LOWNESS FOR GENERICITY

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ABSTRACT. We study lowness for genericity. We show that there exists no Turing degree which is low for 1-genericity and all of computably traceable degrees are low for weak 1-genericity.

1. INTRODUCTION AND NOTATIONS

We study the lowness properties for genericity. In [9], Nitzpon introduced the definition of lowness for 1-genericity.

Definition 1.1 (Nitzpon [9]). *For any $n > 0$,*

(1) *A real x is low for n-genericity if it is noncomputable and every n-generic real is n-x-generic. A Turing degree is low for n-genericity if it (only) contains a real which is low for n-genericity.*
(2) *A real x is low for weak n-genericity if it is noncomputable and every n-generic real is n-x-generic. A Turing degree is low for weak n-genericity if it (only) contains a real which is low for weak n-genericity.*

The motivation of studying lowness for genericity is from the randomness theory. For more information, please see [1]. Obviously, every low for 1-genericity should be "very low". Nitzpon show some properties of the degrees low for 1-genericity. But we prove that there is no such a degree. Nitzpon also conjectured that hyperimmune free degrees are the same as the degrees low for weak 1-genericity. We prove that every computably traceable degree is low for weak 1-genericity and so partially answer his conjecture.

1.1. **Notation.** We follow the standard notation. We list some notations below. For other terminology, we refer the reader to Jockusch [2] and Soare [10].

In this paper, a real means an element in Cantor space 2^ω. Sometimes we say sets replacing reals. The basic open sets in Cantor space are $I_\sigma = \{x \in 2^\omega | x \in 2^\omega \text{ and } \sigma \prec x\}$. The measure of I_σ is $2^{-|\sigma|}$ where $|\sigma|$ is the length of σ. We use $\mu(A)$ to denote the measure of set A. For a

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set $S \subseteq 2^{<\omega}$, define $V_S = \{x \in 2^\omega | \exists \sigma \in S(x \succ \sigma)\}$. A set $S \subset 2^{<\omega}$ is prefix free if $\forall \sigma \in S \forall \tau \in S(\sigma \preceq \tau \implies \sigma = \tau)$.

We use lightface $a, b, c, x, y, z \ldots$ to denote reals and boldface $\mathbf{f}, \mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}, \mathbf{z} \ldots$ to denote Turing degrees.

Definition 1.2. *A set $S \subseteq 2^{<\omega}$ is dense if for every $\sigma \in 2^{<\omega}$, there is a finite string $\tau \in S$ so that $\tau \succ \sigma$.*

Given a set $S \subseteq 2^{<\omega}$, define $[S] = \{f \in 2^\omega | \exists^\infty n(f \upharpoonright n \in S)\}$.

Definition 1.3. *Given a string $\sigma \in 2^\omega$, a real x and a set $S \subseteq 2^{<\omega}$.*

(1) *$\sigma \Vdash x \in S$ if $\sigma \prec x$ and $\sigma \in S$.*
(2) *$\sigma \Vdash x \notin S$ if $\sigma \prec x$ and $\forall \tau \succeq \sigma(\tau \notin S)$.*

Definition 1.4. *Given reals x, y and a number $n \geq 1$,*

(1) *x is n-y-generic if for every $\Sigma^0_n(y)$ set $S \subseteq 2^{<\omega}$, there is a string $\sigma \prec x$ so that either $\sigma \Vdash x \in S$ or $\sigma \Vdash x \notin S$.*
(2) *x is weakly n-y-generic if for every dense $\Sigma^0_n(y)$ set $S \subseteq 2^{<\omega}$, there is a string $\sigma \prec x$ so that $\sigma \Vdash x \in S$.*

2. A genericity version of van Lambalgen theorem

In this section, we prove an interesting proposition which is a genericity version of van Lambalgen theorem [12]. The following lemma is clear.

Lemma 2.1. *For any real x, numbers $m, n > 0$, a real y is $n+m$-x-generic iff y is n-x^m-generic.*

We prove the following main result in this section.

Proposition 2.2. *For any $n \geq 1$, a real $x = x_1 \oplus x_2$ is n-generic iff x_1 is n-generic and x_2 is n-x_1-generic.*

Proof. Suppose x_1 is n-generic and x_2 is n-x_1-generic. Then for any Σ^0_n set $S \subseteq 2^{<\omega}$, define $T(x_1) = \{\tau | \exists \sigma \prec x_1(\sigma \oplus \tau \in S)\}$. Then $T(x_1)$ is a $\Sigma^0_n(x_1)$ set.

Case(1). $\exists \tau \prec x_2(\tau \Vdash x_2 \notin T(x_1))$. Pick up such a τ. Define a Σ^0_n set $T' = \{\sigma | \exists \tau'(\sigma \oplus \tau' \succ x \upharpoonright 2|\tau| \text{ and } \sigma \oplus \tau' \in S)\}$. Then $T' \cap x_1 = \emptyset$ (Otherwise, say $\sigma' \prec x_1$. Then there must a $\tau' \succ \tau$ so that $\sigma' \oplus \tau' \in T'$ and so $\tau' \in T(x_1)$, contradiction). So there is a $\sigma' \prec x_1$ so that $\sigma' \Vdash x_1 \notin T'$. Set $\mu = x \upharpoonright 2|\sigma'|$. Then $\mu \Vdash x \notin S$.

Case(2). Otherwise. $\exists \tau \prec x_2(\tau \in T(x_1))$. So there is a $\sigma \prec x_1$ so that $\sigma \oplus \tau \in S$.

So x is n-generic.

Suppose $x = x_1 \oplus x_2$ is n-generic. Then it is clear that x_1 is n-generic. If x_2 is not n-x_1-generic, then there is a $\Sigma^0_n(x_1)$ set $S \subseteq 2^{<\omega}$ so that $\forall \sigma \prec x_2(\sigma \not\Vdash x_2 \in S \text{ and } \sigma \not\Vdash x_2 \notin S)$. So there is a Σ^0_n formula Φ so that $S = \{\sigma | \exists n(\Phi(x_1 \upharpoonright n, \sigma))\}$. Define $T = \{\sigma \oplus (\tau^\frown 0^{(|\sigma| - |\tau|)}) || \tau| \leq |\sigma| \text{ and } \Phi(\sigma, \tau)\}$. Clearly T is Σ^0_n. Then $\forall \sigma \oplus \tau \prec x(\sigma \oplus \tau \notin T)$

(otherwise, $\exists \tau \prec x_2(\tau \in S)$). But since $S = \{\sigma | \exists n(\Phi(x_1 \upharpoonright n, \sigma))\}$, it is clear that $\forall \sigma \oplus \tau \prec x \exists \sigma' \oplus \tau' \succ \sigma \oplus \tau(\sigma' \oplus \tau' \in T)$. So x is not n-generic, contradiction. $\square$

3. Lowness for generic reals

We first give some definitions.

Definition 3.1. (1) *Given a real $x = \{n_0 < n_1 < n_2 < \ldots\}$, define its principal function $p_x(m) = n_m$.*
(2) *A function f majorizes a real x if $\forall n(f(n) > p_x(n))$.*
(3) *Given a real y, a real x is y-hyperimmune if no y-computable function f majorizes x. Particularly, we say x is hyperimmune if it is 0-hyperimmune.*
(4) *Given a degree $\mathbf{y}$, a degree $\mathbf{x}$ is $\mathbf{y}$-hyperimmune if it contains a $\mathbf{y}$-hyperimmune real. Otherwise, it is $\mathbf{y}$-hyperimmune-free. Particularly, we say $\mathbf{x}$ is hyperimmune-free if it is 0-hyperimmune-free.*
(5) *We say that a degree $\mathbf{x}$ is computably traceable iff there is a computable bounding function h, such that for all $f \leq_T x$, there is a computable array $\{s_n \in \mathcal{P}_{<\omega}(\omega) | n \in \mathbb{N}\}$, where $\mathcal{P}_{<\omega}(\omega) = \{s \subset \omega || s| < \infty\}$, such that for all n,*
 (a) $|s_n| < h(n)$
 (b) $f(n) \in s_n$.

It is clear that every computably traceable degree is hyperimmune. By an observation of Terwijn and Zambella [11], there are $2^{\aleph_0}$ many computably traceable degrees. But there exists a hyperimmune-free degree which is not computably traceable [11]. Actually, there is a Π^0_1 class which only contains 1-random reals, so, by hyperimmune-free base theorem [4], there is a hyperimmune-free 1-random real. It is not hard to see that no random real can be computably traceable.

For $n > 1$, there does exist some degrees which are low for n-genericity.

Lemma 3.2. *For any $n \geq 1$, every $\mathbf{L_n}$ degree is low for $n+1$-genericity.*

Proof. If $\mathbf{x^n} = \mathbf{0^n}$ then for any real y, by Lemma 2.1, y is $n+1$-generic $\Leftrightarrow$ y is 1-$\mathbf{0^n}$-generic $\Leftrightarrow$ y is 1-$\mathbf{x^n}$-generic $\Leftrightarrow$ y is $n+1$-$\mathbf{x}$-generic. $\square$

For $n = 1$, the situation is totally different. Jockusch and Shore prove the following lemma in [3].

Lemma 3.3 (Jockusch and Shore [3])**.** *For any noncomputable real x and $y \geq_T x \oplus \emptyset'$, there is a 1-generic real g so that $g \oplus x \equiv_T y$.*

The following theorem is somewhat surprising.

Corollary 3.4. *There is no real low for 1-genericity.*

Proof. For any noncomputable real x, by Lemma 3.3, there is a 1-generic real g so that $g \oplus x \equiv_T x'$. So $(g \oplus x)' \equiv_T x'' >_T x' \equiv_T g \oplus x'$. But obviously, if g is 1-x-generic, then $(g \oplus x)' \equiv_T g \oplus x'$. Hence g can not be 1-x-generic. □

We turn to lowness for weak 1-genericity. Kurtz proved the following theorem.

Theorem 3.5 (Kurtz [5]). *Weakly 1-generic degrees are exactly hyperimmune degrees.*

So we have the following corollary.

Corollary 3.6 (Nitzpon [9]). *If x is low for weak 1-genericity, then x is hyperimmune free.*

Proof. Suppose x is low for weak 1-genericity. If x is not hyperimmune free, then by Theorem 3.5, there is a weakly 1-generic real $g \equiv_T x$. So g can not be weakly 1-x-generic, contradiction. So x is hyperimmune free. □

We say a Turing degree $\mathbf{x}$ is *low for hyperimmune degrees* if every hyperimmune degree is $\mathbf{x}$-hyperimmune degree. Although it is not hard to see that every hyperimmune-free degree is low for hyperimmune degrees(see [9]), it is not obvious whether there exists a degree low for weak 1-genericity. Since every noncomputable Δ^0_2 degree is hyperimmune [7], no noncomputable Δ^0_2 degree is low for weak 1-genericity. Nitzpon conjectured that the hyperimmune free degrees are the same as the degrees low for weak 1-genericity. We show that every computably traceable degree is low for weak 1-genericity and so partially answer his conjecture. We need a useful lemma.

Lemma 3.7 (Terwijn and Zambella [11]). *Let h be any computable nondecreasing function with $\lim_n h(n) = \infty$. Then if x is computably traceable, it is computably traceable with the bound h.*

Theorem 3.8 (with Miller). *Every computably traceable degree is low for weak 1-genericity.*

Proof. Fix a computably traceable real x. By Lemma 3.7, the computable function $h(n) = n$ is a bound. It suffices to prove that for every $\Sigma^0_1(x)$ dense set S^x_e, there is a Σ^0_1 dense set T_e so that $V_{T_e} \subseteq V_{S^x_e}$. Given a set $S \subseteq 2^{<\omega}$, we say $T \subseteq 2^{<\omega}$ is dense in S if for every $\sigma \in S$ there is a $\tau \in T$ so that $\tau \succeq \sigma$. Define $J_n = \{\sigma||\sigma| = n\}$. The idea is: for every n and every $\sigma \in J_n$, we want to uniformly computably put a finite string $\tau \succ \sigma$ in T_e so that $\tau \succeq \mu$ for some $\mu \in S^x_e$. To do that, by the bounding function, there are at most n many finite strings $\{\sigma_i\}_{i<n}$ so that $S^{\sigma_i}_e$ is dense in J_n for every $i < n$. We will effectively compute a very long possible initial segment of x and force $S^{\sigma_i}_e$ to produce a large number of finite strings so that $\bigcap_{i<n} V_{S^{\sigma_i}_e}$ contains a subinterval of I_σ

for every $\sigma \in J_n$. If we succeed, we put these subintervals into T. We formalize our intuition by the following proof.

From now on, we fix the number e and dense set S_e^x. We use $S_e^\sigma[t]$ to denote the enumeration of S_e^σ at stage t . Define functions f, g as follows:

$\hat{f}(n) = \min\{t | S_e^{x\upharpoonright t}[t] \text{ is dense in } J_n\}$,

$\hat{g}(n) = \max\{|\sigma| | \sigma \in S_e^{x\upharpoonright \hat{f}(n)}[\hat{f}(n)]\}$.

Since S_e^x is dense, both $\hat{f}, \hat{g}$ are x-computable. Since x is hyperimmune-free, there are two computable functions f, g so that for all n, $f(n) \geq \hat{f}(n)$ and $g(n) \geq \hat{g}(n)$. Note for every n, $S_e^{x\upharpoonright f(n)}[f(n)]$ is dense in J_n. Define a computable function $G : (\mathbb{N} \cup \{-1\}) \times \mathbb{N} \mapsto \mathbb{N}$ so that for all n, $G(-1, n) = n$, $G(0, n) = f(n)$ and $G(m+1, n) = f(g^{(m)}(n))$ where $m \in \mathbb{N}$. So the sequence $\{x \upharpoonright G(n, n)\}_n$ is uniformly x-computable. Since x is computably traceable with bounding function h, there is a computable sequence $\{R(n)\}_{n\in\mathbb{N}}$ with $R(n) \in (2^{<\omega})^{<\omega}$ for every n so that for all $n \in \mathbb{N}$, $x \upharpoonright G(n, n) \in R(n)$ and $|R(n)| \leq h(n) = n$. Note $S_e^{x\upharpoonright G(m,n)}[G(m, n)]$ is dense in $J_{G(m-1,n)}$ for every $m < n$. We construct a c.e. dense set T so that $V_T \subseteq V_{S_e^x}$ step by step.

At step 0, define $T = \emptyset$.

At step $n + 1$, suppose $R(n) = \{\sigma_i\}_{i<n}$. There are $n + 1$ many substeps to construct a sequence $\tau_0^i \preceq \tau_1^i \preceq \dots \preceq \tau_{n-1}^i$ so that $\{\tau_m^i\}_{i<2^n}$ is dense in J_n for every $m < n$ and $\tau_m^i \in S_e^{\sigma_{k_m}\upharpoonright G(k_m,n)}[G(m, n)]$ for every $i < 2^n$ where k_m is the maximal number no larger than m so that $S_e^{\sigma_{k_m}\upharpoonright G(k_m,n)}[G(k_m, n)]$ is dense in $J_{G(k_m-1,n)}$. Define $\{\tau_{-1}^i\}_{i<2^n} = J_n$.

Substep m ($0 \leq m < n$), check whether $S_e^{\sigma_m\upharpoonright G(m,n)}[G(m, n)]$ is dense in $J_{G(m-1,n)}$. If not, then it is not hard to see that $\sigma_m \not\prec x$. Define $\tau_m^i = \tau_{m-1}^i$ for every $i < 2^n$ and go to next substep. Otherwise, for every $i < 2^n$, find a $\tau_m^i \in S_e^{\sigma_m\upharpoonright G(m,n)}[G(m, n)]$ so that $\tau_m^i \succeq \tau_{m-1}^i$. By induction, there must exist such strings.

At substep n, put $\{\tau_{n-1}^i\}_{i<2^n}$ into T.

For every n and $m < n$, $S_e^{x\upharpoonright G(m,n)}[G(m, n)]$ is dense in $J_{G(m-1,n)}$. So there exists at least one substep at which we extend J_n. So T is dense in J_n. Thus T is a c.e. dense set. Obviously, by the construction, $V_T \subseteq V_{S_e^x}$.

$\square$

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