Characterizing nonstandard randomness notions via Martin-Löf randomness

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Three ideas

To be random, a real must

1. have fairly general property;
2. non-compressible;
3. non-predicable.
Three methods

Based on the ideas above, there are three methods to define randomness. They are:

1. measure theory argument;
2. Kolmogorov complexity;
3. martingale.
A Martin-Löf test is a uniformly c.e. sequence of open sets \( \{ U_n \}_{n \in \omega} \) so that \( \mu(U_n) < 2^{-n} \) for every \( n \).

A real \( x \) is **Martin-Löf random** if \( x \notin \bigcap_{n \in \omega} U_n \) for every Martin-Löf test \( \{ U_n \}_{n \in \omega} \).

There is a universal Martin-Löf test.
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Algorithmic randomness

A Turing machine $M$ is prefix free if for any finite strings $\sigma \prec \tau$, either $M(\sigma)$ or $M(\tau)$ is undefined. Given a prefix free Turing machine $M$, the Kolmogorov complexity of a finite string $\sigma$, $K_M(\sigma)$, is the length of the shortest input $\tau$ so that $\sigma = M(\tau)$. There is an optimal prefix free Turing machine $U$. A real $x$ is algorithmic random if there is a constant $c$ so that $\forall n (K_U(x \upharpoonright n) \geq n - c)$. 
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A martingale $d$ is a function from $2^{<\omega}$ to $\mathbb{R}^+ \cup \{0\}$ so that for any $\sigma$, $d(\sigma) = \frac{d(\sigma^0)+d(\sigma^1)}{2}$.

A c.e. martingale $d$ is a martingale $d$ so that the set $\{(q,\sigma) \in \mathbb{Q} \mid q < d(\sigma)\}$ is a c.e. set.

A real $x$ is random, then it cannot be predicated by any effective strategy. Or for any c.e. martingale $d$, $\lim_{n \to \infty} d(x \upharpoonright n) < \infty$.

There is an optimal c.e. martingale $d$. 

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A martingale \( d \) is a function from \( 2^{<\omega} \) to \( \mathbb{R}^+ \cup \{0\} \) so that for any \( \sigma \), \( d(\sigma) = \frac{d(\sigma^0)+d(\sigma^1)}{2} \).

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There is an optimal c.e. martingale \( d \).
Connecting these methods

**Theorem (Schnorr)**

For any real $x$, the following are equivalent:

1. for any Martin-Löf test $\{U_n\}_{n \in \omega}$, $x \notin \bigcap_n U_n$;
2. there is a constant $c$ so that $\forall n (K(x \upharpoonright n) \geq n - c)$;
3. $\lim_{n \to \infty} d(x \upharpoonright n) < \infty$ for any c.e. martingale $d$. 

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Characterizing nonstandard randomness notions via Martin-Löf randomness
Given two random reals, can we compare their randomness? Or how to measure randomness for a random real? A usual way is relativizing randomness notions. For example, a real $x$ is Martin-Löf random relativized to some real $z$ if $x$ passes all the Martin-Löf tests relativized to $z$. We use $\text{ML}(z)$ to denote this.
Measure randomness

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Given two reals $x$ and $y$, $x$ is $LR$-reducible to $y$, writing to $x \leq_{LR} y$, if $\text{ML}(L) \subseteq \text{ML}(x)$.

$LR$-reduction is an arithmetical definable way to measure the power compressing randomness information.

**Theorem (Kjors-Hassen, Miller and Solomon)**

$x \leq_{LR} y$ if and only if $\exists c \forall n (K^y(n) \leq K^x(n) + c)$. 

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Background

- On nonstandard randomness notions
- Two general methods to characterize nonstandard randomness notions
- The story of $\emptyset'$-Schnorr randomness
- Some remarks on other randomness notions
**LR-reduction**

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Nice properties of Martin-Löf randomness

Theorem (van-Lambalgen)

$x \oplus y$ is Martin-Löf random if and only if $y$ is Martin-Löf random and $x$ is Martin-Löf random relativized to $y$.

Let $x \geq_K y$ if there is a constant $c$ so that

$$\forall n (K(x \upharpoonright n) \geq K(y \upharpoonright n) - c).$$

Theorem (Miller and Yu)

For any oracle $z \geq_T \emptyset'$, if $x \geq_K y$ and $y$ is Martin-Löf random relativized to $z$, then $x$ is already Martin-Löf random relativized to $z$. 
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Power vs Randomness

Theorem (Miller and Yu)

1. For any real \( z \), if \( x \leq_T y \) are both Martin-Löf random and \( y \) is Martin-Löf random relativized to \( z \), then so is \( x \);

2. If \( y \) is Martin-Löf random and \( x \geq_K y \), then \( x \leq_{LR} y \).

Theorem (Stephan)

If \( x \) is Martin-Löf random but not computing the halting problem, then \( x \) cannot compute a complete extension of Peano Axioms.

All these justify that more random means less power.
Given a universal prefix free Turing machine $U$, the Chaitin’s $\Omega = \sum_{\sigma} U(\sigma) \downarrow 2^{|\sigma|}$ is Martin-Löf random.

**Theorem (Kučera; Gacs)**

For any $z \geq_{\mathbf{T}} \emptyset'$, there is a Martin-Löf random real $x \equiv_{\mathbf{T}} z$. 

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**Some flaws**
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Given a universal prefix free Turing machine $U$, the Chaitin's
$\Omega = \sum U(\sigma) \downarrow 2^{\|\sigma\|}$ is Martin-Löf random.

Theorem (Kučera; Gacs)

For any $z \geq_T \emptyset'$, there is a Martin-Löf random real $x \equiv_T z$. 
A real $x$ is weakly-2-random if for all generalized Martin-Löf test \( \{ U_n \}_{n \in \omega} \), $x \notin \bigcap_n U_n$.

A real $x$ is $\emptyset'$-Schnorr-random if for all $\emptyset'$-Schnorr test \( \{ U_n \}_{n \in \omega} \), $x \notin \bigcap_n U_n$.

... All these randomness notions are called nonstandard randomness.
A real $x$ is weakly-2-random if for all generalized Martin-Löf test \( \{ U_n \}_{n \in \omega} \), $x \notin \bigcap_n U_n$.

A real $x$ is $\emptyset'$-Schnorr-random if for all $\emptyset'$-Schnorr test \( \{ U_n \}_{n \in \omega} \), $x \notin \bigcap_n U_n$.

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All these randomness notions are called nonstandard randomness.
To be a reasonable randomness notion

Given a randomness notion $R$, $x \in R$ and $y \geq_K x$, then $y \in R$. 
Can these nonstandard randomness notions be defined in terms of Martin-Löf randomness?
Given a nonstandard randomness notion $R$, we use $\mathcal{F}(R)$ to denote the collection of all the classes $R$'s which have the property that for every real $z$, $z \in R$ if and only if for every real $x \in R$, $z \in \text{ML}(x)$.

Let $\Pi(R) = \bigcup_{R \in \mathcal{F}(R)} R$.

Note that $\Pi(R) \in \mathcal{F}(R)$. 

\[ \Pi(R) = \bigcup_{R \in \mathcal{F}(R)} R. \]
Definition

Given a nonstandard randomness notion $R$, we use $\mathcal{G}(R)$ to denote the collection of all the classes $R$’s which have the property that for every real $z$, $z \in R$ if and only if there exists some real $x \in R$, $z \in \text{ML}(x)$.

Let $\Sigma(R) = \bigcup_{R \in \mathcal{G}(R)} R$.

Note that $\Sigma(R) \in \mathcal{G}(R)$. 

**Σ-type**
Connect the two types

Proposition

Suppose that both $\Sigma(R)$ and $\Pi(R)$ exist, then for any real $x$, $x \in \Sigma(R)$ if and only if for every real $y \in \Pi(R)$, $y \leq_{LR} x$. 
If there is some $R \in \mathcal{F}(R)$ such that every $x \in R$ is Martin-Löf random, then we may obtain a Kolmogorov complexity characterization of $R$ and conclude that $R$ is $K$-upward closed.

Same for $\mathcal{G}(R)$
Two problems

1. Both $\Sigma(R)$ and $\Pi(R)$ seem are rather complicated, they do not appear to be second order arithmetical definable;

2. They may not exist at all.

$\Pi(W^2R, ML)$ does not exist.
Two problems

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$\Pi(W2R, ML)$ does not exist.
Low degrees

Let $Low$ denote the collection of the class of reals $x$ so that $x' \equiv_T \emptyset'$. So $x$ does not add power for its relativized halting problem.
On $\mathcal{F}(\text{Sch}(\emptyset'))$

**Theorem**

$\text{Low} \in \mathcal{F}(\text{Sch}(\emptyset'))$.

**Proof.**

A typical finite injury argument.

So $\mathcal{F}(\text{Sch}(\emptyset'))$ is not empty. Hence $\Pi(\text{Sch}(\emptyset'))$ exists.
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Random reals in $\mathcal{F}(\text{Sch}(\emptyset'))$

**Theorem**

$$\text{Low} \cap \text{ML} \in \mathcal{F}(\text{Sch}(\emptyset')).$$ 

**Proof.**

This is proved by a coding-decoding argument. We use a Kučera-Gacs coding and effective forcing to do the code, then use a finite injury to decode the coding construction.

Thus there is a natural Kolmogorov complexity characterization for $\text{Sch}(\emptyset')$. 

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**Σ-type characterization**

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Let $\mathcal{B}_L = \{ y \mid \exists x (x \in \text{Low} \land y \leq_{LR} x) \}$.

**Theorem**

$\mathcal{B}_L = \Pi(Sch(\emptyset'))$.

**Proof.**

This is proved by a forcing argument.

Thus $\Pi(Sch(\emptyset'))$ is arithmetical definable.
On $\Pi(\text{Sch}(\emptyset'))$

Let $\mathcal{B}L = \{y \mid \exists x (x \in \text{Low} \land y \leq_{LR} x)\}$.

**Theorem**

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Thus $\Pi(\text{Sch}(\emptyset'))$ is arithmetical definable.
Theorem

\[ \Sigma(Sch(\emptyset')) \] exits.

Proof.

This is proved by the low random theorem and a generalized van-Lambalgen theorem due to Miyabe.

It can be shown that \[ \Sigma(Sch(\emptyset')) \] is arithmetical definable by using a result due to Barmpalias, Miller and Nies.
Theorem

$\Sigma(Sch(\emptyset'))$ exits.

Proof.

This is proved by the low random theorem and a generalized van-Lambalgen theorem due to Miyabe.

It can be shown that $\Sigma(Sch(\emptyset'))$ is arithmetical definable by using a result due to Barmpalias, Miller and Nies.
We don’t know whether $\Sigma(W2R)$ exists.

For the other randomness notions...

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