{i\in\omega}$ of A. Suppose the recursive oracle functional Φ_0 satisfies $\Phi_0^y = 2^{<\omega}$ for all reals y. We construct a sequence of perfect trees step by step. At step n, we construct a perfect tree $T_n \leq_{\mathrm{T}} \oplus_{i<n} x_i$ and a finite string σ_n so that $T_n \upharpoonright |\sigma_n| = \{\tau | \tau \preceq \sigma_n\}$, $|\sigma_n| > n$ and there is a recursive oracle functional Φ_{e_n} so that for each $y \in [T_n]$, $\Phi_{e_n}^y = T_n$.

Construction

At step 0, define $T_0 = 2^{<\omega}$ and $\sigma_0 = \emptyset$.

At step $n+1$, there are three substeps:

Substep 1 (Coding x). For each k: If $x(n) = 0$, define $\sigma_{n+1,0}^k = R_1(T_n^{L_k(T_n)})$; otherwise, define $\sigma_{n+1,0}^k = L_1(T_n^{R_k(T_n)})$. Define $T_{n+1,0,k} = T_n^{\sigma_{n+1,0}^k}$. Hence there is a recursive oracle functional Φ_{i_k} such that for each $y \in T_{n+1,0,k}$, $\Phi_{i_k}^y = T_{n+1,0,k}$. Note that the function $k \mapsto i_k$ is recursive. It follows from the Recursion Theorem that there is a number k_0 such that $\Phi_{i_{k_0}}^y = \Phi_{k_0}^y$ for all $y \in [T_n^{\sigma_{n+1,0}^{k_0}}]$. Fix this k_0 and define $\sigma_{n+1}^0 = \sigma_{n+1,0}^{k_0}$, and $T_{n+1,0} = T_{n+1,0,k_0}$.

Substep 2 (Coding x_n). For each k: Define $T_{n+1,1,k} = \mathcal{T}(R_1(T_{n+1,0}^{L_k(T_{n+1,0})}), x_n, T_{n+1,0})$. By the discussion above, there are recursive functions f, g so that $\Phi_{f(k)}^y = x_n$ and $\Phi_{g(k)}^y = T_{n+1,1,k}$ for all $y \in [T_{n+1,1,k}]$. By the Recursion Theorem, there is a k_1 such that $\Phi_{k_1}^y = \Phi_{g(k_1)}^y = T_{n+1,1,k_1}$ for all $y \in [T_{n+1,1,k_1}]$. Define $\sigma_{n+1,1} = R_1(T_{n+1,0}^{L_{k_1}(T_{n+1,0})})$ and $T_{n+1,1} = T_{n+1,1,k_1}$.

Substep 3 (Forcing a minimal cover). This is the only place where z'' is used.

Case(1) : There exists $\sigma \in T_{n+1,1}$ and a number i_σ so that for all $m > i_\sigma$, $\tau_1, \tau_2 \succeq \sigma$, $\{\tau_1, \tau_2\} \subset T_{n+1,1} \wedge \Phi_{n+1}^{\tau_1}(m)\downarrow \wedge \Phi_{n+1}^{\tau_2}(m)\downarrow \implies \Phi_{n+1}^{\tau_1}(m) = \Phi_{n+1}^{\tau_2}(m)$. Choose the least such $\sigma \in T_{n+1,1}$ (in the sense of the coding of strings). Define $S = T_{n+1,1}^{\sigma}$. For each k, define $S_k = S^{R_1(S^{L_k(S)})}$. By the Recursion Theorem again, there is a k_2 so that $\Phi_{k_2}^y = S^{R_1(S^{L_{k_2}(S)})}$ for all $y \in [S^{R_1(S^{L_{k_2}(S)})}]$. Define $\sigma_{n+1} = R_1(S^{L_{k_2}(S)})$ and $T_{n+1} = S^{\sigma_{n+1}}$.

Case(2) : Otherwise. For each k, define $S_k = T_{n+1,1}^{R_1(T_{n+1,1}^{L_k(T_{n+1,1})})}$. Then we can S_k-recursively find a sub-perfect tree P_k of S_k such that for all $y \in [P_k]$, Φ_n^y is total and for all $\tau_0, \tau_1 \in P_k$, if $\tau_0 | \tau_1$, then there must be some i so that for all reals $z_0 \succ \tau_0$ and $z_1 \succ \tau_1$, $\Phi_n^{z_0}(i)\downarrow \neq \Phi_n^{z_1}(i)\downarrow$. By the Recursion Theorem, there is a k_2 such that $\Phi_{k_2}^y = P_{k_2}$ for all $y \in [P_{k_2}]$. Let $T_{n+1} = P_{k_2}$ and $\sigma_{n+1} = R_1(T_{n+1,1}^{L_{k_2}(T_{n+1,1})}) \in T_{n+1}$.

Let $e_{n+1} = k_3$. Then $\Phi_{e_{n+1}}^y = T_{n+1}$ for all $y \in [T_{n+1}]$.

This completes the construction at step $n+1$.

Note that by induction on n, $T_{n+1} \leq_{\mathrm{T}} \oplus_{i<n+1} x_i$.

Finally, let $z = \bigcup_n \sigma_n$.

By the usual arguments (see [8]), one can show that z is a minimal cover of A since $T_{n+1} \leq_{\mathrm{T}} \oplus_{i<n+1} x_i$ for each n. We show that $z'' \geq_{\mathrm{T}} x$. We prove this by

using induction on n to show that $x(n)$ may be z''-uniformly computed. To do this, we show that the index e_n is uniformly computed from z''.

At step $n+1$, by inductive hypothesis, we can z''-recursively find e_n. By the construction, we have $\Phi^y_{e_n} = T_n$ for all $y \in [T_n]$. We can z-recursively find an initial segment of z which is $R_1(T_n^{L_{k_0}(T_n)})$ or $L_1(T_n^{R_{k_0}(T_n)})$ for some k_0. In the first case, $x(n) = 0$. In the second case, $x(n) = 1$. Note that $\Phi^z_{k_0}$ is the perfect tree in Substep 1 of the construction. We can z-recursively find k_1 so that $R_1((\Phi^z_{k_0})^{L_{k_1}(\Phi^z_{k_0})}) \preceq z$. Then $\Phi^z_{k_1} = \mathcal{T}(R_1((\Phi^z_{k_0})^{L_{k_1}(\Phi^z_{k_0})}), x_n, \Phi^z_{k_0}) = T_{n+1,1}$. We use z'' to decide which case $\Phi^z_{k_1}$ is in. In case (1), we z''-recursively find the least $\sigma \in T_{n+1,1}$ with the required property. Let $S = T^{\sigma}_{n+1,1}$. Then we z-recursively find the k_2 so that $\Phi^z_{k_2} = S^{R_1(S^{L_{k_2}(S)})}$. So $e_{n+1} = k_2$ and $T_{n+1} = S^{R_1(S^{L_{k_2}(S)})} = \Phi^z_{k_2}$. In Case (2), we z-recursively find the k_2 so that $R_1(T_{n+1,1}^{L_{k_2}(T_{n+1,1})}) \preceq z$. Then by the construction, $e_{n+1} = k_2$ and $T_{n+1} = \Phi^z_{k_2}$.

Thus $z'' \geq_{\mathrm{T}} x$ and the proof of Lemma 3.2 is complete. ⊣

Remark. We conjecture that z'' may be replaced by z' in Lemma 3.2.

Corollary 3.3. *(ZF+DC) Assume that A is a countable set of reals. There is a real x so that for each $y \geq_{\mathrm{T}} x$, there is a minimal cover z of A so that $z'' \equiv_{\mathrm{T}} y$.*

Proof. Assume A is a countable set of reals. Fix an enumeration $\{x_i\}_i$ of A. The set

$$\begin{aligned} B &= \{y|(\exists z)(z \text{ is a minimal cover of } A \wedge z'' \equiv_{\mathrm{T}} y)\} \\ &= \{y|(\exists e)(\forall i)(\forall j)[(\Phi^y_e \text{ is total} \wedge (\Phi^y_e)'' \equiv_{\mathrm{T}} y \wedge x_i \leq_{\mathrm{T}} \Phi^y_e \wedge \\ &\quad (\Phi_j^{\Phi^y_e} \text{ is total } \wedge (\forall k)(x_k \leq_{\mathrm{T}} \Phi_j^{\Phi^y_e}) \Longrightarrow \Phi^y_e \leq_{\mathrm{T}} \Phi_j^{\Phi^y_e})]\} \end{aligned}$$

is a Borel set. By Lemma 3.2, for each real x, there is a real $y \in B$ so that $y \geq_{\mathrm{T}} x$. By Borel determinacy (Martin [10]), there is a real x so that $\{y|y \geq_{\mathrm{T}} x\} \subseteq B$. ⊣

To show the main result, we first construct a Π^1_1 maximal chain in the Turing degrees under the assumption $V = L$.

The proof of the following lemma depends heavily on the results of Boolos and Putnam [3]. Call a set $E \subseteq \omega \times \omega$ an arithmetical copy of a structure $(S, \in)$ if there is a 1-1 function $f : S \to \omega$ so that for all $x, y \in S$, $x \in y$ if and only if $(f(x), f(y)) \in E$. In ([3]) it is proved that if $(L_{\alpha+1} \setminus L_\alpha) \cap 2^\omega \neq \emptyset$ then there is an arithmetical copy $E_\alpha \in L_{\alpha+1}$ of $(L_\alpha, \in)$ so that any $x \in (L_{\alpha+1} \setminus L_\alpha) \cap 2^\omega$ is arithmetical in E_α (i.e. E_α is a master code for α in the sense of Jensen [7]). Moreover, each $z \in L_\alpha \cap 2^\omega$ is one-one reducible to E_α. Since $E_\alpha \subset \omega \times \omega$, it may be viewed as a real. Note that for each constructibly countable β, there is an $\alpha > \beta$ such that $(L_{\alpha+1} \setminus L_\alpha) \cap 2^\omega \neq \emptyset$. We will be considering sets $A \subseteq \alpha \times \omega$. It will be convenient to identify A with an α-sequence $\{A_\gamma | \gamma < \alpha\}$ of reals, where $A_\gamma = \{(\gamma, n)|(\gamma, n) \in A\}$.

THEOREM 3.4. *Assume $V = L$. There is a Π^1_1 maximal chain in the Turing degrees of order type ω_1.*

PROOF. By the Gandy-Spector theorem, a set A of reals is Π^1_1 if and only if there is a Σ_0-formula φ such that

$$y \in A \Leftrightarrow (\exists x \in L_{\omega_1^y}[y])(L_{\omega_1^y}[y] \models \varphi(x, y)),$$

where ω_1^y is the least ordinal α bigger than ω so that $L_\alpha[y]$ is admissible (see Theorem 3.1 Chapter IV [2]).

Our proof combines Corollary 3.3 and van Engelen et al's argument [5]. Based on the paper [5], Miller [11] provided a general machinery to construct a Π^1_1 set satisfying some particular properties. However, the presentation is sketchy and incomplete. We give a detailed argument here where it pertains to the theorem at hand.

Assuming $V = L$, we define a function F on $\omega_1 \times \bigcup_{\alpha<\omega_1} \mathcal{P}(\alpha \times \omega)$ as follows:

For each $\alpha < \omega_1$, $A \subseteq \alpha \times \omega$, we define $F(\alpha, A)$ to be the real y such that there exists a lexicographically least triple (β, E, e_0) (where the ordering on the second coordinate is $<_L$) satisfying the following properties :

1. There is a function $h \in L_\beta$ which maps ω onto α, and $(L_{\beta+1} \setminus L_\beta) \cap 2^\omega \neq \emptyset$;
2. $E \in L_{\beta+1}$ is an arithmetical copy of $(L_\beta, \in)$ with the properties mentioned before the statement of Theorem 3.4;
3. y is a minimal cover of the set of reals $\{x_\beta \mid \beta < \alpha \wedge \forall n(x_\beta(n) = A(\beta, n))\}$;
4. $y'' \equiv_T E$ and finally
5. $y = \Phi^E_{e_0}$.

We show that F is a total function.

For each (α, A), we show that there exists a y such that $F(\alpha, A) = y$. It suffices to show that (β, E, e_0) exists. Then by the fact that the lexicographical order is a well ordering, there must be a least one and this will yield the y needed to define $F(\alpha, A)$. Fix a real x for A as in Corollary 3.3, the base of a cone of Turing degrees that are double jumps of minimal covers of A. Since $V = L$, there is a $\beta > \alpha$ such that $x \in L_\beta$, $(L_{\beta+1} \setminus L_\beta) \cap 2^\omega \neq \emptyset$ and there is a function h_α mapping ω onto α. By the discussion above, there is an arithmetical copy $E \subseteq \omega \times \omega$ in $L_{\beta+1}$ such that $E >_T x$. By Corollary 3.3 and the choice of x, there is a minimal cover y of A so that $y'' \equiv_T E$ and $y = \Phi^E_{e_0}$ for some e_0. Obviously, $L_{\beta+1} \in L_{\omega_1^y}[y]$. By the absoluteness of $<_L$, it is easy to see that $F(\alpha, A)$ is defined. Note that $F(\alpha, A)$ depends essentially on A since A is a sequence of reals of length α.

Moreover, for such A's one can verify using the absoluteness of $<_L$ that there is a Σ_0 formula $\varphi(\alpha, A, z, y)$ such that

$$F(\alpha, A) = y \Leftrightarrow L_{\omega_1^{(A,y)}}[A, y] \models \exists z \exists h(\varphi(\alpha, A, z, y) \wedge h \text{ is a function from } \omega \text{ onto } \alpha).$$

Thus we can perform transfinite induction on countable ordinals to construct a maximal chain of Turing degrees of order type ω_1. But care has to be exercised here since in general sets constructed this way are Σ_1 over L_{ω_1}, i.e. Σ^1_2 and not necessarily Π^1_1.

Define $G(\alpha) = y$ if and only if

$$\alpha < \omega_1^y \wedge \exists f(f \in (2^\omega)^{\alpha+1} \wedge f \in L_{\omega_1^y}[y] \wedge f(\alpha) = y \wedge \forall \beta(\beta < \alpha \Longrightarrow f(\beta) = F(\beta, \{(\gamma, n) | n \in f(\gamma) \wedge \gamma < \beta\}))).$$

Since $L_{\omega_1^y}[y]$ is admissible, G is Σ_1-definable. In other words, $G(\alpha) = y$ if and only if there is a function $f : \alpha + 1 \to 2^\omega$ with $f \in L_{\omega_1^y}[y]$ such that

$$L_{\omega_1^y}[y] \models ((\exists s)(\forall \beta \leq \alpha)(\exists z \in s)\varphi(\beta, \{(\gamma, n)|n \in f(\gamma) \wedge \gamma < \beta\}, z, f(\beta))) \wedge f(\alpha) = y.$$

Define the range of G to be T. Then $y \in T$ if and only if there exists an ordinal $\alpha < \omega_1^y$ and a function $f : \alpha + 1 \to 2^\omega$ with $f \in L_{\omega_1^y}[y]$ such that

$$L_{\omega_1^y}[y] \models ((\exists s)(\forall \beta \leq \alpha)(\exists z \in s)\varphi(\beta, \{(\gamma, n)|n \in f(\gamma) \wedge \gamma < \beta\}, z, f(\beta))) \wedge f(\alpha) = y.$$

So T is Π_1^1.

All that remains is to show that G is a well-defined total function on ω_1. This can be done using the same argument as that for showing the recursion theorem over admissible structures (see Barwise [2]). The only difficult part is to argue, as was done earlier, that the function f defined above exists. We leave this to the reader.

Thus T is a chain of order type ω_1. To see that it is a maximal chain, let x be a real which is T-comparable with all members of T. Select the least α such that $G(\alpha) \geq_{\mathrm{T}} x$. Then $x \geq_{\mathrm{T}} G(\beta)$ for all $\beta < \alpha$. Since $G(\alpha)$ is a minimal cover of $\{G(\beta)|\beta < \alpha\}$, we have $G(\alpha) \equiv_{\mathrm{T}} x$. Thus T is a Π_1^1 maximal chain. ⊣

We arrive at the following characterization:

THEOREM 3.5. *Assume $ZF + DC$. The following statements are equivalent:*
1. $(\omega_1)^L = \omega_1$*;*
2. *There exists a Π_1^1 maximal chain in the Turing degrees.*
3. *There exists a Π_1^1 uncountable chain in the Turing degrees.*

PROOF. (1) $\Longrightarrow$ (2): Suppose $(\omega_1)^L = \omega_1$. Fix the Π_1^1 set T as in Theorem 3.4. Since the statement "T is a chain" is Π_2^1 and $L \models T$ is a chain, T is a chain in the real world V. Since T is uncountable in L and $(\omega_1)^L = \omega_1$, T is uncountable. Thus if x is a real so that $\{x\} \cup T$ is a chain, then $x <_T y$ for some $y \in T$ so that $x \in L$. Since $L \models T$ is a maximal chain, T is a maximal chain in V.

(2) $\Longrightarrow$ (3): This is Obvious.

(3) $\Longrightarrow$ (1): Suppose T is a Π_1^1 uncountable chain in the Turing degrees. By Lemma 2.1, T is a thin set. Solovay [16] proved that if T is a thin Π_1^1 set, then $T \subseteq L$, and $(T)^L = T \cap L = T$. Thus $T \subset L_{(\omega_1)^L}$. Since T is uncountable, $(\omega_1)^L = \omega_1$. ⊣

Now Theorem 3.5 may be relativized to any real a. To do this one first observes that an analog of the Boolos-Putnam theorem [3] on arithmetic copies holds, so that if $L_{\alpha+1}[a] \setminus L_\alpha[a] \neq \emptyset$, then there is an $E_\alpha \in L_{\alpha+1}[a] \cap 2^{\omega \times \omega}$ in which every real in $L_{\alpha+1}[a]$ is a-arithmetical (i.e. arithmetical in $E_\alpha \oplus a$). This provides the setting for establishing a relativized version of Theorem 3.4, namely if $V = L[a]$, then there is a $\Pi^1_1[a]$ maximal chain in the Turing degrees of order type ω_1. With this, one derives a relativized version of Theorem 3.5, where ω_1^L is replaced by $\omega_1^{L[a]}$, and Π^1_1 by $\Pi^1_1[a]$. This leads to the following corollary showing that bold face Π^1_1 maximal chains play a critical role in the existence problem of maximal chains, and gives an answer to the second question posed at the beginning of this paper.

Corollary 3.6. *The following statements are equiconsistent:*

(1) *$ZFC + I$;*
(2) *$ZF + DC +$ "There exists no $\underset{\sim}{\mathbf{\Pi}}^1_1$maximal chain in the Turing degrees".*
(3) *$ZF + AC +$ "There exists no $\underset{\sim}{\mathbf{\Pi}}^1_1$maximal chain in the Turing degrees".*

Proof. (1)$\Longrightarrow$ (2). Assume that $ZFC+I$ is consistent. Then by Proposition 2.2, $ZF + DC +$ "There exists no $\underset{\sim}{\mathbf{\Pi}}^1_1$maximal chain in the Turing degrees" is consistent.

(2)$\Longrightarrow$ (1). Assume that $ZF + DC +$ "There exists no $\underset{\sim}{\mathbf{\Pi}}^1_1$ maximal chain in the Turing degrees" is consistent. By the observation above on relativizing Theorem 3.5, the existence of a $\Pi^1_1[a]$ maximal chain of Turing degrees is equivalent to $\omega_1^{L[a]} = \omega_1$. Thus if there is no $\underset{\sim}{\mathbf{\Pi}}^1_1$ maximal chain in the Turing degrees, then $\omega_1^{L[a]} < \omega_1$ for all reals a. This implies that $ZFC + I$ is consistent.

(3)$\Longrightarrow$ (2). Obvious.

(1)$\Longrightarrow$ (3). Assume that $ZFC + I$ is consistent. Then $ZFC +$ "ω_1 is inaccessible in L" is consistent (by Levy collapse). So there is a ZFC model $\mathcal{M}$ so that

$$\mathcal{M} \models \forall x \in 2^\omega(\omega_1^{L[x]} < \omega_1).$$

By the relativized version of Theorem 3.5, there is no $\underset{\sim}{\mathbf{\Pi}}^1_1$ maximal chain in the Turing degrees in $\mathcal{M}$. ⊣

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