A NEW PROOF OF FRIEDMAN'S CONJECTURE

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ABSTRACT. We give a new proof of Friedman's conjecture that every uncountable Δ^1_1 set of reals has a member of each hyperdegree greater than or equal to the hyperjump.

1. Introduction

In 1975, Friedman [2] conjectured that every uncountable Δ^1_1 set of reals has a member of each hyperdegree greater than or equal to the hyperdegree of Kleene's $\mathcal{O}$ (which we also refer to as the hyperjump). According to Friedman, the conjecture had two motivations.

The first was to study basis theorems for Δ^1_1 sets of reals. A basis theorem is a kind of theorem which states that some classes of sets of reals must contain a "nice" real. For example, Gandy's basis theorem says that every Σ^1_1 set of reals must contain a real x with $x <_h \mathcal{O}$. The motivation for studying basis theorems is to pin down the complexity of some mathematical statements. Basis theorems have been applied widely in recursion theory and effective descriptive set theory. For example, a classical application of basis theorems in effective descriptive set theory is the proof of Shoenfield's absoluteness theorem.

It is well known that the set $A = \{x \mid x \not\geq_h \mathcal{O}\}$ is an uncountable Σ^1_1 set. Sometimes Σ^1_1 sets are not "well behaved" since some sophisticated Σ^1_1 sets such as A do not contain an upper cone of hyperdegrees which is irregular. We may ask whether this already happens for Δ^1_1 sets. Friedman's conjecture refutes the possibility of such a kind of irregular property for Δ^1_1 sets. Furthermore, according to Friedman's conjecture, there are only two types of Δ^1_1 sets of reals. They either only contain hyperarithmetic reals, or contain a real in each hyperdegree greater than or equal to the degree of $\mathcal{O}$.

The second motivation is to know whether a result of Feferman, that there exists an uncountable Σ^1_1 antichain of hyperdegrees, remains true for Δ^1_1 sets (A proof can be found in [11]). Feferman's result is another kind of irregular property for Σ^1_1 sets. It means that a Σ^1_1 set can be "spread out". Friedman's conjecture refutes this possibility for Δ^1_1 sets.

Eventually the conjecture was confirmed by Martin [7] (and independently, by Friedman himself). The idea of Martin's ingenious proof is the following. For every

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uncountable Δ^1_1 set of reals, he can find a real coding a special function h. Then, using h, he defined a $\Sigma^1_1(h)$ set of functions in which every function grows fast. Using these facts, he was able to prove that the hyperjump is a $\Sigma^1_1(h)$ singleton. Thus the hyperjump is hyperaithmetic in h. All these were compressed in a one page proof, which is highly dense and not easy to understand.

In addition to Friedman's original motivations, we have an further one. Recently higher randomness was developed by Chong, Hjorth, Nies, Yu et al (see [5] and [1]). Higher randomness theory requires many techniques in higher recursion theory developed in [10]. One of the major questions in this area is whether Π^1_1-randomness is different from strong Π^1_1-ML-randomness (these notions can be found in Chapter 9 of [9]). This question can be reduced to a conjecture which says that there is a strongly Π^1_1-ML-random real x so that $x \geq_h \mathcal{O}$. Obviously we need a powerful coding argument to confirm the conjecture. But Martin's proof, while ingenious, seems not flexible enough. Our proof is a finite injury argument which is natural and should be easier to understand. We also hope that our method can eventually be used to prove the higher randomness conjecture.

We assume that the reader has some basic knowledge of higher recursion theory. [10] is a good resource.

2. The proof of the conjecture

Suppose that $A \subseteq 2^\omega$ is an uncountable Δ^1_1 set. There is a recursive function $j : \omega^\omega \to 2^\omega$ and a Π^0_1 set $B \subseteq \omega^\omega$ so that j is function from B onto A and 1-1 restricted to B. That is there is a recursive oracle function Φ (see [8]) so that

(1) For any $f \in \omega^\omega$, $\Phi^f = j(f)$;
(2) For any $f \in B$, $\Phi^f \in A$;
(3) For any $x \in A$, there is exactly one $f \in B$ so that $\Phi^f = x$.

Then for every $x \in B$, $j(x) \equiv_h x$ and for every $x \in A$, there exists some $y \in B$ so that $y \equiv_h j(y) = x$. So it suffices to show that B has a member of each hyperdegree greater than or equal to the hyperjump. Since B is a Π^0_1 class, there is a recursive tree $T \subseteq \omega^{<\omega}$ so that $B = [T] = \{y \mid \forall n(y \upharpoonright n \in T)\}$. We prove a slightly stronger result.

Theorem 2.1 (Martin and Friedman). *For any Σ^1_1 tree T_1 which has uncountably many infinite paths, $[T_1]$ has a member of each hyperdegree greater than or equal to the hyperjump.*

The rest of this section is devoted to proving Theorem 2.1.

Let $T_1 \subseteq \omega^{<\omega}$ be a Σ^1_1-tree having a non-hyperaithmetic path. Then there is a Σ_1 enumeration of the nodes of $\omega^{<\omega} - T_1$ in $L_{\omega_1^{\mathrm{CK}}}$. We use $T_1[\beta]$ to denote the tree $T_1[\beta]$ enumerated up to the β-th stage.

Fix a recursive tree $T_0 \subseteq \omega^{<\omega}$ having an infinite path but without any hyperarithmetic infinite path. Let f be the leftmost path of T_0. Then $f \equiv_h \mathcal{O}$.

Lemma 2.2. *If $[\sigma] \cap T_1$ has no non-hyperarithmetic infinite path, then there must be some stage $\alpha < \omega_1^{\mathrm{CK}}$ witnessing that. In other words, there is a $\Sigma_1(L_{\omega_1^{\mathrm{CK}}})$ partial* *function* $H : \omega^{<\omega} \to \{0\}$ *so that* $H(\sigma)$ *is undefined if and only if* $T_1 \cap [\sigma] = \{\tau \in T_1 \mid \tau \succ \sigma\}$ *contains a non-hyperarithmetic infinite path.*

Proof. Suppose that $\sigma \in T_1$ for which $[\sigma] \cap T_1$ has no non-hyperarithmetic infinite path. Then we can do a Cantor-Bendixon construction as follows:

Let $<_{KB}$ be the Kleene-Brouwer order over $\omega^{<\omega}$. At any stage $\beta < \omega_1^{\mathrm{CK}}$, we construct a subtree T_1^β. T_1^β has the same enumeration as $T_1[\beta]$ except those finite strings not in T_1^β. In other words, we make sure $T_1^\alpha = T_1^\alpha \cap T_1[\alpha] \subseteq T_1[\alpha]$ for any $\alpha < \omega_1^{\mathrm{CK}}$.

At stage 0, let $T_1^0 = T_1[0]$.

At stage $\beta + 1 < \omega_1^{\mathrm{CK}}$, for any $\tau \succ \sigma$, if

1. either there exists an order preserving (in the $<_{KB}$ sense) function $f \in L_\beta$ so that $f : [\tau] \cap T_1^\beta \to \beta$; or
2. there exists a function $f \in L_\beta$ so that $[\tau] \cap T_1^\beta = \{f \upharpoonright (|\tau| + i) \mid i \in \omega\}$,

then we let $T_1^{\beta+1} = T_1^\beta - [\tau]$ and claim that τ is cut from T_1 at stage β.

By the usual Cantor-Bendixon proof, if $[\sigma] \cap T_1$ has no non-hyperarithmetic infinite path, then every $\tau \succ \sigma$ is cut at some stage β. Then by the admissibility of ω_1^{CK}, there must be some stage $\alpha < \omega_1^{\mathrm{CK}}$ such that every $\tau \succ \sigma$ is cut at α. Then we define $H(\sigma) = 0$. $\square$

So H can be viewed as a Σ_1 partial enumeration function in $L_{\omega_1^{\mathrm{CK}}}$. Then we call a node $\sigma \in \omega^{<\omega}$ as a *splitting node in the tree* T_1 if there are two numbers $i \neq j$ so that both $H(\sigma^\frown i)$ and $H(\sigma^\frown j)$ are undefined. For $\alpha < \omega_1^{\mathrm{CK}}$, we call a node $\sigma \in \omega^{<\omega}$ as a *splitting node in the tree* $T_1[\alpha]$ if there are two numbers $i \neq j$ so that both $H(\sigma^\frown i)$ and $H(\sigma^\frown j)$ are undefined at stage α.

By the same method, there is a $\Sigma_1(L_{\omega_1^{\mathrm{CK}}})$ partial function $K : \omega^{<\omega} \to \{0\}$ so that $K(\sigma)$ is undefined if and only if there exists an infinite path in $T_0 \cap [\sigma] = \{\tau \in T_0 \mid \tau \succ \sigma\}$. Note that this is different from H.

Then we call a node $\sigma \in \omega^{<\omega}$ as a *dead node in* T_0 if $K(\sigma)$ is defined. For $\alpha < \omega_1^{\mathrm{CK}}$, we call a node $\sigma \in \omega^{<\omega}$ is a *dead node in* $T_0[\alpha]$ if $K(\sigma)$ is defined at stage α.

Now for any real $x \geq_h \mathcal{O}$ in 2^ω, we build a real $z \in [T_1]$ as follows:

At stage 0, let $\sigma_0 = \emptyset$.

At stage $s+1$, let $\sigma' \succ \sigma_s$ be the leftmost splitting node of T_1 extending σ_s so that there are $f(s)$ many splitting nodes between σ_s and σ'. Let $\sigma'' \succ \sigma'$ be the second-leftmost splitting node extending σ' in T_1. Set $\sigma_{s+1} \succ \sigma''$ to be the $x(s)+1$-th-leftmost splitting node extending σ'' in T_1.

Let $z = \bigcup_{s \in \omega} \sigma_s$.

We prove that $x \in L_{\omega_1^z}[z]$. Actually we show that there is a $\Sigma_1(L_{\omega_1^z}[z])$ (and so $\Delta_1(L_{\omega_1^z}[z])$)-definable increasing sequence ordinals $\{\alpha_s\}_{s \in \omega}$ below ω_1^{CK} so that $x \in L_{(\bigcup_s \alpha_s)+1}[z]$. Then $\bigcup_s \alpha_s + 1 < \omega_1^z$.[1] So $x \in L_{\omega_1^z}[z]$.

[1] One may conjecture that $\beta = \bigcup_s \alpha_s + 1 < \omega_1^{\mathrm{CK}}$. This is not true. Suppose that this is true. Let $A = \{g \in [T_1] \mid L_\beta[g] \models \exists f (f \text{ is the leftmost path in } [T_0])\}$. Then A is a non-empty Σ_1^1 set. By the Gandy's basis theorem, there is a real $g \in A$ so that $\omega_1^g = \omega_1^{\mathrm{CK}}$. But for any f,

The idea is to decode the construction in $L_{\omega_1^z}[z]$.

Before proceeding the formal decoding proof, we give a sketch of the idea behind it. Essentially we want to use z to decode the n-th value of the leftmost path f of T_0 for any $n \in \omega$. First we give an example for $n = 0$. The idea is: at any stage α, we consecutively count, from the root of $T_1[\alpha]$, how many times z goes the left way in $T[\alpha]$. Say l_α-times. Then we guess $l_\alpha \in \omega$ should be the value of $f(0)$. Of course we maybe wrong. To avoid those naive mistakes, we also need to check whether $f(0)$ is l_α at stage α. In other words, l_α is not a dead node in $T_0[\alpha]$ but l' is dead for every $l'_\alpha < l_\alpha$. If not, then we need to go to a bigger stage $\alpha' > \alpha$ to check this fact. By the coding procedure, there must be some stage $\alpha < \omega_1^{\mathrm{CK}}$ so that l_α matches the value of $f(0)$ at stage α. Let α_1 be the first such stage. But we still cannot make sure whether l_{α_1} is the real $f(0)$. Both l_{α_1} and $f(0)$ may change later. But this cannot happen infinitely often since otherwise either l_α goes to infinity or $l_\alpha = l$ infinitely often for some finite number l. In the both cases, since "σ is a dead node in T_1" is a Σ_1 statement, z would be the leftmost path in T_1, a contradiction. Hence, we make mistakes at most finitely many times. So l_α will be stable. For a general number n, to decide $f(n)$, we iterate the construction for $f(0)$. This is a typical finite injury argument. The full approximation is that: at the decoding stage s, once we find that the finite sequence $(0, l_\alpha^0)^\frown(1, l_\alpha^1)^\frown...(n, l_\alpha^n)$ matches with $f \upharpoonright n$ at stage α, we set α_s to be α. They both may change later. But this happens at most finitely often. So l_α^n will be stable. Take γ be the least upper bound of $\{\alpha_s\}_{s\in\omega}$. Obviously $\gamma < \omega_1^z$ and the limit of $\{f_{\alpha_s}\}_{s\in\omega}$ is an infinite path in $T_0[\gamma]$. Since $[T_0] = [T_0[\gamma]]$, it is also an infinite path in T_0. More than that, it is the leftmost path. So z hyperarithmetically compute the leftmost path of T_0. Once this is done, then one can show, by an easy zig-zag coding method, that $z \equiv_h x$.

Let us turn to the formal proof.

We define $\alpha_s, \sigma_s^n, f_s(n)$ as follows. What we try to do is to make that $\lim_s f_s(n) = f(n)$.

At stage 0, let $\alpha_0 = f_0(n) = 0$ for every n and $\sigma_0^0 = \emptyset$. We claim that 0 receives attention.

Before proceeding the construction, we introduce some notations. At any stage s, if $j \leq s$ receives attention, then we will introduce some notations α_s, $f_s(j)$ and σ_s^j for j. At stage s, we say that z *is correct up to* j *at* α if for every $0 < i \leq j$,

(1) $f_s \upharpoonright (i+1)$ is not a dead node but every σ at the left of $f_s \upharpoonright (i+1)$ is dead in $T_0[\alpha]$; and
(2) There exists a $\sigma' \prec \sigma_s^i$ which is the leftmost splitting node of $T_1[\alpha]$ so that σ' have $f_s(i)$ many splitting nodes extending σ_s^{i-1} in $T_1[\alpha]$; and
(3) For the σ' above, there is a node $\sigma'' \prec \sigma_s^i$ extending σ' which is the second-leftmost next splitting node extending σ' in $T_1[\alpha]$ so that σ_s^i is the next splitting node extending σ'' in $T_1[\alpha]$

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if "$L_\beta[g] \models f$ is the leftmost path in $[T_0]$", then f is the leftmost path in $[T_0]$. So $\omega_1^g > \omega_1^{\mathrm{CK}}$, a contradiction. Harrington even proved that $g^{(\beta)} \not\geq_T \emptyset^{(\beta)}$ for some Π_1^0-tree T_1. See Theorem 3.2.

At stage $s+1$. Check whether j is not correct up to j at α_s+1. Let j be the least such number. Then initialize all the $j'>j$ and keep all the parameters for $j'<j$ unchanging at stage $s+1$.

Case(1). j has not received any attention after initialized. Let $\alpha_{s+1}=\alpha_s+1$, $\sigma_s^j \succ \sigma_s^{j-1}$ be the next splitting node in $T_1[\alpha_{s+1}]$ and $f_{s+1}(j)=0$ and claim that j receives attention at stage $s+1$.

Case(2). Otherwise. We search a stage $\alpha>\alpha_s$ below ω_1^{CK} so that at stage s, z is correct up to $j-1$ at α and there exists some $\sigma \prec z$ extending σ_s^{j-1} so that there exists the least l for which :

(1) $f_s \upharpoonright j^\frown(j,l)$ is not a dead node but for every $l'<l$, $f_s \upharpoonright j^\frown(j,l')$ is dead in $T_0[\alpha]$; and
(2) There exists a $\sigma' \prec \sigma$ which is the leftmost splitting node of $T_1[\alpha]$ so that σ' has l many splitting nodes extending σ_s^{j-1} in $T_1[\alpha]$; and
(3) For the σ' above, there is a node $\sigma'' \prec \sigma$ extending σ' which is the second-leftmost next splitting node extending σ' in $T_1[\alpha]$ so that σ is the next splitting node extending σ'' in $T_1[\alpha]$

If during the search, z is always correct up to $j-1$, then by the construction of z, we can find such an α. If so, then let $\alpha_{s+1}=\alpha$, $f_{s+1}(j)=l$ and $\sigma_s^j=\sigma$ and claim that j receives attention at stage $s+1$. Otherwise, z is not correct up to $j-1$ at α, then we go to $j-1$ and do the search again. But this only happens finitely many times since z is always correct up to 0 at any α.

This finishes the construction.

Let $\gamma=\bigcup_{s\in\omega}\alpha_s$.

Obviously $\{\alpha_s\}_{s\in\omega}$ is a $\Delta_1(L_{\omega_1^z}[z])$ sequence. So $\gamma<\omega_1^z$.

Lemma 2.3. *For every j, there exists a stage s_j so that for every $t\geq s_j$, z is correct up to j at α_t.*

Proof. Otherwise, let j be the least number so that for every s there is some $t\geq s$ so that z is not correct up to j at α_t. Let s_j be the least stage so that for every $t\geq s_n$, z is correct up to $j-1$ at α_t.

For any stage $t>s_j$, we let σ_t' be the σ' as defined in the construction for j. In other words, $\sigma_t' \prec \sigma$ is the leftmost splitting node of $T_1[\alpha_t]$ so that σ_t' has $f_t(j)$ many splitting nodes extending σ_s^{j-1} in $T_1[\alpha_t]$. Note that for any $t\geq s_j$, $f_{t+1}(j)\geq f_t(j)$. So $|\sigma_{t+1}'|\geq|\sigma_t'|$ for any $t>s_j$. Since for any σ, z cannot be the leftmost path of $T_1\cap[\sigma]$, $\lim_t \sigma_t'$ exists. Let $\sigma'=\lim_t\sigma_t'$ and $t_1>s_j$ so that $\sigma_t'=\sigma'$ for any $t\geq t_1$. Then, by the construction, at any stage $t>t_1$, $f_t(j)=f_{t_1}(j)$. Since for any σ, z cannot be the leftmost path of $T_1\cap[\sigma]$, there must be some stage $t_2\geq t_1$ so that for any $t\geq t_2$, $\sigma_t^j=\sigma_{t_2}^j$.

Then for any stage $t\geq t_2$, z is correct up to j at α_t. □

By Lemma 2.3, we may let $\hat{f}(j)=\lim_t f_t(j)$. Obviously, $\hat{f}\in L_{\gamma+1}[z]$. Note that for any j, $T_0\supseteq\{\sigma \mid \forall k\leq j(\sigma(k)=\hat{f}(k))\}$. So $\hat{f}$ is an infinite path in T_0. By the construction, $\hat{f}=f$. So $f\in L_{\gamma+1}[z]$. Since $f\geq_h \mathcal{O}$, we have that $\mathcal{O}\leq_h z$ and $\omega_1^{\mathrm{CK}}<\omega_1^z$. Then z can easily decode the coding construction. Thus $z\geq_h x$.

If $x \geq_h \mathcal{O}$, then x can also easily decode the construction. Hence $x \equiv_h z$.

3. SOME REMARKS

There are several ways to generalize Friedman's conjecture.

The first one that was announced in [3] is due to Harrington.

Theorem 3.1 (Harrington)**.** *Suppose A is a Π^1_1 set of reals, and α is the least admissible ordinal so that there is some real $x \in A$ such that $\omega_1^x = \alpha$ and $x \notin L_{\omega_1^x}$. Then, for any y with $\omega_1^y > \alpha$, there exists a real $x \in A$ so that $x \equiv_h y$.*

By Gandy's basis theorem, Friedman's conjecture is an immediate consequence of Harrington's theorem. However, the proof of Theorem 3.1 is extremely sophisticated and has never been published. Actually, our proof uses some ideas from Harrington's proof of Theorem 3.1. Another interesting conclusion of Theorem 3.1 is that $\mathbf{\Pi^1_1}$-hyper-determinacy is equivalent to that for every real x, there is a non-x-constructible real y. So, in the consistency strength sense, $\mathbf{\Pi^1_1}$-hyper-determinacy is much weaker than Π^1_1-Turing determinacy, of which the consistency strength is equivalent to the existence of $0^\sharp$.

After Friedman's conjecture had been settled, it was conjectured that for any recursive tree T in $\omega^{<\omega}$ having a non-hyperaithmetic infinite path, the Turing jumps of the infinite paths in T range over an upper cone of Turing degrees. This conjecture was refuted by Harrington (a proof can be found in [4]). He proves the following result.

Theorem 3.2 (Harrington)**.** *There is a recursive tree $T \subseteq 2^{<\omega}$ in which there are uncountably many infinite paths such that for any real $x \in [T]$ and ordinal $\alpha < \omega_1^{\mathrm{CK}}$, $x^{(\alpha)} \not\geq_T \emptyset^{(\alpha+1)}$, where $x^{(\alpha)}$ is the α-th Turing jump of x.*

The third way to generalize Friedman's conjecture is to look "higher up". Woodin prove the following remarkable result (for more information about Q-theory, see [6]).

Theorem 3.3 (Woodin)**.** *Assuming $\mathbf{\Delta}^1_{2n+1}$-Determinacy, if $A \subseteq 2^\omega$ is a Δ^1_{2n+1} set without a Q_{2n+1}-member, then for every $x \geq_{2n+1} y^0_{2n+1}$, there is a real $z \in A$ for which $z \equiv_{2n+1} x$ (where y^0_{2n+1} denotes the Q_{2n+1}-jump).*

The last way is to partially relativize the conjecture.

Question 3.4. *Is it true that for any real x and $\Delta^1_1(x)$ set A of reals, if there is a real $y \in A$ with $y >_h x$, then for every real $z \geq_h \mathcal{O}^x$, there is a real $y \in A$ with $y \equiv_h z$?*

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