MEASURE THEORY ASPECTS OF LOCALLY COUNTABLE ORDERINGS

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Abstract. We prove that for any locally countable $\mathbf{\Sigma}^1_1$ partial order $\mathbb{P} = \langle 2^\omega, \leq_P\rangle$, there exists a nonmeasurable antichain in $\mathbb{P}$. Some applications of the result are also presented.

1. Introduction

We say $\mathbb{P} = \langle P, \leq_P\rangle$ is a *partial order* if $\leq_P$ is a reflexive transitive binary relation on P.

Definition 1.1. *A partial order $\mathbb{P} = \langle P, \leq_P\rangle$ is locally countable if for every $p \in P$, $|\{q \in P | q \leq_P p\}| \leq \aleph_0$.*

Sacks [14] initiated the study of locally countable partial orders. He conjectured that every locally countable partial order on 2^ω can be embedded into the Turing degrees [14]. In this paper, we will give some structure theorems for such partial orders. Particularly, we are concerned with the possible size of chains and antichains in such partial orders. Given a partial order $\mathbb{P} = \langle P, \leq\rangle$, we say that a non-empty set $X \subseteq P$ is a *chain* in $\mathbb{P}$ if for any two elements x, y in X, either $x \leq y$ or $y \leq x$ and we say a non-empty set $X \subseteq P$ is an *antichain* in $\mathbb{P}$ if for any two different elements x, y in X, $x \not\leq y$. One would not expect that there are any nice structure theorems for arbitrary locally countable partial orders within ZFC. Most of them are independent of ZFC (we will give the reason in the following sections). So we are concerned only with some "well-behaved" orders, say Borel orders. By the work due to Friedman [3], Harrington and Shelah [5], many pathologies are avoided when we consider Borel orderings. $\mathbb{P} = \langle 2^\omega, \leq_P\rangle$ is said to be *thin* if there is no antichain which is a perfect set. Harrington and Shelah proved the following theorem.

Theorem 1.2 (Harrington and Shelah [5])**.** *If $\mathbb{P} = \langle 2^\omega, \leq_P\rangle$ is a thin Borel order, then*

(1) *for some $\alpha < \omega_1$ there is an order preserving Borel function $f : 2^\omega \mapsto 2^\alpha$ (where 2^α is ordered lexicographically);*
(2) *2^ω can be written as a countable union of Borel chains.*

An immediate consequence of Theorem 1.2 is that every Borel locally countable partial order $\mathbb{P} = \langle 2^\omega, \leq_P\rangle$ is not thin.[1] It means that there are some large size

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[1]This can be seen as follows. By (2) in Theorem 1.2, there is an uncountable Borel chain in $\langle 2^\omega, \leq_P\rangle$. Since $\leq_P$ is locally countable, there must be an ω_1-chain. This contradicts (1) in Theorem 1.2. One also can deduce the result from Theorem 1.2 and Proposition 5.4.

antichains in the Borel locally countable partial orders. Although it means that Borel locally countable partial orderings can have many antichains, it is natural to ask whether they can have large measure. Studying measure theoretical properties of partial orders is a topic in descriptive set theory. In this paper, we give an almost complete measure theoretical description for the locally countable partial orderings. The main result is that for any locally countable $\mathbf{\Sigma}^1_1$ partial order $\mathbb{P} = \langle 2^\omega, \leq_P\rangle$, there exists a nonmeasurable antichain in $\mathbb{P}$.[2] Some applications of the result are also presented.

The article is organized as follows. In Section 2, we prove some easy results about the chains in locally partial orders. Section 3 is the main part of this paper, in which we consider the antichains in locally partial orders and prove the main result in this section. In Section 4, we consider some specific locally countable partial orders and present some applications of the results in Section 3. Finally, we mention some questions in Section 5.

To prove these results it is useful to look instead at the lightface refinements. One can easily relativize our proofs to show the results on the boldface cases.

Our notations are relatively standard. We list some notations below. For more information on randomness theory, descriptive set theory and recursion theory, please refer to [2], [6], [9], [13] and [16].

A real is an element in Cantor space 2^ω. For $F \subseteq 2^{<\omega}$, define $[F] = \{x \in 2^\omega \mid \exists \sigma \in F(\sigma \prec x)\}$. We use μ to denote Lebesgue measure.

Definition 1.3 (Martin-Löf [10])**.**

(1) *Given a real x, a $\Sigma^0_n(x)$ Martin-Löf test is a computable collection $\{V_n : n \in \mathbb{N}\}$ of $\Sigma^0_n(x)$ sets such that $\mu(V_n) \leq 2^{-n}$.*
(2) *Given a real x, a real y is said to pass the $\Sigma^0_n(x)$ Martin-Löf test if $y \notin \bigcap_{n\in\omega} V_n$.*
(3) *Given a real x, a real y is said to be n-x-random if it passes all $\Sigma^0_n(x)$ Martin-Löf tests.*

Obviously for any real z, $\mu(\{x|x \text{ is } 1\text{-}z\text{-random}\}) = 1$.

Definition 1.4. *Given a string $\sigma \in 2^{<\omega}$, a real x and a set $S \subseteq 2^{<\omega}$.*

(1) *$\sigma \Vdash x \in S$ if $\sigma \prec x$ and $\sigma \in S$.*
(2) *$\sigma \Vdash x \notin S$ if $\sigma \prec x$ and $\forall \tau \succeq \sigma(\tau \notin S)$.*

Definition 1.5. *Given reals x, y and a number $n \geq 1$, x is n-y-generic if for every $\Sigma^0_n(y)$ set $S \subseteq 2^{<\omega}$, there is a string $\sigma \prec x$ so that either $\sigma \Vdash x \in S$ or $\sigma \Vdash x \notin S$.*

It is easy to see that no 1-generic real is 1-random.

We list some results in randomness theory which we will need later.

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[2]The referee suggested a more general result. Call a binary relation P on 2^ω *locally countable* if for each $y \in 2^\omega$ there are at most countably many $x \in 2^\omega$ such that $P(x, y)$ holds. Call a set $X \subseteq 2^\omega$ *P-independent* if there do not exist distinct $x, y \in X$ such that $P(x, y)$ holds. The referee suggested that if P is locally countable $\mathbf{\Sigma}^1_1$ binary relation on 2^ω, then there is a nonmeasurable P-independent set $X \subseteq 2^\omega$. There are two ways to prove the result. One is to directly modify the proof of Theorem 3.1. Another one is to extend P to be a partial order. To do that, we define $x \leq_Q y$ iff $x = y$ or there are finitely many reals $\{z_0, z_1, ..., z_{n+1}\}$ so that $P(x, z_0) \wedge P(z_0, z_1), ..., P(z_n, z_{n+1}) \wedge P(z_{n+1}, y)$. It is easy to see that $\leq_Q$ is a $\mathbf{\Sigma}^1_1$ locally countable partial ordering if $\leq_P$ is a $\mathbf{\Sigma}^1_1$ locally countable binary relation.

Theorem 1.6 (van Lambalgen [17]). *For any number $n > 0$ and real $x = x_0 \oplus x_1$ (or $x = x_1 \oplus x_0$), x is n-random if and only if x_0 is n-random and x_1 is n-x_0-random.*

Note that by Theorem 1.6, if $x = x_0 \oplus x_1$ is n-random, then $x_0 <_T x$. Moreover, given two reals x and y, if x is n-y-random and y is n-random then y is n-x-random.

Theorem 1.7 (Kurtz [8], Kautz [7]). [3] *For every 2-random real x, there is a 1-generic real y so that $y <_T x$.*

2. Chains in locally countable partial orders

In this section, we consider chains in locally countable partial orders. We need lots of facts from [13] and [16]. For reals $x, y \in 2^\omega$, we say that x is *hyperarithmetic in y* ($x \leq_h y$) if x is $\Delta^1_1(y)$ definable. Note that $\leq_h$ is a Π^1_1-relation. The following theorem plays a critical role in this paper. The proof can be found in Theorem 6.2 III [16].

Theorem 2.1 (Harrison [4]). *For any real z and countable $\Sigma^1_1(z)$ set $Z \subset 2^\omega$, if $x \in Z$, then $x \leq_h z$.*

So we have the following corollary.

Corollary 2.2. *If $\mathbb{P} = \langle 2^\omega, \leq_P\rangle$ is a Σ^1_1 locally countable partial order, then for any $x, y \in 2^\omega$, $x \leq_P y$ implies $x \leq_h y$.*

Sacks proved the following lemma.

Lemma 2.3 (Sacks [15]). *If x is not Δ^1_1, then $\mu(\{y | x \leq_h y\}) = 0$.*

Hence we have the following proposition.

Proposition 2.4. *For any locally countable Σ^1_1 partial order $\mathbb{P} = \langle 2^\omega, \leq_P\rangle$, every chain in $\mathbb{P}$ has measure 0.*

Proof. Given a chain $X \subseteq 2^\omega$ in $\mathbb{P}$. Note that for any $x \in 2^\omega$, the set $\{y | y \leq_P x\}$ is $\Sigma^1_1(x)$. Hence by Corollary 2.2, for any $x, y \in X$, either $x \leq_h y$ or $y \leq_h x$. If X contains only Δ^1_1 reals, then $\mu(X) = 0$. Otherwise, fix a non-Δ^1_1 real $x \in X$, and then $X \subseteq \{y | y \leq_h x\} \cup \{y | x \leq_h y\}$. The set $\{y | y \leq_h x\}$ is countable and by Lemma 2.3, the set $\{y | x \leq_h y\}$ is of measure 0. So $\mu(X) = 0$. □

One may ask whether Proposition 2.4 is still true when "Σ^1_1" is omitted. The answer is independent of ZFC.

Proposition 2.5.
1. *Assume $ZFC + V = L$. There is a locally countable Δ^1_2 partial order $\mathbb{P} = \langle 2^\omega, \leq_P\rangle$ so that there is a chain in $\mathbb{P}$ which has measure 1.*
2. *Assume $ZFC + MA_{\aleph_1}$. For every locally countable partial order $\mathbb{P} = \langle 2^\omega, \leq_P\rangle$, every chain has measure 0.*

Proof. For (1), take $\mathbb{P} = \langle 2^\omega, \leq_L\rangle$. Then $\leq_L$ is a Δ^1_2 well order of 2^ω of which the order type is ω_1. For (2), since every chain in any locally countable partial order has size at most $\aleph_1$, it has measure 0. □

[3] Kautz claimed that 2-randomness can be replaced with weak 2-randomness. It's incorrect. For more details, see [2].

3. Antichains in locally countable partial orders

In this section, we prove the following theorem, which is our main result.

Theorem 3.1. *For any locally countable $\mathbf{\Sigma}^1_1$ partial order $\mathbb{P} = \langle 2^\omega, \leq_P\rangle$, there exists a nonmeasurable antichain in $\mathbb{P}$.*

The proof of Theorem 3.1 is lengthy. We decompose it into a sequence of lemmas. Without loss of generality, we assume $\leq_P$ is Σ^1_1. It is routine to obtain the bold-face version by relativization.

Some facts about higher randomness theory are necessary. Very little is known in this area.

Definition 3.2. *Given a real z and a number $n \geq 1$, a real $x \in 2^\omega$ is $\Delta^1_n(z)$-random if $x \notin A$ for each $\Delta^1_n(z)$ set A for which $\mu(A) = 0$.*

The following lemma says that there is no large size antichain in some "well-behaved" locally partial orders and motivates the main theorem.

Lemma 3.3. *If $X \subseteq 2^\omega$ and $\mu(X) > 0$, then there are two reals $x, y \in X$ so that $y <_T x$. Furthermore, there are two reals $x, y \in X$ so that $y \leq_1 x$ and $x \not\leq_T y$.*

Proof. [4] Suppose $X \subseteq 2^\omega$ and $\mu(X) > 0$. Without loss of generality, we assume that X is contains only 1-random reals. Define a set $X^* = \{x | \exists y \in X \exists n \forall m > n(x(m) = y(m))\}$. Since X is measurable, X^* is measurable. Then, by Kolmogorov's 0-1 law (see [1]), $\mu(X^*) = 1$. Define a set $Y = \{x | x = x_0 \oplus x_1 \wedge x_0 \in X^*)\}$. Then Y is measurable and $\mu(Y) = \mu(X^*) = 1$. Define $Z = \{x \in 2^\omega | x = x_0 \oplus x_1 \ \wedge \ x_0 <_T x\}$. By Theorem 1.6, Z contains all 1-random reals. So $\mu(Z) = 1$. Hence $\mu(X \cap Y \cap Z) > 0$. Take a real $x \in X \cap Z$ for which $x = x_0 \oplus x_1$ and $x_0 \in X^*$. There is a real $y \in X$ such that y is different from x_0 at only finitely many bits. Since both x_0 and x_1 are 1-random, they both are infinite and co-infinite. Then it is easy to see that $y \leq_1 x$. Obviously $x \not\leq_T y$ since $x \not\leq_T x_0$. $\square$

Since Turing reducibility implies h-reducibility, we have the following corollary.

Corollary 3.4. *If X is an antichain in $\langle 2^\omega, \leq_T\rangle$ (or in $\langle 2^\omega, \leq_h\rangle$) and is measurable, then $\mu(X) = 0$.*

Hence there is no antichain of positive measure in any locally countable partial order.

We say that a predicate P is $d{-}\Sigma^1_1$ if there is a Π^1_1 predicate R and a Σ^1_1 predicate R so that $P(z, i)$ iff $R(z, i) \wedge S(z, i)$ for each real z and number i. A set $A \subseteq 2^\omega$ is $d - \Sigma^1_1$ if the predicate "$x \in A$" is $d - \Sigma^1_1$.

From now on, we fix a standard enumeration $\{A_i\}_{i\in\omega}$ of Σ^1_1 sets and an enumeration $\{2^\omega - A_i\}_{i\in\omega}$ of Π^1_1-sets. By the index, we can get a d-Σ^1_1 enumeration $\{A_i - A_j\}_{\langle i,j\rangle\in\omega}$ of $d - \Sigma^1_1$ sets. The following proposition can be found in [16] (Exercise 1.11.IV).

Proposition 3.5 (Sacks [16])**.** *The index set $\{\langle i, j\rangle | \mu(A_i) > r_j\}$ is Π^1_1, where A_i ranges over Π^1_1 sets and r_j ranges over rationals.*

Proof. (sketch) Since no proof is found in the literature, we sketch a proof here. By the relativized Spector-Gandy theorem, $A \subseteq 2^\omega$ is Π^1_1 iff there is a Δ_0 formula

[4] This proof combines some ideas from Jockusch and the referee.

$\varphi(x,y)$ so that $x \in A$ iff $L_{\omega_1^x}[x] \models \exists y\varphi(x,y)$. Since for almost every real x, $\omega_1^x = \omega_1^{CK}$(Corollary 1.6.IV.), for almost every real x, $x \in A$ iff $L_{\omega_1^{CK}}[x] \models \exists y\varphi(x,y)$. For each $\alpha < \omega_1^{CK}$, define a Δ_1^1-set $A_\alpha = \{x | L_\alpha[x] \models \exists y\varphi(x,y).\}$. If $\omega_1^x = \omega_1^{CK}$, then $x \in A$ iff $x \in A_\alpha$ for some $\alpha < \omega_1^{CK}$. Since $A_\alpha \subseteq A_\beta$ for $\alpha < \beta < \omega_1^{CK}$, $\mu(A) = \sup_{\alpha<\omega_1^{CK}} \mu(A_\alpha)$. So $\mu(A) > r$ iff $\exists n(n \in \mathcal{O}_1 \ \wedge \ \mu(A_{|n|}) > r)$ which is Π_1^1 by Theorem 1.3.IV [16] where $\mathcal{O}_1$ is the standard path through $\mathcal{O}$ as defined in Theorem 2.4.III [16]. $\square$

Corollary 3.6. *The predicate, $\mu(A \cap B) > r$, is Δ_2^1, where A ranges over Π_1^1 sets, B ranges over Σ_1^1 sets and r ranges over rationals. In other words, the set $\{\langle i,j\rangle | \mu(C_i) > r_j\}$ is Δ_2^1, where C_i ranges over $d{-}\Sigma_1^1$ sets, r_j ranges over rationals.*

Proof. $\mu(A \cap B) = \mu(A) - \mu(A \cap (2^\omega - B))$. So it suffices to show that $\mu(A) - \mu(A \cap C) > r$ is Δ_2^1 where A, C range over Π_1^1 sets and r ranges over rationals. It is easy to see that $\mu(A) - \mu(A \cap C) > r$ if and only if there is a rational p so that $\mu(A) > p + r$ and $\mu(A \cap C) \leq p$. By Proposition 3.5, "$\mu(A) > p + r$" is Π_1^1 and "$\mu(A \cap C) \leq p$" is Σ_1^1. So the predicate "$\mu(A) - \mu(A \cap C) > r$" is Δ_2^1. $\square$

Given two predicates $P(y^*,i), Q(y^*,i)$ for which $\forall y^*\forall i\neg(P(y^*,i) \ \wedge \ Q(y^*,i))$ and a real x (or a string $\sigma \in 2^{<\omega}$) , we use $\Sigma(P,Q,y^*,i) \leftrightarrow x(i)$ (or $\Sigma(P,Q,y^*,i) \leftrightarrow \sigma(i)$) to denote:

$$(x(i) = 0 \rightarrow P(y^*,i)) \ \wedge \ (x(i) = 1 \rightarrow Q(y^*,i))$$

(or $(\sigma(i) = 0 \rightarrow P(y^*,i)) \ \wedge \ (\sigma(i) = 1 \rightarrow Q(y^*,i)))$.

It is easy to see the following fact.

Lemma 3.7. *If both P and Q are $d{-}\Sigma_1^1$, then the sets $\{y^* \mid \forall i \leq n(\Sigma(P,Q,y^*,i) \leftrightarrow \sigma(i))\}$ are uniformly $d - \Sigma_1^1$ where $\sigma \in 2^{<\omega}$ and $n \in \omega$.*

Proof. Note that

$$\{y^* \mid \forall i \leq n(\Sigma(P,Q,y^*,i) \leftrightarrow \sigma(i))\} = \bigcap_{\sigma(i)=0 \wedge i \leq n} \{y^* | P(y^*,i)\} \cap \bigcap_{\sigma(i)=1 \wedge i \leq n} \{y^* | Q(y^*,i)\}.$$

$\square$

Note that every $d - \Sigma_1^1$ set is measurable.

Lemma 3.8. *For any reals $x \leq_h y$, there is a Π_1^1 predicate $P(y^*,i)$ and a $d - \Sigma_1^1$-predicate $Q(y^*,i)$ so that $\forall i(\Sigma(P,Q,y,i) \leftrightarrow x(i))$ and $\forall y^*\forall i\neg(P(y^*,i) \ \wedge \ Q(y^*,i))$.*

Proof. Since $x \leq_h y$, there are two Π_1^1 predicates $R(y^*,i), S(y^*,i)$ so that $x(i) = 0$ iff $R(y,i)$ iff $\neg S(y,i)$. Define $P = R$ and $Q = S \wedge \neg R$. It is easy to see that $\Sigma(P,Q,y,i) \leftrightarrow x(i)$ and $\forall y^*\forall i\neg(P(y^*,i) \ \wedge \ Q(y^*,i))$. $\square$

The main ideas of the argument used in the following two lemmas are from [12].

Lemma 3.9. *If $x \in 2^\omega$ is Δ_2^1-random, then for any Π_1^1 predicate $P(y^*,i)$ and $d{-}\Sigma_1^1$ predicate $Q(y^*,i)$ which satisfy $\forall y^*\forall i\neg(P(y^*,i) \ \wedge \ Q(y^*,i))$, there is a constant c so that*

$$\forall n(\mu(\{y^* \in 2^\omega | \forall i \leq n(\Sigma(P,Q,y^*,i) \leftrightarrow x(i))\}) \leq 2^{-n+c}).$$

Proof. Uniformly define a family $\{\mathcal{V}_\sigma\}_{\sigma\in 2^{<\omega}}$ of $d-\Sigma^1_1$ classes by $\mathcal{V}_\sigma = \{y^* \in 2^\omega \mid \forall i < |\sigma|(\Sigma(P,Q,y^*,i) \leftrightarrow \sigma(i))\}$. Since $\forall y^*\forall i\neg(P(y^*,i) \wedge Q(y^*,i))$, if σ and τ are incompatible strings, then $\mathcal{V}_\sigma \cap \mathcal{V}_\tau = \emptyset$. Now for each $i \in \omega$, define

$$F_i = \{\sigma \in 2^{<\omega} \mid \mu(\mathcal{V}_\sigma) > 2^{-|\sigma|+i}\}.$$

Note, by Corollary 3.6, that the sets $F_i \subseteq 2^{<\omega}$ are uniformly Δ^1_2. Define $\mathcal{G}_i = [F_i]$. Note the set $\mathcal{G} = \bigcap_{i\in\omega}\mathcal{G}_i$ is Δ^1_2. We claim that $\mu(\mathcal{G}_i) \leq 2^{-i}$. Assume not. Then there is a prefix-free set $D \subseteq F_i$ such that $\mu([D]) > 2^{-i}$. For distinct $\sigma,\tau \in D$, we have $\mathcal{V}_\sigma \cap \mathcal{V}_\tau = \emptyset$. Therefore,

$$\begin{aligned}\mu(\{y^* \in 2^\omega \mid \exists\sigma \in D\forall i \leq n(\Sigma(P,Q,y^*,i) \leftrightarrow \sigma(i))\})&\\ = \sum_{\sigma\in D}\mu(\mathcal{V}_\sigma) > \sum_{\sigma\in D} 2^{-|\sigma|+i} = 2^i\sum_{\sigma\in D}2^{-|\sigma|}&\\ = 2^i\mu([D]) > 2^i2^{-i} = 1.&\end{aligned}$$

This is a contradiction, so $\mu(\mathcal{G}_i) \leq 2^{-i}$. Therefore, $\mathcal{G}$ is Δ^1_2 and of measure 0. Now let $x \in 2^\omega$ be Δ^1_2-random. Then $x \notin \mathcal{G}_c$ for some c. In other words,

$$\forall n(\mu(\{y^* \in 2^\omega | \forall i \leq n(\Sigma(P,Q,y^*,i) \leftrightarrow x(i))\}) \leq 2^{-n+c})$$

which completes the proof. □

Lemma 3.10. *For any real z and Δ^1_2-random real x, if x is not 1-z-random and $y \geq_h x$, then y is not $\Delta^1_2(z)$-random.*

Proof. Take reals $x, y \in 2^\omega$ such that x is Δ^1_2-random and $x \leq_h y$. Since x is $\Delta^1_1(y)$ definable, by Lemma 3.8, there is a Π^1_1 predicate $P(y^*,i)$ and $d-\Sigma^1_1$-predicate $Q(y^*,i)$ so that $\Sigma(P,Q,y,i) \leftrightarrow x(i)$ and $\forall y^*\forall i\neg(P(y^*,i) \wedge Q(y^*,i))$. So by Lemma 3.9, there is a constant c so that

$$\forall n(\mu(\{y^* \in 2^\omega | \forall i \leq n(\Sigma(P,Q,y^*,i) \leftrightarrow x(i))\}) \leq 2^{-n+c}).$$

For every $\sigma \in 2^{<\omega}$, uniformly define a $d-\Sigma^1_1$-set

$$F_\sigma = \{y^* \in 2^\omega | \forall i \leq |\sigma|(\Sigma(P,Q,y^*,i) \leftrightarrow \sigma(i))\}.$$

Define $G_\sigma = F_\sigma$ if $\mu(F_\sigma) \leq 2^{-|\sigma|+c}$ and $G_\sigma = \emptyset$ otherwise. Note that by Lemma 3.6, $\{G_\sigma\}_{\sigma\in 2^{<\omega}}$ is a Δ^1_2-collection of $d-\Sigma^1_1$ sets and $G_\sigma = F_\sigma$ if $\sigma \prec x$.

Now suppose $z \in 2^\omega$ such that x is not 1-z-random. Then there is a computable collection of $\Sigma^0_1(z)$ sets $\{V_i\}_{i\in\omega}$ so that $\mu(V_i) \leq 2^{-i}$ for every i and $x \in \bigcap_{i\in\omega} V_i$. Fix a uniformly c.e. collection of z-c.e. prefix free sets $\{\hat{V}_i\}_{i\in\omega}$ so that $[\hat{V}_i] = V_i$ for each i. Define $H_i = \bigcup_{\sigma\in\hat{V}_{i+c}} G_\sigma$. Then

$$\mu(H_i) \leq \sum_{\sigma\in\hat{V}_{i+c}}\mu(G_\sigma) \leq \sum_{\sigma\in\hat{V}_{i+c}} 2^{-|\sigma|+c} = 2^c\cdot\sum_{\sigma\in\hat{V}_{i+c}}2^{-|\sigma|} = 2^c\cdot\mu(V_{i+c}) \leq 2^{-i}.$$

Since $\{H_i\}_{i\in\omega}$ is a $\Delta^1_2(z)$ sequence of $d-\Sigma^1_1$ sets, $H = \bigcap_{i\in\omega} H_i$ is a $\Delta^1_2(z)$ set and $\mu(H) = 0$. But for each i, there is a $\sigma \in \hat{V}_i$ for which $\sigma \prec x$ and so, by Lemma 3.9, $F_\sigma = G_\sigma \subseteq H_i$. Hence $y \in F_\sigma \subseteq H_i$ for each i. Thus $y \in H$. So y is not $\Delta^1_2(z)$-random. □

Given a set $X \subseteq 2^\omega$, define $\mathcal{U}_h(X) = \{y | \exists x \in X(x \leq_h y)\}$.

Lemma 3.11. *Suppose $X \subset 2^\omega$ contains only Δ^1_2-random reals. If $\mu(X) = 0$, then $\mu(\mathcal{U}_h(X)) = 0$.*

Proof. If $\mu(X) = 0$, then there exists a sequence of open sets $\{U_i\}_{i\in\omega}$ with $\mu(U_i) \leq 2^{-i}$ for all $i \in \omega$ so that $X \subseteq \bigcap_{i\in\omega} U_i$. Obviously, there exists a real z so that the sequence $\{U_i\}_{i\in\omega}$ is uniformly z-computable. Thus $\{U_i\}_{i\in\omega}$ is a $\Sigma^0_1(z)$ Martin-Löf test. So X does not contain any 1-z-random real. By Lemma 3.10, $\mathcal{U}_h(X)$ does not contain any $\Delta^1_2(z)$-random real. Hence $\mu(\mathcal{U}_h(X)) = 0$. □

Lemma 3.12. *There exists a nonmeasurable antichain in $\langle 2^\omega, \leq_h\rangle$.*

Proof. Define $\mathcal{R} = \{x | x \text{ is } \Delta^1_2\text{-random.}\}$. Take a maximal set $X \subset \mathcal{R}$ so that $\forall x \in X \forall y \in X(x \neq y \implies \forall z(z \leq_h x \ \wedge \ z \leq_h y \implies z \text{ is not } \Delta^1_2\text{-random}))$. By maximality of X, if $z_0 \in \mathcal{R}-X$, then there exists a real $x_0 \in X$ and a real $y_0 \in \mathcal{R}$ so that $y_0 \leq_h x_0$ and $y_0 \leq_h z_0$. Note that X is an antichain in the h-degrees. Define $\mathcal{D}_x = \{y | y \leq_h x \text{ and } y \in \mathcal{R}\}$ for every $x \in X$. So there exists an enumeration $\{d^x_e\}_{e\in\omega}$ of $\mathcal{D}_x$. Define $\mathcal{D}_e = \{d^x_e | x \in X\}$. Note that $\mathcal{D}_e$ is an antichain for every $e \in \omega$. So by Corollary 3.4, $\mathcal{D}_e$ is either nonmeasurable or is of measure 0. Since $\mathcal{R} \subseteq \bigcup_{e\in\omega} \mathcal{U}_h(\mathcal{D}_e)$, there exists a number e so that $\mathcal{U}_h(\mathcal{D}_e)$ is either nonmeasurable or $\mu(\mathcal{U}_h(\mathcal{D}_e)) > 0$. In both cases, by Lemma 3.11, $\mathcal{D}_e$ is nonmeasurable. □

Finally, we can prove Theorem 3.1

Proof. (of Theorem 3.1). Suppose $\leq_P$ is a Σ^1_1-relation. By Lemma 3.12, there is a nonmeasurable antichain A in $\langle 2^\omega, \leq_h\rangle$. By Corollary 2.2, for any $x, y \in 2^\omega$, $x \leq_P y$ implies $x \leq_h y$. Hence A is also an antichain in $\mathbb{P}$. □

Comparing with Proposition 2.5, we have the following proposition.

Proposition 3.13. *Assume $ZFC + V = L$. There is a locally countable Δ^1_2 partial order on 2^ω in which every antichain has size 1.*

Proof. If $V = L$, then define $\mathbb{P} = \langle 2^\omega, \leq_L\rangle$. $\mathbb{P}$ is a Δ^1_2 well order of which the order type is ω_1. □

Theorem 3.14. *Assume $ZFC + MA_{\aleph_1}$. If a partial order $\mathbb{P} = \langle 2^\omega, \leq\rangle$ is locally countable, then there exists a nonmeasurable antichain in $\mathbb{P}$.*

It is not as easy to show Theorem 3.14 as Proposition 2.5. We need some definitions.

Definition 3.15.
(1) *A partial order $\mathbb{P} = \langle P, \leq\rangle$ is* special *if there is a sequence $\{X_\alpha\}_{\alpha<\omega_1}$ so that $P = \bigcup_{\alpha<\omega_1} X_\alpha$ and for every $\alpha < \omega_1$, X_α is an antichain.*
(2) *A partial order $\mathbb{P} = \langle P, \leq\rangle$ is* top closed *if there is an antichain $T \subseteq P$ for which $P = \{q | \exists p \in T(q \leq p)\}$. We say T is the* top *of P.*

We prove a structure theorem for locally countable partial orders.

Lemma 3.16. *Every locally countable, top closed partial order is special.*

Proof. Given a locally countable, top closed partial order $\mathbb{P} = \langle P, \leq\rangle$ for which $|P| = \kappa$. Without loss of generality, we assume $\kappa \geq \aleph_0$. There is an antichain $T = \{x_i\}_{i<\kappa}$ which is the top of P. Define $A_i = \{y | y \leq x_i \text{ and } y \notin \bigcup_{j<i} A_j.\}$ for every $i < \kappa$. Note that $P = \bigcup_{i<\kappa} A_i$. Since A_i is countable for every $i < \kappa$, set $A_i = \{y_{i,n}\}_{n\in\omega}$. We decompose P into ω_1-many antichains $\{B_s\}_{s<\omega_1}$ step by step.

At step 0. Select $B_0 = T$.

At step $s > 0$. We decompose the construction of B_s into κ-many substeps.

Substep $\alpha < \kappa$. There are two cases:

(1) $\forall y(y \in A_\alpha - \bigcup_{t<s} B_t)\exists z \in B_{s,<\alpha}(z \leq y \text{ or } z \geq y)$. Go to the next substep.
(2) Otherwise, select the least number n for which $y_{\alpha,n} \in A_\alpha - \bigcup_{t<s} B_t$ and $\forall z \in B_{s,<\alpha}(z \nleq y_{\alpha,n} \text{ and } z \ngeq y_{\alpha,n})$. Put $y_{\alpha,n}$ into $B_{s,\alpha}$.

Define $B_s = \bigcup_{\alpha<\kappa} B_{s,\alpha}$.

Obviously, $B_s \cap B_t = \emptyset$ if $s \neq t$. By the construction, B_s is an antichain for every $s < \omega_1$. It remains to show $P = \bigcup_{s<\omega_1} B_s$.

Suppose not. Select the least $i < \kappa$ so that there exists an element $y \in A_i$ but $y \notin \bigcup_{s<\omega_1} B_s$. Choose the least n so that $y_{i,n} \in A_i - \cup_{s<\omega_1} B_s$. Note that for every $j < i$, $y_{i,n} \nleq x_j$. Since there are at most $\aleph_0$ many elements in A_i, there must be a stage $s_0 < \omega_1$ so that $A_i - \bigcup_{s<\omega_1} B_s = A_i - \bigcup_{s<s_0} B_s$ and $y_{i,k} \in \cup_{s<s_0} B_s$ for all $k < n$. So at any stage $t > s_0$, we always put some element $z \in \bigcup_{j<i} A_j$ with $z < y_{i,n}$ into B_t. Thus there are $\aleph_1$-many elements below $y_{i,n}$, a contradiction. □

Lemma 3.17. *Every locally countable partial order is a union of $\aleph_1$-many locally countable, top closed partial orders.*

Proof. Consider a locally countable partial order $\mathbb{P} = \langle P, \leq\rangle$. Take a maximal antichain A_0. We decompose P into $\aleph_1$-many sets.

Define $B_0 = \{y | \exists x \in A_0(y \leq x)\}$.

For $\alpha < \omega_1$, define A_α to be a maximal antichain in $\langle P - \bigcup_{\beta<\alpha} B_\beta, \leq\rangle$. Define $B_\alpha = \{y | \exists x \in A_\alpha(y \leq x)\}$.

So $\langle B_\alpha, \leq\rangle$ is a locally countable, top closed partial order for every $\alpha < \omega_1$. It remains to show $P = \bigcup_{\alpha<\omega_1} B_\alpha$.

For any $y \in P$, there are at most $\aleph_0$-many elements below it. But if $y \notin \bigcup_{\alpha<\omega_1} B_\alpha$, then for every $\alpha < \omega_1$, there exists an element $x \in A_\alpha$ so that $y \geq x$, a contradiction. So $y \in \bigcup_{\alpha<\omega_1} B_\alpha$. □

We have the following structure theorem about locally countable partial orders.

Proposition 3.18. *Every locally countable partial order is special.*

Proof. By Lemma 3.17, every locally countable partial order is a union of $\aleph_1$-many locally countable, top closed partial orders. By Lemma 3.16, every locally countable, top closed partial order is special. So every locally countable partial order is special. □

Proof. (of Theorem 3.14) By Proposition 3.18, 2^ω is a union of $\aleph_1$-many antichains. By $MA_{\aleph_1}$, there must be an antichain which either is nonmeasurable or has positive measure. In the second case, we can select a nonmeasurable sub-antichain. □

4. Some specific locally countable partial orders

In this section, we consider some specific locally countable partial orders.

The first one is related to the Turing degrees and was the original motivation of this paper.

Corollary 4.1. *There exists a nonmeasurable antichain in $\langle 2^\omega, \leq_T\rangle$.*

We present some other properties related to the antichains in the Turing degrees. Since every real is Turing equivalent to a non-1-random real, not every maximal antichain in $\langle 2^\omega, \leq_T\rangle$ is nonmeasurable. After we proved Corollary 4.1, Jockusch asked the following question :" Is it true that for every measurable antichain X of $\langle 2^\omega, \leq_T\rangle$ that $X \cup \{x\}$ is an antichain for almost every real x ?". We give a negative answer. Given a set $X \subseteq 2^\omega$, define $\mathcal{U}(X) = \{y | \exists x \in X(x \leq_T y)\}$.

Proposition 4.2. *There exists an antichain X in $\langle 2^\omega, \leq_T\rangle$ for which $\mu(X) = 0$ and either $\mathcal{U}(X)$ is nonmeasurable or $\mu(\mathcal{U}(X)) > 0$.*

Proof. Take a maximal set A which contains only 2-random reals so that

$$\forall x, y \in A \forall g(x \neq y \ \wedge \ g \text{ is 1-generic} \ \wedge \ g \leq_T x \implies g \not\leq_T y).$$

Define $\mathcal{G}^x = \{g | g \leq_T x \ \wedge \ g \text{ is 1-generic.}\}$. Then there is an enumeration $\{g_e^x\}_{e\in\omega}$ of $\mathcal{G}^x$. Define $\mathcal{G}_e = \{g_e^x | x \in A\}$. Then since no 1-generic real can be 1-random, $\mu(\mathcal{G}_e) = 0$ and $\mathcal{G}_e$ is an antichain for every e. We have the following claim:

Claim: $\bigcup_{e\in\omega} \mathcal{U}(\mathcal{G}_e)$ contains all 2-random reals.

Proof. Suppose not. There is a 2-random real $r \notin \bigcup_{e\in\omega} \mathcal{U}(\mathcal{G}_e)$. By Theorem 1.7, every 2-random real bounds a 1-generic real. So $\mathcal{G}^r \neq \emptyset$ and $\mathcal{G}^r \cap \bigcup_{x\in A} \mathcal{G}^x = \mathcal{G}^r \cap \bigcup_{e\in\omega} \mathcal{G}_e = \emptyset$. By maximality of A, $r \in A$ and so $r \in \bigcup_{e\in\omega} \mathcal{U}(\mathcal{G}_e)$, a contradiction. □

So there must be some e so that either $\mathcal{U}(\mathcal{G}_e)$ is nonmeasurable or $\mu(\mathcal{U}(\mathcal{G}_e)) > 0$. □

Another application is related to the K-degrees. For any $\sigma \in 2^{<\omega}$, we use $K(\sigma)$ to denote the prefix free Kolmogorov complexity of σ. We say that a real x is *K-reducible to y* ($x \leq_K y$) if there is a constant c so that $\forall n(K(x \upharpoonright n) \leq K(y \upharpoonright n) + c)$ (for more details, see [2]). Miller [11] proved that for every 3-random real x, $|\{y | y \geq_K x\}| = \aleph_0$. Set $x \leq_P y$ iff $x = y$ or both x, y are 3-random and $x \geq_K y$. By Miller's result, $\mathbb{P} = \langle 2^\omega, \leq_P\rangle$ is a Δ^1_1 locally countable partial order. So there is a nonmeasurable antichain in $\mathbb{P}$. But the collection of non-3-random reals is of measure 0. So we have the following result.

Corollary 4.3. *There is a nonmeasurable antichain in $\langle 2^\omega, \leq_K\rangle$.*

5. Some questions

We raise some open questions in this section.

The first question is connected with Proposition 4.2.

Question 5.1 (Jockusch)**.** *Is there an antichain X with $|X| > 1$ in $\langle 2^\omega, \leq_T\rangle$ for which $\mu(X) = 0$ and $\mu(\mathcal{U}(X)) = 1$?*

We remark that it suffices to construct an antichain X so that $\mu(\mathcal{U}(X)) = 1$ since we have the following proposition.

Proposition 5.2. *For any antichain X in $\langle 2^\omega, \leq_T\rangle$, if $\mathcal{U}(X)$ is measurable, then $\mu(X) = 0$.*

Proof. Suppose X is an antichain in $\langle 2^\omega, \leq_T\rangle$ and $\mu(\mathcal{U}(X)) > 0$. Then there exists a $\mathbf{\Sigma}^0_2$ set $Y \subseteq \mathcal{U}(X)$ for which $\mu(Y) = \mu(\mathcal{U}(X)) > 0$. Define $Z = \{z \in Y | \forall y(y \in Y \implies y \not<_T z)\}$. Z is $\mathbf{\Pi}^1_1$ and so measurable. By Lemma 3.3, $\mu(Z) = 0$. Note that $\mu(\mathcal{U}(X) - Y) = 0$ and $X - Z \subseteq \mathcal{U}(X) - Y$. So $\mu(X - Z) = 0$. Hence $\mu(X) = \mu(Z) = 0$. □

We say that a set $X \subset 2^\omega$ is a *quasi-antichain* in the Turing degrees if it satisfies the following properties:

(1) $\exists x \in X \exists y \in X (x \not\equiv_T y)$.
(2) $\forall x \in X \forall y (x \equiv_T y \to y \in X)$.
(3) $\forall x \in X \forall y \in X (x \not\equiv_T y \to x \not\leq_T y)$.

It is not hard to see that there is a nonmeasurable quasi-antichain in the Turing degrees using Lemma 3.3 and Corollary 4.1.

Question 5.3 (Jockusch)**.** *Is every maximal quasi-antichain in the Turing degrees nonmeasurable?*

We say that a partial order $\mathbb{P} = \langle 2^\omega, \leq_P \rangle$ is *locally null* if for every x, $\mu(\{y | y \leq_P x\}) = 0$. We have the following proposition.

Proposition 5.4. *For any measurable locally null partial order $\mathbb{P} = \langle 2^\omega, \leq_P \rangle$, every chain in $\mathbb{P}$ has measure 0.*

Proof. Define $A = \{\langle x, y \rangle | x \leq_P y\}$. Then A is measurable. For every y, the set $A_y = \{x | x \leq_P y\}$ is of measure 0. According to Fubini's theorem, $\mu(A) = 0$. By Fubini's theorem again, for almost every real x, the set $A^x = \{y | x \leq_P y\}$ is of measure 0. Set $B = \{x | \mu(A^x) = 0\}$. Note that $\mu(B) = 1$.

Now take any chain X in $\mathbb{P}$. If $X \cap B = \emptyset$, then $\mu(X) = 0$. Otherwise, fix a real $x \in X \cap B$. Then $X \subseteq A_x \cup A^x$. Both A_x and A^x are of measure 0. Hence $\mu(X) \leq \mu(A_x) + \mu(A^x) = 0$. So $\mu(X) = 0$. □

Since every Π^1_1 set is measurable, by Proposition 5.4, every chain in any Π^1_1 locally countable partial order is of measure 0. So the remaining question is:

Question 5.5. *Is it true that for every locally countable Π^1_1 partial order $\mathbb{P} = \langle 2^\omega, \leq_P \rangle$, there exists a nonmeasurable antichain in $\mathbb{P}$?*

The difficulty to give a positive answer to Question 5.5 is that we cannot control the complexity of some predicates as we do in Section 3.

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