A NEW PROOF OF FRIEDMAN’S CONJECTURE

LIANG YU

Abstract. We give a new proof of Friedman’s conjecture that every uncountable \( \Delta^1_1 \) set of reals has a member of each hyperdegree greater than or equal to the hyperjump.

1. Introduction

In 1975, Friedman [2] conjectured that every uncountable \( \Delta^1_1 \) set of reals has a member of each hyperdegree greater than or equal to the hyperdegree of Kleene’s \( \mathcal{O} \) (which we also refer to as the hyperjump). According to Friedman, the conjecture had two motivations.

The first was to study basis theorems for \( \Delta^1_1 \) sets of reals. A basis theorem is a kind of theorem which states that some classes of sets of reals must contain a “nice” real. For example, Gandy’s basis theorem says that every \( \Sigma^1_1 \) set of reals must contain a real \( x \) with \( x <_h \mathcal{O} \). The motivation for studying basis theorems is to pin down the complexity of some mathematical statements. Basis theorems have been applied widely in recursion theory and effective descriptive set theory. For example, a classical application of basis theorems in effective descriptive set theory is the proof of Shoenfield’s absoluteness theorem.

It is well known that the set \( A = \{ x \mid x \not\geq_h \mathcal{O} \} \) is an uncountable \( \Sigma^1_1 \) set. Sometimes \( \Sigma^1_1 \) sets are not “well behaved” since some sophisticated \( \Sigma^1_1 \) sets such as \( A \) do not contain an upper cone of hyperdegrees which is irregular. We may ask whether this already happens for \( \Delta^1_1 \) sets. Friedman’s conjecture refutes the possibility of such a kind of irregular property for \( \Delta^1_1 \) sets. Furthermore, according to Friedman’s conjecture, there are only two types of \( \Delta^1_1 \) sets of reals. They either only contain hyperarithmetic reals, or contain a real in each hyperdegree greater than or equal to the degree of \( \mathcal{O} \).

The second motivation is to know whether a result of Feferman, that there exists an uncountable \( \Sigma^1_1 \) antichain of hyperdegrees, remains true for \( \Delta^1_1 \) sets (A proof can be found in [11]). Feferman’s result is another kind of irregular property for \( \Sigma^1_1 \) sets. It means that a \( \Sigma^1_1 \) set can be “spread out”. Friedman’s conjecture refutes this possibility for \( \Delta^1_1 \) sets.

Eventually the conjecture was confirmed by Martin [7] (and independently, by Friedman himself). The idea of Martin’s ingenious proof is the following. For every...
Uncountable $\Delta^1_1$ set of reals, he can find a real coding a special function $h$. Then, using $h$, he defined a $\Sigma^1_1(h)$ set of functions in which every function grows fast. Using these facts, he was able to prove that the hyperjump is a $\Sigma^1_1(h)$ singleton. Thus the hyperjump is hyperarithmetically in $h$. All these were compressed in a one page proof, which is highly dense and not easy to understand.

In addition to Friedman’s original motivations, we have another one. Recently higher randomness was developed by Chong, Hjorth, Nies, Yu et al. (see [5] and [1]). Higher randomness theory requires many techniques in higher recursion theory developed in [10]. One of the major questions in this area is whether $\Pi^1_1$-randomness is different from strong $\Pi^1_1$-ML-randomness (these notions can be found in Chapter 9 of [9]). This question can be reduced to a conjecture which says that there is a strongly $\Pi^1_1$-ML-random real $x$ so that $x \equiv_h O$. Obviously we need a powerful coding argument to confirm the conjecture. But Martin’s proof, while ingenious, seems not flexible enough. Our proof is a finite injury argument which is natural and should be easier to understand. We also hope that our method can eventually be used to prove the higher randomness conjecture.

We assume that the reader has some basic knowledge of higher recursion theory. [10] is a good resource.

2. The proof of the conjecture

Suppose that $A \subseteq 2^\omega$ is an uncountable $\Delta^1_1$ set. There is a recursive function $j : \omega^\omega \rightarrow 2^\omega$ and a $\Pi^1_1$ set $B \subseteq \omega^\omega$ so that $j$ is function from $B$ onto $A$ and 1-1 restricted to $B$. That is there is a recursive oracle function $\Phi$ (see [8]) so that

1. For any $f \in \omega^\omega$, $\Phi^f = j(f)$;
2. For any $f \in B$, $\Phi^f \in A$;
3. For any $x \in A$, there is exactly one $f \in B$ so that $\Phi^f = x$.

Then for every $x \in B$, $j(x) \equiv_h x$ and for every $x \in A$, there exists some $y \in B$ so that $y \equiv_h j(y) = x$. So it suffices to show that $B$ has a member of each hyperdegree greater than or equal to the hyperjump. Since $B$ is a $\Pi^1_1$ class, there is a recursive tree $T \subseteq \omega^{<\omega}$ so that $B = [T] = \{y \mid \forall n(y \upharpoonright n \in T)\}$. We prove a slightly stronger result.

**Theorem 2.1** (Martin and Friedman). For any $\Sigma^1_1$ tree $T_1$ which has uncountably many infinite paths, $[T_1]$ has a member of each hyperdegree greater than or equal to the hyperjump.

The rest of this section is devoted to proving Theorem 2.1.

Let $T_1 \subseteq \omega^{<\omega}$ be a $\Sigma^1_1$-tree having a non-hyperarithmetical path. Then there is a $\Sigma_1$ enumeration of the nodes of $\omega^{<\omega} - T_1$ in $L_{\omega_1^{CK}}$. We use $T_1[\beta]$ to denote the tree $T_1[\beta]$ enumerated up to the $\beta$-th stage.

Fix a recursive tree $T_0 \subseteq \omega^{<\omega}$ having an infinite path but without any hyperarithmetical infinite path. Let $f$ be the leftmost path of $T_0$. Then $f \equiv_h O$.

**Lemma 2.2.** If $[\sigma] \cap T_1$ has no non-hyperarithmetical infinite path, then there must be some stage $\alpha < \omega_1^{CK}$ witnessing that. In other words, there is a $\Sigma_1(L_{\omega_1^{CK}})$ partial
function $H : \omega^{< \omega} \to \{0\}$ so that $H(\sigma)$ is undefined if and only if $T_1 \cap [\sigma] = \{ \tau \in T_1 \mid \tau \succ \sigma \}$ contains a non-hyperarithmetic infinite path.

Proof. Suppose that $\sigma \in T_1$ for which $[\sigma] \cap T_1$ has no non-hyperarithmetic infinite path. Then we can do a Cantor-Bendixon construction as follows:

Let $<KB$ be the Kleene-Brouwer order over $\omega^{< \omega}$. At any stage $\beta < \omega_1^{CK}$, we construct a subtree $T_1^{\beta}$. $T_1^{\beta}$ has the same enumeration as $T_1^{[\beta]}$ except those finite strings not in $T_1^{\beta}$. In other words, we make sure $T_1^{\alpha} = T_1^{\alpha} \cap T_1[\alpha] \subseteq T_1[\alpha]$ for any $\alpha < \omega_1^{CK}$.

At stage 0, let $T_0 = T_1[0]$.

At stage $\beta + 1 < \omega_1^{CK}$, for any $\tau \succ \sigma$, if

(1) either there exists an order preserving (in the $<KB$ sense) function $f \in L_\beta$ so that $f : [\tau] \cap T_1^{\beta} \to \beta$; or

(2) there exists a function $f \in L_\beta$ so that $[\tau] \cap T_1^{\beta} = \{ f \upharpoonright (|\tau| + i) \mid i \in \omega \}$, then we let $T_1^{\beta+1} = T_1^{\beta} - [\tau]$ and claim that $\tau$ is cut from $T_1$ at stage $\beta$.

By the usual Cantor-Bendixon proof, if $[\sigma] \cap T_1$ has no non-hyperarithmetic infinite path, then every $\tau \succ \sigma$ is cut at some stage $\beta$. Then by the admissibility of $\omega_1^{CK}$, there must be some stage $\alpha < \omega_1^{CK}$ such that every $\tau \succ \sigma$ is cut at $\alpha$. Then we define $H(\sigma) = 0$. \hfill \Box

So $H$ can be viewed as a $\Sigma_1$ partial enumeration function in $L_{\omega_1^{CK}}$. Then we call a node $\sigma \in \omega^{< \omega}$ as a splitting node in the tree $T_1$ if there are two numbers $i \neq j$ so that both $H(\sigma \upharpoonright i)$ and $H(\sigma \upharpoonright j)$ are undefined. For $\alpha < \omega_1^{CK}$, we call a node $\sigma \in \omega^{< \omega}$ as a splitting node in the tree $T_1[\alpha]$ if there are two numbers $i \neq j$ so that both $H(\sigma \upharpoonright i)$ and $H(\sigma \upharpoonright j)$ are undefined at stage $\alpha$.

By the same method, there is a $\Sigma_1(L_{\omega_1^{CK}})$ partial function $K : \omega^{< \omega} \to \{0\}$ so that $K(\sigma)$ is undefined if and only if there exists an infinite path in $T_0 \cap [\sigma] = \{ \tau \in T_0 \mid \tau \succ \sigma \}$.

Note that this is different from $H$.

Then we call a node $\sigma \in \omega^{< \omega}$ as a dead node in $T_0$ if $K(\sigma)$ is defined. For $\alpha < \omega_1^{CK}$, we call a node $\sigma \in \omega^{< \omega}$ is a dead node in $T_0[\alpha]$ if $K(\sigma)$ is defined at stage $\alpha$.

Now for any real $x \geq 0$ in $2^\omega$, we build a real $z \in [T_1]$ as follows:

At stage 0, let $\sigma_0 = \emptyset$.

At stage $s + 1$, let $\sigma' \succ \sigma_s$ be the leftmost splitting node of $T_1$ extending $\sigma_s$ so that there are $f(s)$ many splitting nodes between $\sigma_s$ and $\sigma'$. Let $\sigma'' \succ \sigma'$ be the second-leftmost splitting node extending $\sigma'$ in $T_1$. Set $\sigma_{s+1} \succ \sigma''$ to be the $x(s) + 1$-th-leftmost splitting node extending $\sigma''$ in $T_1$.

Let $z = \bigcup_{s \in \omega} \sigma_s$.

We prove that $x \in L_{\omega_1^{CK}}[z]$. Actually we show that there is a $\Sigma_1(L_{\omega_1^{CK}}[z])$ (and so $\Delta_1(L_{\omega_1^{CK}}[z])$)-definable increasing sequence ordinals $\{\alpha_s\}_{s \in \omega}$ below $\omega_1^{CK}$ so that $x \in L(\bigcup_{s \in \omega} \alpha_s + 1)[z]$. Then $\bigcup_{s \in \omega} \alpha_s + 1 < \omega_1^{CK}$. So $x \in L_{\omega_1^{CK}}[z]$.\footnote{One may conjecture that $\beta = \bigcup_{s \in \omega} \alpha_s + 1 < \omega_1^{CK}$. This is not true. Suppose that this is true. Let $A = \{ g \in [T_1] \mid L_\beta[g] = 3f(f$ is the leftmost path in $[T_0]) \}$. Then $A$ is a non-empty $\Sigma_1$ set. By the Gandy’s basis theorem, there is a real $g \in A$ so that $\omega_1^T = \omega_1^{CK}$. But for any $f$,}
The idea is to decode the construction in $L_{\omega_1^1}[z]$. Before proceeding the formal decoding proof, we give a sketch of the idea behind it. Essentially we want to use $z$ to decode the $n$-th value of the leftmost path $f$ of $T_0$ for any $n \in \omega$. First we give an example for $n = 0$. The idea is: at any stage $\alpha$, we consecutively count, from the root of $T_1[\alpha]$, how many times $z$ goes the left way in $T[\alpha]$. Say $l_\alpha$-times. Then we guess $l_\alpha \in \omega$ should be the value of $f(0)$. Of course we maybe wrong. To avoid those naive mistakes, we also need to check whether $f(0)$ is $l_\alpha$ at stage $\alpha$. In other words, $l_\alpha$ is not a dead node in $T_0[\alpha]$ but $l'$ is dead for every $l' < l_\alpha$. If not, then we need to go to a bigger stage $\alpha' > \alpha$ to check this fact. By the coding procedure, there must be some stage $\alpha < \omega_1^{CK}$ so that $l_\alpha$ matches the value of $f(0)$ at stage $\alpha$. Let $\alpha_1$ be the first such stage. But we still cannot make sure whether $l_{\alpha_1}$ is the real $f(0)$. Both $l_{\alpha_1}$ and $f(0)$ may change later. But this cannot happen infinitely often since otherwise either $l_\alpha$ goes to infinity or $l_\alpha = l$ infinitely often for some finite number $l$. In the both cases, since “$\sigma$ is a dead node in $T_1$” is a $\Sigma_1$ statement, $z$ would be the leftmost path in $T_1$, a contradiction. Hence, we make mistakes at most finitely many times. So $l_\alpha$ will be stable. For a general number $n$, to decide $f(n)$, we iterate the construction for $f(0)$. This is a typical finite injury argument. The full approximation is that: at the decoding stage $s$, once we find that the finite sequence $(0, l_\alpha^n)^\frown (1, l_\alpha^n)^\frown \cdots ^\frown (n, l_\alpha^n)$ matches with $f \upharpoonright n$ at stage $\alpha$, we set $\alpha_s$ to be $\alpha$. They both may change later. But this happens at most finitely often. So $l_\alpha^n$ will be stable. Take $\gamma$ be the least upper bound of $\{\alpha_s\}_{s \in \omega}$. Obviously $\gamma < \omega_1$ and the limit of $\{f_{\alpha_s}\}_{s \in \omega}$ is an infinite path in $T_0[\gamma]$. Since $[T_0] = [T_0[\gamma]]$, it is also an infinite path in $T_0$. More than that, it is the leftmost path. So $z$ hyperarithmetically compute the leftmost path of $T_0$. Once this is done, then one can show, by an easy zig-zag coding method, that $z \equiv_b x$.

Let us turn to the formal proof.

We define $\alpha_s, \sigma_s^n, f_s(n)$ as follows. What we try to do is to make that $\lim_n f_s(n) = f(n)$.

At stage 0, let $\alpha_0 = f_0(n) = 0$ for every $n$ and $\sigma_0^0 = \emptyset$. We claim that 0 receives attention.

Before proceeding the construction, we introduce some notations. At any stage $s$, if $j \leq s$ receives attention, then we will introduce some notations $\alpha_s, f_s(j)$ and $\sigma_s^j$ for $j$. At stage $s$, we say that $z$ is correct up to $j$ at $\alpha$ if for every $0 < i \leq j$,

1. $f_s \upharpoonright (i + 1)$ is not a dead node but every $\sigma$ at the left of $f_s \upharpoonright (i + 1)$ is dead in $T_0[\alpha]$; and

2. There exists a $\sigma' \prec \sigma_s^i$ which is the leftmost splitting node of $T_1[\alpha]$ so that $\sigma'$ have $f_s(i)$ many splitting nodes extending $\sigma_s^{i-1}$ in $T_1[\alpha]$; and

3. For the $\sigma'$ above, there is a node $\sigma'' \prec \sigma_s^i$ extending $\sigma'$ which is the second-leftmost next splitting node extending $\sigma'$ in $T_1[\alpha]$ so that $\sigma_s^i$ is the next splitting node extending $\sigma''$ in $T_1[\alpha]$.

if “$L_\beta[g] \models f$ is the leftmost path in $[T_0]$”, then $f$ is the leftmost path in $[T_0]$. So $\omega_1^\beta > \omega_1^{CK}$, a contradiction. Harrington even proved that $\eta^{(3)} \not\leq_T \emptyset^{(3)}$ for some $\Pi_3^0$-tree $T_1$. See Theorem 3.2.
At stage $s + 1$. Check whether $j$ is not correct up to $j$ at $\alpha_s + 1$. Let $j$ be the least such number. Then initialize all the $j' > j$ and keep all the parameters for $j' < j$ unchanging at stage $s + 1$.

Case (1). $j$ has not received any attention after initialized. Let $\alpha_{s+1} = \alpha_s + 1$, $\sigma'_s > \sigma'^{-1}_s$ be the next splitting node in $T_1[\alpha_{s+1}]$ and $f_{s+1}(j) = 0$ and claim that $j$ receives attention at stage $s + 1$.

Case (2). Otherwise. We search a stage $\alpha > \alpha_s$ below $\omega_1^{CK}$ so that at stage $s$, $z$ is correct up to $j - 1$ at $\alpha$ and there exists some $\sigma < z$ extending $\sigma'^{-1}_s$ so that there exists the least $l$ for which:

1. $f_s \upharpoonright j^- (j, l)$ is not a dead node but for every $l' < l$, $f_s \upharpoonright j^-(j, l')$ is dead in $T_0[\alpha]$; and
2. There exists a $\sigma' \prec \sigma$ which is the leftmost splitting node of $T_1[\alpha]$ so that $\sigma'$ has $l$ many splitting nodes extending $\sigma'^{-1}_s$ in $T_1[\alpha]$; and
3. For the $\sigma'$ above, there is a node $\sigma'' \prec \sigma$ extending $\sigma'$ which is the second-leftmost next splitting node extending $\sigma'$ in $T_1[\alpha]$ so that $\sigma$ is the next splitting node extending $\sigma''$ in $T_1[\alpha]$.

If during the search, $z$ is always correct up to $j - 1$, then by the construction of $z$, we can find such an $\alpha$. If so, then let $\alpha_{s+1} = \alpha$, $f_{s+1}(j) = l$ and $\alpha'_s = \sigma$ and claim that $j$ receives attention at stage $s + 1$. Otherwise, $z$ is not correct up to $j - 1$ at $\alpha$, then we go to $j - 1$ and do the search again. But this only happens finitely many times since $z$ is always correct up to $0$ at any $\alpha$.

This finishes the construction.

Let $\gamma = \bigcup_{s \in \omega} \alpha_s$. Obviously $\{\alpha_s\}_{s \in \omega}$ is a $\Delta_1(L_{\omega_1^z})$ sequence. So $\gamma < \omega_1^z$.

**Lemma 2.3.** For every $j$, there exists a stage $s_j$ so that for every $t \geq s_j$, $z$ is correct up to $j$ at $\alpha_t$.

**Proof.** Otherwise, let $j$ be the least number so that for every $s$ there is some $t \geq s$ so that $z$ is not correct up to $j$ at $\alpha_t$. Let $s_j$ be the least stage so that for every $t \geq s_n$, $z$ is correct up to $j - 1$ at $\alpha_t$.

For any stage $t > s_j$, we let $\sigma'_t$ be the $\sigma'$ as defined in the construction for $j$. In other words, $\sigma'_t \prec \sigma$ is the leftmost splitting node of $T_1[\alpha_t]$ so that $\sigma'_t$ has $f_t(j)$ many splitting nodes extending $\sigma'^{-1}_s$ in $T_1[\alpha_t]$. Note that for any $t \geq s_j$, $f_{t+1}(j) \geq f_t(j)$. So $|\sigma'_{t+1}| \geq |\sigma'_t|$ for any $t > s_j$. Since for any $\sigma$, $z$ cannot be the leftmost path of $T_1 \cap [\sigma]$, $\lim_t \sigma'_t$ exists. Let $\sigma' = \lim_t \sigma'_t$ and $t_1 > s_j$ so that $\sigma'_t = \sigma'$ for any $t \geq t_1$. Then, by the construction, at any stage $t > t_1$, $f_t(j) = f_{t_1}(j)$. Since for any $\sigma$, $z$ cannot be the leftmost path of $T_1 \cap [\sigma]$, there must be some stage $t_2 \geq t_1$ so that for any $t \geq t_2$, $\sigma'_t = \sigma'_{t_2}$.

Then for any stage $t \geq t_2$, $z$ is correct up to $j$ at $\alpha_t$. \hfill $\Box$

By Lemma 2.3, we may let $\hat{f}(j) = \lim_t f_t(j)$. Obviously, $\hat{f} \in L_{\omega_{\gamma+1}^z}$. Note that for any $j$, $T_0 \supseteq \{\sigma \mid \forall k \leq j(\sigma(k) = \hat{f}(k))\}$. So $\hat{f}$ is an infinite path in $T_0$. By the construction, $\hat{f} = f$. So $f \in L_{\omega_{\gamma+1}^z}$. Since $f \geq_h \mathcal{O}$, we have that $\mathcal{O} \leq_h z$ and $\omega_1^{CK} < \omega_1^z$. Then $z$ can easily decode the coding construction. Thus $z \geq_h x$. 


If $x \geq_h \mathcal{O}$, then $x$ can also easily decode the construction. Hence $x \equiv_h z$.

3. SOME REMARKS

There are several ways to generalize Friedman’s conjecture.

The first one that was announced in [3] is due to Harrington.

**Theorem 3.1** (Harrington). Suppose $A$ is a $\Pi^1_1$ set of reals, and $\alpha$ is the least admissible ordinal so that there is some real $x \in A$ such that $\omega^x_1 = \alpha$ and $x \not\in L_{\omega^x_1}$. Then, for any $y$ with $\omega^y_1 > \alpha$, there exists a real $x \in A$ so that $x \equiv_h y$.

By Gandy’s basis theorem, Friedman’s conjecture is an immediate consequence of Harrington’s theorem. However, the proof of Theorem 3.1 is extremely sophisticated and has never been published. Actually, our proof uses some ideas from Harrington’s proof of Theorem 3.1. Another interesting conclusion of Theorem 3.1 is that $\Pi^1_1$-hyper-determinacy is equivalent to that for every real $x$, there is a non-$x$-constructible real $y$. So, in the consistency strength sense, $\Pi^1_1$-hyper-determinacy is much weaker than $\Pi^1_1$-Turing determinacy, of which the consistency strength is equivalent to the existence of $0^\#$. After Friedman’s conjecture had been settled, it was conjectured that for any recursive tree $T$ in $\omega^{<\omega}$ having a non-hyperaithmetic infinite path, the Turing jumps of the infinite paths in $T$ range over an upper cone of Turing degrees. This conjecture was refuted by Harrington (a proof can be found in [4]). He proves the following result.

**Theorem 3.2** (Harrington). There is a recursive tree $T \subseteq 2^{<\omega}$ in which there are uncountably many infinite paths such that for any real $x \in [T]$ and ordinal $\alpha < \omega^{CK}_1$, $x^{(\alpha)} \not\leq_T 0^{(\alpha+1)}$, where $x^{(\alpha)}$ is the $\alpha$-th Turing jump of $x$.

The third way to generalize Friedman’s conjecture is to look “higher up”. Woodin proves the following remarkable result (for more information about $Q$-theory, see [6]).

**Theorem 3.3** (Woodin). Assuming $\Delta^1_{2n+1}$-Determinacy, if $A \subseteq 2^{\omega}$ is a $\Delta^1_{2n+1}$ set without a $Q_{2n+1}$ member, then for every $x \geq_{2n+1} y_{2n+1}^0$ there is a real $z \in A$ for which $z \equiv_{2n+1} x$ (where $y_{2n+1}^0$ denotes the $Q_{2n+1}$-jump).

The last way is to partially relativize the conjecture.

**Question 3.4.** Is it true that for any real $x$ and $\Delta^1_1(x)$ set $A$ of reals, if there is a real $y \in A$ with $y >_h x$, then for every real $z \geq_h \mathcal{O}^x$, there is a real $y \in A$ with $y \equiv_h z$?

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Institute of Mathematical Science, Nanjing University, P.R. of China 210093

State Key Laboratory for Novel Software Technology at Nanjing University, Nanjing University, P.R. of China 210093

E-mail address: yuliang.nju@gmail.com