Recursion Theory in the Constructible Universe

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What is recursion theory?

A possible definition: A theory of studying algorithm and sets computable by an algorithm.

Wait a minute. What is algorithm? What does it mean "a set computable by an algorithm"?

A description: An algorithm is a procedure which can be executed by a machine step by step. An algorithm computable set say $A$ is a set for which each statement "$x \in A$" can be decided by the algorithm during finite steps by inputting $x$.
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A description:
An algorithm is a procedure which can be executed by a *machine* step by step.
An algorithm computable set say \( A \) is a set for which there is an algorithm so that each statement “\( x \in A \)” can be decided by the algorithm during *finite steps* by inputting \( x \).
Two examples

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2. On the general sets. A machine can be generated by Gödel operations. Then we have constructible sets.
Part I: Recursion theory.
Primitive recursive functions

A function $f$ is primitive if it is directly obtainable from constant functions, successor functions and identity functions via the following procedure:

1. Composition;

2. Induction:
   \[ f(0, x, y) = g(x, \overline{y}); \quad f(z + 1) = h(z, f(z, x, \overline{y}), x, \overline{y}). \]
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Recursive functions

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We introduce an operator $\mu$, i.e. a minimalization operator.

A partial recursive function $p$ is a function which can be written as $p(\vec{x}) = y \iff g(\mu z f(\vec{x}, z) = 1) = y$ where both $f$ and $g$ are recursive.

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However:
Partial Turing computable = partial recursive.
A set $A$ is recursive if there is a total recursive function $f$ so that $f(n) = 0 \Leftrightarrow n \in A$.
A set $A$ is recursively enumerable if there is a partial recursive function $p$ so that $p(n) = 0 \Leftrightarrow n \in A$
Relativization

There is an effective way to code binary strings into natural numbers.
A partial function \( p(\sigma, n) \) is consistent if for all \( \sigma \preceq \tau \),
\( p(\sigma, n) \simeq p(\tau, n) \).
A partial \( p^A(n) \) is \( A \)-partial recursive if there is a partial recursive consistent function \( p(\sigma, n) \) so that \( p^A(n) = m \) if and only if there is a \( \sigma \prec A \) so that \( p(\sigma, n) = m \).

There is a universal partial recursive consistent function
\( p(\sigma, e, n) \) in the sense that for every set \( A \) and \( A \)-partial recursive function \( p^A \), there is an index \( e \) so that \( p^A_e(n) = m \) if and only if there is some \( \sigma \prec A \) so that \( p(\sigma, e, n) = m \).

In this sense, we have a uniformly enumeration for all \( A \)-partial recursive functions, i.e. \( \{ \Phi^A_e \}_e \).
So we have a uniform enumeration \( \{ W^A_e \}_e \) of \( A \)-r.e. sets.
An \( A \)-recursive set can be defined by an obvious way.
A \leq_T B \text{ if } A \text{ is } B-\text{recursive.}

For a set A, the Turing degree \( a = \{ B \mid A \equiv_T B \} \).

The Turing jump of A, written to \( A' \), is the Turing degree of the \( A \)-halting problem \( K^A = \{ e \mid \Phi^A_e(e) \downarrow \} \).

**Theorem (Kleene, Post)**

There are two set \( A, B \leq_T \emptyset' \) so that \( A \not\leq_T B \) and \( B \not\leq_T A \).
Natural Turing degrees

Are there natural Turing degrees between $\emptyset$ and $\emptyset'$? Which can be formalized as following:

- Positive conjecture:
  - Conjecture (Sacks): There is an index $e$ so that $1^e$ for all set $A$, $A < T^W_A < 2^e < T^A'$;
  - If $A \equiv_T B$, then $W^A_e \equiv_T W^B_e$ (degree invariant).

Sacks' conjecture essentially says that there is a natural way to find a degree between $\emptyset$ and $\emptyset'$. 
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Positive conjecture:

Conjecture (Sacks)

There is an index $e$ so that

1. for all set $A$, $A <_T W^A_e <_T A'$;
2. if $A \equiv_T B$, then $W^A_e \equiv_T W^B_e$ (degree invariant).

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Martin’s conjecture

Negative conjecture:

**Conjecture (Martin)**

Assume $ZF + AD + DC$.

1. If $f$ is degree invariant and $f(X) \not\geq_T X$ on a cone (i.e. on some $\{Y \mid Y \geq_T Z\}$), then $f$ is a constant on a cone.

2. For those $f$’s which are degree invariant and $f(X) \geq_T X$ on a cone, $\leq_m = \{(f, g) \mid \exists Z \forall X \geq_T Z(f(X) \leq_T g(X))\}$ is a prewellordering on these functions so that the successor operator $S(f(X))$ is $(f(X))'$ for all $X$. 
To generalize recursiveness from sets of numbers to sets of reals. A relation \( P(f, n) \subseteq \omega^\omega \times \omega \) is partial recursive if there is an index \( e \) so that \( \forall f \forall n (R(f, n) \iff \Phi^f_e(n) = 0) \).

It also can be written as \( \exists m R(f, m, n) \) for some primitive recursive relation \( R \) which means there is a index \( e \) so that for every \( f \) and \( m, n \), \( \Phi^f_e(m, n) \) convergent.
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To give more intuition, just fixing $n = 0$. Then a partial recursive set of reals is exactly an open set with an effective enumeration.
A hierarchy of sets

A relation $P(f, m)$ is $\Sigma_1^0$ if it is partial recursive;
$P(f, m)$ is $\Pi_n^0$ if its complement is $\Sigma_n^0$;
$P(f, m)$ is $\Sigma_{n+1}^0$ if there is a $\Pi_n^0$ relation $R(f, m, k)$ so that
$P(f, n) \iff \exists k R(f, m, k)$;
$P(f, m)$ is $\Sigma_1^1$ if there is a $\Pi_1^1$ relation $R(f, g, m)$ so that
$P(f, m) \iff \exists g R(f, g, m)$;
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$P(f, n) \iff \exists g R(f, g, m)$. 
A $\Pi^1_1$ complete set

Let $V_e = \{ \sigma \mid \exists n < |\sigma| (\Phi_e(\sigma, n)[|\sigma|] \downarrow) \}$. Note that $\omega^{<\omega} \setminus V_e$ is a tree $T_e$.
Define $e \in WF_0$ iff $\forall f \exists n = \langle n_0, n_1 \rangle (\Phi_e(f \upharpoonright n_0, n_1) \downarrow)$. In other words, $[T_e] = \{ f \mid \forall n (f \upharpoonright n \in T_e) \}$ is an empty set. Or $e \in WF_0$ if and only if $T_e$ is a well founded tree.

Theorem

Every $\Pi^1_1$ set of numbers is 1-1 reducible to $WF_0$.
Proof.
Fix a $\Pi^1_1$ set $U_e = \{ m \mid \forall f (\Phi_f(e)(m) = 0) \}$. There is a 1-1 recursive function $h_e$ with $\Phi_f(e)(m) = 0$ $\iff \exists n (\Phi_f(h_e)(m)(n) \downarrow)$. 
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Given $\sigma, \tau \in \omega^{<\omega}$, $\sigma <_{KB} \tau$ iff $\sigma \succ \tau$ or there exists $n$ so that $\sigma(n) < \tau(n)$ but $\sigma \upharpoonright n = \tau \upharpoonright n$.

Define $e \in WO_0$ iff $<_{KB}$ is a well ordering on $V_e$.

$e \in WO_0$ if and only if $e \in WF_0$. 
Definition

A well-ordering $<_o$ on $\omega$ is defined by transfinite induction as follows:

- $0 <_o 1$;
- $(\forall n) n <_o 2^n$;
- $(\forall n) \Phi_e(n) <_o \Phi_e(n + 1) \implies (\forall n) \Phi_e(n) <_o 3 \cdot 5^e$.

$O$ is the field of $<_o$.

An ordinal $\alpha$ is constructive if it is isomorphic to $O \upharpoonright n = \{ m | m <_o n \}$ for some $n \in O$. 
Kleene’s $\mathcal{O}$ II

$\mathcal{O}$ is constructed in a very uniform way.

**Proposition**

*There are two recursive functions $p$ and $q$ so that for all $n \in \mathcal{O}$,*

1. $W_{p(n)} = \{ m \mid m <_\mathcal{O} n \};$
2. $W_{q(n)} = \{ \langle i, j \rangle \mid i <_\mathcal{O} j <_\mathcal{O} n \}.$
Theorem (Kleene)

Every $\Pi^1_1$ set of numbers is 1-1 reducible to $O$.

Proof.

We do an effective transfinite induction to reduce $WF_0$ to $O$. If $\sigma \in V_e$ and we have already $f_e(\sigma^-) \in O$, then we intend to define $f_e(\sigma)$ to be “the sum of $f(\tau)$ for $\tau \prec \sigma$”. The point is that we can do this by recursion theorem.

Then we can define $g(e)$ to be “the sum of $f_e$”.

Argue $e \in WF_0$ iff $g(e) \in O$. $\square$
$\Sigma_1^1$-boundedness

**Theorem (Kleene)**

If $A$ is $\Sigma_1^1$ and many-1 reducible to $\mathcal{O}$ via $f$, then there is a notation $n \in \mathcal{O}$ so that $f(A) \subseteq \mathcal{O}_n = \{m \mid m \in \mathcal{O} \wedge |m| < |n|\}$.

**Proof.**

Otherwise, $\mathcal{O}$ would be $\Sigma_1^1$.

As a conclusion, every $\Delta_1^1$ set is 1-1 reducible to some $\mathcal{O}_n$. In other words, every $\Delta_1^1$ set can be totally enumerated at some constructive ordinal stage.
Recursive ordinals

Let $\omega_1^{CK}$ be the least ordinal which cannot be an order type of a recursive well ordering.

By Kleene’s theorem, $\omega_1^{CK} = |\mathcal{O}|$, i.e. the length of $\mathcal{O}$ under $\prec_{\mathcal{O}}$.

$\omega_1^{CK}$ is the least ordinal which is inaccessible by a recursive procedure.
We can define transfinite jumps along $\mathcal{O}$.

**Definition**

An $H$-set is a set $H_n$ for some $n \in \mathcal{O}$ as following.

\[
H_1 = \emptyset,
\]

\[
H_{2n} = (H_n)',
\]

\[
H_{3.5^e} = \{\langle m, n \rangle | m \in H_{\Phi_e(n)}\}.
\]
$\Pi^0_2$-singletons

**Theorem**

There exists a $\Pi^0_2$ predicate $H(n, x)$ so that

$$\forall n \in \mathcal{O}(\exists!x H(n, x) \land H(n, H_n)).$$

**Proof.**

If $A$ is a $\Pi^0_2$ singleton, too is $A'$. Moreover, the index of $A'$ can be found uniformly.

Then by a usual effective transfinite induction. \(\square\)
Theorem

There exists a $\Pi^0_2$ predicate $H(n, x)$ so that

$$\forall n \in \mathcal{O} (\exists! x H(n, x) \land H(n, H_n)).$$

Proof.

If $A$ is a $\Pi^0_2$ singleton, too is $A'$. Moreover, the index of $A'$ can be found uniformly.
Then by a usual effective transfinite induction.

It can be shown that for different $m, n \in \mathcal{O}$, if $|m| = |n|$, then $H_n \equiv_T H_m$.
So for $\alpha < \omega_1^{CK}$, we just simply use $\emptyset^\alpha$ to denote $\alpha$-th jump.
A set $A$ of numbers is hyperarithmetic if $A \leq_T \emptyset^\alpha$ for some $\alpha < \omega_1^{CK}$.
The set $\{A \mid A \in \Delta^1_1\}$ is $\Pi^1_1$.

**Theorem (Kleene)**

A is hyperarithmetic iff $A$ is $\Delta^1_1$.

**Proof.**

“$\Rightarrow$”: By the closure property of arithmetical operators.
“$\Leftarrow$”: Every $\Delta^1_1$ set can be 1-1 reduced to $\mathcal{O}$ with a bound.
Relativization

One also can relativize definitions before. For every set $A$, we can define the least non $A$-recursive ordinal to be $\omega_1^A$. We use $A^{(\alpha)}$ to denote the $\alpha$-th jump for $\alpha < \omega_1^A$. We also have a $\Pi^1_1(A)$ set $O^A$ to which every $\Pi^1_1(A)$ set is 1-1 reducible.

We say that $A$ is hyperarithmetic reducible to $B$, written to $A \leq_h B$, if there is a $B$-recursive ordinal so that $A \leq_T B^\alpha$. Relativizing Kleene’s result, we have that $A \leq_h B$ iff $A$ is $\Delta^1_1(B)$ definable. Moreover $A \leq_h B$ is a $\Pi^1_1$ relation.
One also can study sets of reals along this way. Define $WF_1 = \{f \mid \forall g \exists n \Phi_{f(0)}^{f^-(n)}(n)\} \downarrow$ where $f^-(n) = f(n + 1)$ for all $n \in \mathbb{N}$. In other words, $f \in WF_1$ iff $f$ codes a $f$-recursive well founded tree.

A relativization to the proof shows that for every $\Pi_1^1$ set $A$ of reals, there is a recursive function $f$ so that $x \in A \iff f(x) \in WF_1$. 
Theorem (Spector, Gandy)

A set $A$ is $\Pi^1_1$ iff there is an arithmetical relation $R$ so that $n \in A \iff \exists Y (Y \in \Delta^1_1 \land R(Y, n))$.

Proof.

“$\Leftarrow$”: Since $\mathcal{O}$ is $\Pi^1_1$-complete, then by $\Delta^1_1$-boundedness.

“$\Rightarrow$”:

$e \in \text{WO}_0 \iff \exists X (X \in \Delta^1_1 \land \sigma <_{\text{KB}} \tau \implies ((X)_{\sigma})' \leq (X)_{\tau})$.  \qed
Spector-Gandy Theorem

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A relativization of Spector-Gandy Theorem says that a set $A \subseteq 2^\omega \times \omega$ is $\Pi^1_1$ if and only if there is an arithmetical relation $R$ so that $(X, n) \in A \iff \exists Y (Y \leq_h X \land R(X, Y, n))$.

Fix $n = 0$, we have a characterization of $\Pi^1_1$ sets of reals which says that for every $\Pi^1_1$ set $A$, $X \in A$ can be witnessed by some $Y \leq_h X$. 
Theorem (Feferman and Spector)

There exists a $\Pi^1_1$ path through $\mathcal{O}$.

Proof.

Define a nonstandard $\mathcal{O}$, $\mathcal{O}^*$ to be
$$\bigcap \{ X \in \Delta^1_1 \mid X \text{ satisfies (1)-(3) of } \mathcal{O} \}.$$ $\mathcal{O}^*$ is $\Sigma^1_1$. So $\mathcal{O}^*$ is a proper end extension of $\supset \mathcal{O}$.

Fix a number $e \in \mathcal{O}^* - \mathcal{O}$. Let $\mathcal{O}_1 = \{ n \mid n <_{\mathcal{O}^*} e \land n \in \mathcal{O} \}$.

Using $\mathcal{O}_n = \{ m \mid m \in \mathcal{O} \land |m| < |n| \}$ is $\Delta^1_1$ to argue that $\mathcal{O}_1$ is the length of $\omega^\text{CK}_1$.

Note that $n \in \mathcal{O}_1 \iff n \in W_{p(e)} \land p \in \mathcal{O}$.

This path provides a uniform way to construct hyperarithmetic sets by effective transfinite induction.
To be continued