DESCRIPTIVE SET THEORETICAL COMPLEXITY OF RANDOMNESS NOTIONS

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Abstract. We study the descriptive set theoretical complexity of various randomness notions.

1. Introduction

The original motivation of this paper is to characterize weakly-2-random reals by the prefix-free Kolmogorov complexity. Since Schnorr characterized Martin-Löf randomness by the prefix-free Kolmogorov complexity, many people thought that every randomness notion should have a characterization by the initial segment complexity. For example, Miller and others obtained a very successful characterization of 2-randomness.

Theorem 1.1 (Miller [8] and [9]; Nies, Stephen and Terwijn [12]). *A real x is 2-random if and only if*

$$\exists c \forall n \exists m (C(x \upharpoonright m) \geq m - c)$$

if and only if

$$\exists c \forall n \exists m > n (K(x \upharpoonright m) \geq m + K(m) - c).$$

Recently, Miller and Yu [10] obtained the following result.

Theorem 1.2 (Miller and Yu [10]). *$x \oplus y$ is random if and only if*

$$\exists c \forall n (K(x \upharpoonright n) + C(y \upharpoonright n) \geq 2n - c).$$

The theorem gives almost all the "relativizable" randomness notions stronger than Martin-Löf randomness unrelativized Kolmogorov complexity characterizations. An important question remaining open is whether there is a Kolmogorov complexity characterization for weak-2-randomness. This question has been tried by many ways. For example, one way is to ask whether there is a sequence of functions $\{f_n\}_{n\in\omega}$ so that for every real x, x is weakly-2-random if and only if $\exists n \forall m \exists k \geq m(K(x \upharpoonright k) \geq k + f_n(k))$? Most of these attempts turned to be some kind of $\mathbf{\Sigma}^0_3$-characterizations for weak-2-randomness. But all of the ways (of course) failed. So people suspected that the collection of weakly-2-random reals is not $\mathbf{\Sigma}^0_3$. We confirm the doubt.

Then we also study the descriptive set theoretical complexity of some other classical randomness notions. Many results have been obtained in [5] by using Wade reductions. Given two sets of reals A and B, A is Wade reducible to B, writing to $A \leq_W B$, if there is a continuous functions $f : 2^\omega \to 2^\omega$ so that for every x, $x \in A$ if and only if $f(x) \in B$. They prove, for example, that the collection of Schnorr random reals is $\mathbf{\Pi}^0_3$-complete (and so non-$\mathbf{\Sigma}^0_3$). Here we give another more direct way, by using forcing argument, to prove our results. One may think that the results in [5] are stronger since what they prove is that the collection of Schnorr random reals is $\mathbf{\Pi}^0_3$-complete. Actually it is not by the following well known descriptive set theory result.

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Theorem 1.3 (Folklore)**.** *For any $\xi < \omega_1$ and $\mathbf{\Sigma}^0_\xi$ (or $\mathbf{\Pi}^0_\xi$) set A, if A is not $\mathbf{\Pi}^0_\xi$ (or $\mathbf{\Sigma}^0_\xi$), then every $\mathbf{\Sigma}^0_\xi$ set is Wade reducible to A.*

Theorem 1.3 is an immediate conclusion of Borel determinacy. Moreover, our technique results are interesting independently. For example, we prove that the forcing notion of Π^0_1-classes with computable positive measures does not produce a Martin-Löf random real.

We also study the complexity of the collection of Δ^1_1-random reals. Sacks essentially proves that the collection of Δ^1_1-random reals is $\mathbf{\Pi}^0_3$. Hjorth and Nies introduced Π^1_1-Martin-Löf randomness in [6], which is an analog to the classical Martin-Löf randomness in higher recursion theory. But it was a very difficult question whether Π^1_1-Martin-Löf randomness is different with Δ^1_1-randomness. The separation of Π^1_1-Martin-Löf randomness from Δ^1_1-randomness was given in [2]. The proof in the paper is fairly involved. Only a sketch was presented there. Now we can give a full proof by a simpler argument. Further more, we have a total characterization where Δ^1_1-randomness is different with Π^1_1-Martin-Löf randomness.

We organize the paper as follows: In section 2, we give some basic definitions. In section 3, we present some easy facts about the descriptive set theoretical complexity of various randomness notions. Most of them are probably known; In section 4, we prove that the collection of weakly-2-random reals is not $\mathbf{\Sigma}^0_3$; In section 5, we prove that the collection of Schnorr random reals is not $\mathbf{\Sigma}^0_3$; In section 6, we prove that the collection of Δ^1_1-random reals is not $\mathbf{\Sigma}^0_3$.

2. Preliminary

A real is Kurtz random if it does not belong to any Π^0_1-null set. Since co-null open Σ^0_1 set is dense, every weakly 1-generic real is Kurtz random.

A Schnorr test is an uniformly c.e. sequence of open sets $\{U_n\}_{n\in\omega}$ so that $\mu(U_n) = 2^{-n}$ for every n. A real x is Schnorr random if for every Schnorr test $\{U_n\}_{n\in\omega}$, $x \notin \bigcap_{n\in\omega} U_n$. This is equivalent to that $x \notin \bigcap_{n\in\omega} U_n$ for any c.e. sequence of open sets $\{U_n\}_{n\in\omega}$ so that $\mu(U_n) = 2^{-f(n)}$ for every n where f is a computable function from ω to $[0,1]$ such that $\lim_{n\to\infty} f(n) = 0$.

A Martin-Löf test is an uniformly c.e. sequence of open sets $\{U_n\}_{n\in\omega}$ so that $\mu(U_n) < 2^{-n}$ for every n. A real x is Martin-Löf random (or 1-random) if for every Martin-Löf test $\{U_n\}_{n\in\omega}$, $x \notin \bigcap_{n\in\omega} U_n$. There exists a universal Martin-Löf test.

A generalized Martin-Löf test is an uniformly c.e. sequence of open sets $\{U_n\}_{n\in\omega}$ so that $\lim_{n\to\infty} \mu(U_n) = 0$ for every n. A real x is weakly-2-random if for every generalized Martin-Löf test $\{U_n\}_{n\in\omega}$, $x \notin \bigcap_{n\in\omega} U_n$. There is no a universal Martin-Löf test. Hirschfedlt and Miller proved the following nice result.

Theorem 2.1 (Hirschfeld and Miller [4])**.** *A real x is weakly-2-random if and only if x is 1-random and does not Turing-compute any non-computable Δ^0_2-real.*

For some information about higher randomness, please see [13], [6] and [2]. A real is Δ^1_1-random if and only if it does not belong to any Δ^1_1-null set. It is essentially due to Sacks [13] that a real x is Δ^1_1-random if and only if for any Δ^1_1-sequence of Δ^1_1 open sets $\{U_n\}_{n\in\omega}$ for which $\lim_{n\to\infty}\mu(U_n) = 0$, $x \notin \bigcap_n U_n$. So the collection of Δ^1_1-random reals is $\mathbf{\Pi}^0_3$.

A Π^1_1-Martin Löf test is a Π^1_1-sequence of Π^1_1-coded open sets $\{U_n\}_{n\in\omega}$ (i.e. the set $\{(n,\sigma) \mid \sigma \in U_n\}$ is Π^1_1) so that $\mu(U_n) < 2^{-n}$ for every n. Hjorth and Nies proved that there is a universal Π^1_1-Martin Löf test. A real is Π^1_1-Martin-Löf random if it does not belong to any Π^1_1-Martin-Löf test. We have the following result.

Theorem 2.2 (Chong, Nies and Yu [2])**.** *If $\omega_1^x = \omega_1^{\mathrm{CK}}$, then x is Δ^1_1-random if and only if x is Π^1_1-Martin-Löf random.*

For an open set U, we also identify it as a set of finite strings. For any finite string $\sigma \in 2^{<\omega}$, we use $[\sigma]$ to denote the open set $\{x \mid x \succ \sigma\}$. For any tree T, we use $[T]$ to denote the closed set $\{x \mid \forall n(x \upharpoonright n \in T)\}$.

For more information about randomness and computability theory, see [11] and [3].

3. Some basic facts

The following facts are immediate and probably known. Many of them can be found in [5]

Proposition 3.1. (1) *The collection of Kurtz random reals is* $\mathbf{\Pi}^0_2$ *but not* Π^0_2*;*

(2) *The collection of Schnorr random reals is* Π^0_3*;*
(3) *The collection of 1-random reals is* Σ^0_2*;*
(4) *The collection of weakly 2-random reals is* $\mathbf{\Pi}^0_3$ *but not* Π^0_3*;*
(5) *The collection of* Δ^1_1*-random reals is* $\mathbf{\Pi}^0_3$*.*

Proof. (1) Obviously the collection of Kurtz random reals K is $\mathbf{\Pi}^0_2$. Suppose that K is Π^0_2. Then there is a recursive set $R \subseteq \omega \times \omega \times 2^{<\omega}$ so that $x \in K$ if and only if $\forall n \exists m R(n, x \upharpoonright m)$. For each n, let $K_n = \{x \mid \exists m R(n, x \upharpoonright m)\}$. Then K_n is Σ^0_1, co-null and $K \subseteq K_n$ for every n. Then it would be easy to computably construct a sequence finite strings $\sigma_0 \prec \sigma_1 ...$ so that $[\sigma_n] \subseteq K_n$ for every n. Then the computable real $x = \bigcup_{n\in\omega} \sigma_n \in \bigcap_{n\in\omega} K_n = K$ would be Kurtz random, a contradiction.

(2) Obviously (see [5]).
(3) Obviously.
(4) Obviously the collection of weakly-2-random reals W is $\mathbf{\Pi}^0_3$. Suppose that K is Π^0_3. Then there is a computable set $R \subseteq \omega\times\omega\times\omega\times 2^{<\omega}$ so that $x \in W$ if and only if $\forall n \exists m \forall j R(n, m, x \upharpoonright j)$. For each n, let $W_n = \{x \mid \exists m \forall j R(n, m, x \upharpoonright j)\}$ and $W_{n,m} = \{x \mid \forall j R(n, m, x \upharpoonright j)\}$. Then K_n is Σ^0_2, co-null and $W \subseteq W_n$ for every n. We $\emptyset'$-computably construct a sequence finite strings $\sigma_0 \prec \sigma_1 ...$ and Π^0_1 positive measure sets $T_0 \supseteq T_1 \supseteq T_2 ...$ so that $\sigma_n \in T_n$ as follows: $\sigma_0 = \emptyset$ and $W_0 = 2^\omega$. Given σ_n and R_n. Since W_{n+1} is co-null, we may $\emptyset'$-computably find the least m so that $T_n \cap W_{n,m} \cap [\sigma_n] = \{x \succ \sigma_n \mid x \in [T_n] \wedge \forall j R(n, m, x \upharpoonright j)\}$ has positive measure. Let $T_{n+1} = T_n \cap W_{n,m} \cap [\sigma_n]$ and σ_{n+1} be a finite string in T_{n+1} extending σ_n. Then the $\emptyset'$-computable real $x = \bigcup_{n\in\omega} \sigma_n \in \bigcap_{n\in\omega} W_n = W$ is weakly-2-random, a contradiction to Theorem 2.1.
(5) Obviously.

□

The results above about descriptive complexity of the collections of Kurtz random and 1-random reals are rigid.

Proposition 3.2. (1) *The collection of Kurtz random reals is not* $\mathbf{\Sigma}^0_2$*;*

(2) *The collection of 1-random reals is not* $\mathbf{\Pi}^0_2$*.*

Proof. (1). Otherwise, there is a sequence closed sets $\{P_n\}_{n\in\omega}$ such that $\bigcup_n P_n$ contains exactly all the Kurtz random reals. Since all the generic reals are Kurtz random, $\bigcup_n P_n$ is comeager. Then there must be some n so that P_n is not meager. Then P_n must contain an interval and so contain a computable real, a contradiction.

(2). Otherwise, there is a sequence open sets $\{U_n\}_{n\in\omega}$ such that $\bigcap_n U_n$ contains exactly all the 1-random reals. Then for every n, $\mu(U_n) = 1$. So every U_n is dense. So every sufficient generic real would belong to $\bigcap_n U_n$. But no 1-generic reals can be random, a contradiction to 2.1. □

The second result above can be found in [5].

4. Weak 2-randomness

In this section, we prove that the collection of weakly 2-random reals is not $\mathbf{\Sigma}^0_3$. We apply a forcing argument.

Definition 4.1. *Define a forcing notion $\mathbb{P} = (\boldsymbol{P}, \leq)$ as follows:*
(1) *$P \in \boldsymbol{P}$ if and only if P is a Π^0_1-class with positive measure;*
(2) *For $P, Q \in \boldsymbol{P}$, $P \leq Q$ if and only if $P \subseteq Q$.*

Let $\{F_m\}_{m\in\omega}$ be an increasing sequence Π^0_1 sets so that $\bigcup_{m\in\omega} F_m$ is of measure 1. Set $C = \bigcup_{m\in\omega} F_m$. Let $\mathcal{D}_C = \{P \mid P \in \boldsymbol{P} \wedge P \subseteq C\}$.

Lemma 4.2. *$\mathcal{D}_C$ is dense.*

Proof. Suppose that $\{F_m\}_{m\in\omega}$ be an increasing sequence Π^0_1 sets so that $\bigcup_{m\in\omega} F_m$ is of measure 1 and $C = \bigcup_{m\in\omega} F_m$. Let $P \in \boldsymbol{P}$. Then there is some big enough m so that $\mu(F_m) > 1 - \frac{\mu(P)}{2}$. So

$$\mu(F_m \cap P) = \mu(F_m) + \mu(P) - \mu(F_m \cup P) > 1 - \frac{\mu(P)}{2} + \mu(P) + 1 = \frac{\mu(P)}{2}.$$

Thus $F_m \cap P \in \mathcal{D}_C$. □

The following lemma is a stronger version of Lemma 2.2 in [1].

Lemma 4.3. *For every computable tree T, there is a generalized Martin-Löf test $\{V_n\}_{n\in\omega}$ so that for any σ, if $[\sigma] \cap [T]$ is not empty, then $[\sigma] \cap [T] \cap \bigcap_n V_n$ is not empty.*

Proof. The idea is to build a uniformly sequence c.e. open sets $\{V_n\}_{n\in\omega}$ densely meeting $[T]$. The method is just like to build a null comeager set. But we may make some mistakes since there is no effective way to predicate whether $[\sigma] \cap [T]$ is not empty. So, at every step, we need to "correct" the construction of the previous steps. But the measure of mistakes will become very small whenever the step is large enough. This is the reason we can make sure that $\{V_n\}_{n\in\omega}$ is a generalized Martin-Löf test.

Fix a computable tree T. So there a computable approximation of computable trees $\{T_s\}_{s\in\omega}$ to T so that

(1) $T_0 = T$;
(2) $T_{s+1} = \{\sigma \mid \sigma \in T \wedge \exists \tau \in 2^{s+1} \cap T(\tau \text{ is compatible with } \sigma)\}$;

Then for every s, $T_{s+1} \subseteq T_s$.

Fix a computable enumeration $\{\sigma_i\}_{i\in\omega}$ of $2^{<\omega}$ and an enumeration of finite string $\{\sigma^{s+1}_i\}_{i\leq 2^{s+1}}$ of 2^{s+1} for each s.

We construct V_n for every n step by step.

At step 0, we put λ into V_0. So the open set $V_0 = 2^\omega$.

At step $s+1$.

Substep 1: We correct $\{V_k\}_{k\leq s}$ step by step.

Substep 1.0: Check whether there is a $\sigma \in T_{s+1} \cap 2^{s+1}$. If so, then do nothing. Otherwise, stop the construction.

Substep 1.k: Check whether there is some $\tau \in V_k$ so that there is no $\nu \in T_{s+1} \cap 2^{s+1}$ so that $\nu \succ \tau$. If so, check whether there is some $\tau' \succ \tau \upharpoonright k$ in $2^{|\tau|}$ so that there is a $\nu \in T_{s+1} \cap 2^{s+1}$ so that $\nu \succ \tau'$. If so, then put τ' into V_j for any $j \leq k$. Otherwise, do nothing.

Substep 2: For every i, check whether there is some $\tau \in T_{s+1}$ extending σ^{s+1}_i: If not, we go to $i+1$; Otherwise, check whether there is some $\tau \in V_s$ so that $\tau \succ \sigma^{s+1}_i$: if yes, then put τ into V_{s+1}; Otherwise, check whether there is some very long $\tau \succ \sigma^{s+1}_i$ in T_{s+1} so that is longer than any finite strings mentioned before. If yes, pick up such a τ and put it into V_{s+1}. Otherwise, do nothing.

Now for any $k \leq s$, check whether there is some $\tau' \in V_k$ compatible with τ. If yes, do nothing; Otherwise, put τ into V_k.

This finishes the construction.

By the construction, for any n, $V_{n+1} \subseteq V_n$.

If $\sigma \in T$ and $[\sigma] \cap [T] \neq \emptyset$, then there is some stage $s_0 \geq |\sigma|$ at which we find some $\sigma_0 \succ \sigma$ so that $\sigma_0 \in T$ and $[\sigma_0] \cap [T] \neq \emptyset$ and put it into $V_{|\sigma|}$. Then there is some larger stage $s_1 \geq |\sigma_0|$ at which we find some $\sigma_1 \succ \sigma_0$ so that $\sigma_1 \in T$ and $[\sigma_1] \cap [T] \neq \emptyset$ and put it into $V_{|\sigma_0|}$, etc. Since $\bigcap_{n\in\omega} = \bigcap_{i\in\omega} V_{|\sigma_i|}$, there real $x = \bigcup_{i\in\omega} \sigma_i \in (\bigcap_{n\in\omega} V_n) \cap T$. In other words, $[\sigma] \cap [T] \cap \bigcap_n V_n$ is not empty.

To see that $\{V_n\}_{n\in\omega}$ is a generalized Martin-Löf test, it is sufficient to show $\lim_{n\to\infty} \mu(V_n) = 0$. For any i, there is a big enough $s > i+1$ so that the open set $E_s = \{\sigma \in 2^s \mid \sigma \in T_s\}$ has measure less than $\mu([T]) + 2^{-i-1}$. Then from the step s of the construction, except the correction substep, we only put a prefix free set of finite strings into V_s. Moreover, except those strings putting at correction substep, for different strings in V_s, they have different lengths greater equal to s. But at the correction substep, by the assumption of E_s, we put at most 2^{-i-1} measure of finite strings into V_s. So

$$\mu(V_s) \leq \sum_{t\geq s} 2^{-t} + 2^{-i-1} = 2^{-s+1} + 2^{-i-1} \leq 2^{-i-1} + 2^{-i-1} = 2^{-i}.$$

Thus $\lim_{n\to\infty} \mu(V_n) = 0$.

□

For any $\mathbf{\Pi}^0_2$ set G, let $\mathcal{D}_G = \{P \mid P \in \boldsymbol{P} \wedge P \cap G = \emptyset\}$.

Lemma 4.4. *If G is a $\mathbf{\Pi}^0_2$ only containing weakly-2-random reals, then the set $\mathcal{D}_G = \{P \mid P \in \boldsymbol{P} \wedge P \cap G = \emptyset\}$ is dense in $\mathbb{P}$.*

Proof. Suppose that G is $\mathbf{\Pi}^0_2$ only containing weakly-2-random reals. Let $\{U_n\}_{n\in\omega}$ be a sequence open sets so that $G = \bigcap_n U_n$. Let $P \in \boldsymbol{P}$. Without loss of generality, we may assume that for any σ, if $[\sigma] \cap P \neq \emptyset$, then $\mu([\sigma] \cap P) > 0$ (since we may assume that P only contains 1-random reals). Then we claim that there is some σ so that $P \cap [\sigma] \cap G = \emptyset$ but $P \cap [\sigma] \neq \emptyset$.

Suppose not. By Lemma 4.3, there is a generalized Martin-Löf test $\{V_n\}_{n\in\omega}$ so that for any σ, if $[\sigma] \cap P$ is not empty, then $[\sigma] \cap P \cap (\bigcap_n V_n)$ is not empty. Then we build a sequence strings $\sigma_0 \prec \sigma_1 ...$ as follows.

Let $\sigma_0 = \emptyset$. Now suppose $[\sigma_i] \cap P \neq \emptyset$. Let $\tau \succ \sigma_i$ so that $[\tau] \cap P \neq \emptyset$ and $[\tau] \cap P \subseteq V_i$. By the property of $\{V_n\}_n$, there exists such a τ. Then by the assumption, let $\sigma_{i+1} \succ \tau$ so that $[\sigma_{i+1}] \cap P \subseteq U_i$.

Let $x = \bigcup_{i\in\omega} \sigma_i$. Then $x \in P \cap (\bigcap_{n\in\omega} U_n) \cap (\bigcap_{n\in\omega} V_n)$. Since $x \in \bigcap_{n\in\omega} V_n$, x is not weakly 2-random which contradicts to that G only contains weakly 2-random reals.

So there is some σ so that $P \cap [\sigma] \cap G = \emptyset$ but $P \cap [\sigma] \neq \emptyset$. Let $Q = P \cap [\sigma]$. Then $Q \in \boldsymbol{P}$ and $Q \leq P$. □

Theorem 4.5. *The collection of weakly 2-random reals is not $\mathbf{\Sigma}^0_3$.*

Proof. Suppose not. Then there is a countable sequence $\mathbf{\Pi}^0_2$ sets $\{G_n\}_n$ such that the set $\bigcup_n G_n$ contains exactly all the weakly-2-random reals. So G_n only contains weakly 2-random reals for every n. Then by Lemma 4.4, for any sufficient generic real g over $\mathbb{P}$, $g \notin G_n$ for any n. By Lemma 4.2, for any sufficient generic real g over $\mathbb{P}$, g is weakly-2-random, a contradiction. □

5. Schnorr randomness

In this section, we give another proof that the collection of Schnorr random reals is not $\mathbf{\Sigma}^0_3$. We use a similar method to the previous section with some modifications.

Definition 5.1. *Define a forcing notion $\mathbb{Q} = (\boldsymbol{Q}, \leq)$ as follows:*
1. *$Q \in \boldsymbol{Q}$ if and only if Q is a Π^0_1-class with some computable positive measure;*
2. *For $P, Q \in \boldsymbol{Q}$, $P \leq Q$ if and only if $P \subseteq Q$.*

For any Schnorr test $\{U_n\}_{n\in\omega}$ with $\mu(U_n) = 2^{-n}$ for every n, set $U = \bigcap_n U_n$. Let $\mathcal{D}_U = \{P \mid P \in \boldsymbol{Q} \wedge P \cap U = \emptyset\}$.

Lemma 5.2. *$\mathcal{D}_U$ is dense.*

Proof. Suppose that $\{U_n\}_{n\in\omega}$ is a Schnorr test with $\mu(U_n) = 2^{-n}$ for every n, $U = \bigcap_n U_n$ and $P \in \boldsymbol{Q}$. Then there is some big enough n so that $\mu(U_n) < \frac{\mu(P)}{2}$. Then the complement $P_0 = 2^\omega - U_n$ has measure greater or equal to $1 - \frac{\mu(P)}{2}$. So $P_0 \cap P$ has measure greater or equal to $\frac{\mu(P)}{2}$. We show that $\mu(P_0 \cap P)$ is a computable real. Both P and P_0 can be represented by computable trees T and T^0 respectively. Since both P and P_0 belong to $\boldsymbol{Q}$, for any i, we may computablely find some big enough s_i such that $\mu((\bigcup_{\sigma\in E_{s_i}}[\sigma]) - P) < 2^{-i-1}$ and $\mu((\bigcup_{\sigma\in E^0_{s_i}}[\sigma]) - P_0) < 2^{-i-1}$ where $E_{s_i} = \{\sigma \in 2^{s_i} \mid \sigma \in T\}$ and $E^0_{s_i} = \{\sigma \in 2^{s_i} \mid \sigma \in T^0\}$. Then

$$\mu((\bigcup_{\sigma\in E_{s_i}\cap E^0_{s_i}}[\sigma]) - (P \cap P_0)) = \mu(((\bigcup_{\sigma\in E_{s_i}\cap E^0_{s_i}}[\sigma]) - P) \cup ((\bigcup_{\sigma\in E_{s_i}\cap E^0_{s_i}}[\sigma]) - P_0)) \leq$$
$$\mu((\bigcup_{\sigma\in E_{s_i}\cap E^0_{s_i}}[\sigma]) - P) + \mu((\bigcup_{\sigma\in E_{s_i}\cap E^0_{s_i}}[\sigma]) - P_0) \leq 2^{-i-1} + 2^{-i-1} = 2^{-i}.$$

So
$$\mu(\bigcup_{\sigma\in E_{s_i}\cap E^0_{s_i}}[\sigma]) - 2^{-i} \leq \mu(P \cap P_0) \leq \mu(\bigcup_{\sigma\in E_{s_i}\cap E^0_{s_i}}[\sigma]).$$

Thus $\mu(P \cap P_0)$ is computable. In other words, $P \cap P_0 \in \boldsymbol{Q}$. □

Now we want to mimic the proof of Lemma 4.4. But there is a problem. In the proof of Lemma 4.4, we can make sure that, for any condition $P \in \boldsymbol{P}$, $\mu([\sigma] \cap P) > 0$ whenever $[\sigma] \cap P$ is not empty. The reason is that we can make sure that P only contains 1-random reals. But every condition $Q \in \boldsymbol{Q}$ contains a computable real. So we have to be more careful.

Lemma 5.3. *For ever computable tree T for which $\mu([T]) > 0$ is computable, there is a Schnorr test $\{V_n\}_{n\in\omega}$ so that for any σ, if $\mu([\sigma] \cap [T]) > 0$, then $\mu([\sigma] \cap [T] \cap V_n) > 0$ for each n.*

Proof. Suppose that T is a computable tree such that $\mu([T]) > 0$ is computable. Then there is a computable function $f : \omega \to \omega$ so that for every s, $\frac{|E_{f(s)}|}{2^{f(s)}} - \mu(T) < 2^{-s}$ where $E_t = \{\sigma \in 2^t \mid \sigma \in T\}$. Fix a computable enumeration $\{\sigma_i\}_{i\in\omega}$ of $2^{<\omega}$ and an enumeration of finite string $\{\sigma^{s+1}_i\}_{i\leq 2^{s+1}}$ of 2^{s+1} for each s. We define $U_0 = \bigcup_s U_0[s]$ as follows:

At step 0, do nothing.

At step $s + 1$. Select the lest index i such that
1. There is no $\tau \succeq \sigma_i$ belonging to $U_0[s]$;
2. $|[\sigma_i] \cap E_{f(s)}| > 2^{f(s)-s+1}$.

Then pick up any $2^{f(s)-s+1}$ many finite strings in $[\sigma_i] \cap E_{f(s)}$ and put them into $U_0[s + 1]$.

Then by the definition of f, $U_0[s + 1] \cap [\sigma_i] \cap [T] \neq \emptyset$. Obviously at any stage $s + 1$, $\mu(U_0[s + 1] - U_0[s]) < 2^{-s+2}$. So $\mu(U_0)$ is computable. Moreover, for any σ, if $\mu([\sigma] \cap [T]) > 0$, then

$\mu([\sigma] \cap [T] \cap U_0) > 0$. If not, pick up the least index i such that $\mu([\sigma_i] \cap [T]) > 0$ but $\mu([\sigma] \cap [T] \cap U_0) = 0$. Then there is a large enough stage s_0 so that for each $j < i$, if $\mu([\sigma_j] \cap [T]) > 0$, then $\mu([\sigma_j] \cap [T] \cap U_0[s_0]) > 0$. Suppose that $\mu([\sigma_i] \cap [T]) > 2^{-k}$, then at any stage $t > s_0 + k$, $|[\sigma_i] \cap E_{f(t)}| > 2^{f(t)-k} > 2^{f(t)-t+1}$. Then we pick up any $2^{f(t)-t+1}$ many finite strings in $[\sigma_i] \cap E_{f(t)}$ and put them into $U_0[t]$. Then $\mu([\sigma_i] \cap [T] \cap U_0[t]) > 2^{-t}$, a contradiction.

Generally, for each n, we define $U_n = \bigcup_s U_n[s]$ as follows:

At step 0, do nothing.

At step $s+1$. Select the lest index i such that

(1) There is no $\tau \succeq \sigma_i$ belonging to $U_0[s]$;
(2) $|[\sigma_i] \cap E_{f(s+n)}| > 2^{f(s+n)-s-n+1}$.

Then pick up any $2^{f(s+n)-s-n+1}$ many finite strings in $[\sigma_i] \cap E_{f(s+n)}$ and put them into $U_n[s+1]$.

By the same argument above, for every s, $\mu(U_n[s+1] - U_n[s]) < 2^{-s-n+2}$. So for any n, $\mu(U_n) < 2^{-n+3}$ is computable. Moreover, for any σ, if $\mu([\sigma] \cap [T]) > 0$, then $\mu([\sigma] \cap [T] \cap U_n) > 0$.

Now define $V_n = \bigcup_{m \geq n} U_m$. Then $\mu(V_n) < 2^{-n+4}$ for each n. Then by an easy calculation, $\{\mu(V_n)\}_{n\in\omega}$ is uniformly computable. Thus $\{V_n\}_{n\in\omega}$ is a schnorr test. By the property of $\{U_n\}_{n\in\omega}$, for any σ and n, if $\mu([\sigma] \cap [T]) > 0$, then $\mu([\sigma] \cap [T] \cap V_n) > 0$. □

For any $\mathbf{\Pi}^0_2$ set G, let $\mathcal{D}_G = \{P \mid P \in \boldsymbol{Q} \wedge P \cap G = \emptyset\}$.

Lemma 5.4. *If G is a $\mathbf{\Pi}^0_2$ only containing Schnorr random reals, then the set $\mathcal{D}_G = \{P \mid P \in \boldsymbol{P} \wedge P \cap G = \emptyset\}$ is dense in $\mathbb{Q}$.*

Proof. Suppose that G is $\mathbf{\Pi}^0_2$ only containing Schnorr random reals. Let $\{U_n\}_{n\in\omega}$ be a sequence open sets so that $G = \bigcap_n U_n$. Let $P \in \boldsymbol{Q}$. Then we claim that there is some σ so that $P \cap [\sigma] \cap G = \emptyset$ but $\mu(P \cap [\sigma]) > 0$.

Suppose not. By Lemma 5.3, there is a Schnorr test $\{V_n\}_{n\in\omega}$ so that for any σ, if $\mu([\sigma] \cap P) > 0$, then $\mu([\sigma] \cap P \cap V_n) > 0$ for each n. Then we build a sequence strings $\sigma_0 \prec \sigma_1 ...$ as follows.

Let $\sigma_0 = \emptyset$. Now suppose $\mu([\sigma_i] \cap P) > 0$. Let $\tau \succ \sigma_i$ so that $\mu([\tau] \cap P) > 0$ and $[\tau] \cap P \subseteq V_i$. By the property of $\{V_n\}_n$, there exists such a τ. Then by the assumption, let $\sigma_{i+1} \succ \tau$ so that $[\sigma_{i+1}] \cap P \cap G \neq \emptyset$. Since G only contains Schnorr random reals, $\mu([\sigma_{i+1}] \cap P \cap U_i) > 0$. Then we may assume that $[\sigma_{i+1}] \cap P \subseteq U_i$ and $\mu([\sigma_{i+1} \cap P]) > 0$.

Let $x = \bigcup_{i\in\omega} \sigma_i$. Then $x \in P \cap (\bigcap_{n\in\omega} U_n) \cap (\bigcap_{n\in\omega} V_n)$. Since $x \in \bigcap_{n\in\omega} V_n$, x is not Schnorr random which contradicts to that G only contains Schnorr random reals.

So there is some σ so that $P \cap [\sigma] \cap G = \emptyset$ but $\mu(P \cap [\sigma]) > 0$. Let $Q = P \cap [\sigma]$. Then $Q \in \boldsymbol{Q}$ and $Q \leq P$. □

Theorem 5.5 (Hitchcock, Lutz and Terwijn [5])**.** *The collection of Schnorr random reals is not $\mathbf{\Sigma}^0_3$.*

Proof. Suppose not. Then there is a countable sequence $\mathbf{\Pi}^0_2$ sets $\{G_n\}_n$ such that the set $\bigcup_n G_n$ contains exactly Schnorr random reals. Then by Lemma 5.2 and Lemma 5.4, for any sufficient generic real g over $\mathbb{Q}$, g is Schnorr random but $g \notin G_n$ for any n, a contradiction. □

We want to point out that the forcing $\mathbb{Q}$ does not produce a 1-random real. To see this, fix a universal Martin-Löf test $\{U_n\}_{n\in\omega}$. For each n, let $\mathcal{D}_n = \{P \in \mathbb{Q} \mid P \subseteq U_n\}$.

Corollary 5.6. *For each n, $\mathcal{D}_n$ is dense.*

Proof. Let $P \in \boldsymbol{Q}$ and $G = 2^\omega - U_n$. Then G is a Π^0_1 class only containing 1-random reals. Then by Lemma 5.4, there is some $Q \leq P$ such that $Q \in \mathcal{D}_n$. □

So if g is sufficient generic over $\mathbb{Q}$, then g is Schnorr random but not 1-random.

6. Δ^1_1-RANDOMNESS

In this section, we prove that the collection of Δ^1_1-random reals is not $\mathbf{\Sigma}^0_3$. Some basic facts in higher randomness theory can be found in [13], [6] and [2].

Definition 6.1. *Define a forcing notion $\mathbb{D} = (\boldsymbol{D}, \leq)$ as follows:*
(1) *$P \in \boldsymbol{D}$ if and only if P is a Δ^1_1, closed set of reals with positive measure;*
(2) *For $P, Q \in \boldsymbol{D}$, $P \leq Q$ if and only if $P \subseteq Q$.*

For any Δ^1_1-sequence of Δ^1_1-open sets $\{U_n\}_{n\in\omega}$ with $\lim_{n\to\infty} \mu(U_n) = 0$, set $U = \bigcap_n U_n$. Let $\mathcal{D}_U = \{P \mid P \in \boldsymbol{D} \wedge P \cap U = \emptyset\}$.

Lemma 6.2. *$\mathcal{D}_U$ is dense.*

Proof. Suppose that $\{U_n\}_{n\in\omega}$ is a Δ^1_1-sequence of Δ^1_1-open sets with $\lim_{n\to\infty} \mu(U_n) = 0$, $U = \bigcap_n U_n$ and $P \in \boldsymbol{D}$. Then there is some big enough n so that $\mu(U_n) < \frac{\mu(P)}{2}$. Then the complement $P_0 = 2^\omega - U_n$ has measure greater or equal to $1 - \frac{\mu(P)}{2}$. So $P_0 \cap P$ is a Δ^1_1, closed set and has measure greater or equal to $\frac{\mu(P)}{2}$. Thus $P \cap P_0 \in \boldsymbol{D}$. □

For any $\mathbf{\Pi}^0_2$ set G, let $\mathcal{D}_G = \{P \mid P \in \boldsymbol{D} \wedge P \cap G = \emptyset\}$.

Lemma 6.3. *If G is a $\mathbf{\Pi}^0_2$ only containing Δ^1_1-random reals, then the set $\mathcal{D}_G = \{P \mid P \in \boldsymbol{D} \wedge P \cap G = \emptyset\}$ is dense in $\mathbb{D}$.*

Proof. Suppose that G is $\mathbf{\Pi}^0_2$ only containing Δ^1_1-random reals. Let $\{U_n\}_{n\in\omega}$ be a sequence open sets so that $G = \bigcap_n U_n$. Let $P \in \boldsymbol{D}$. Then there is a hyperaithmetic real x so that P is $\Pi^0_1(x)$. Without loss of generality, we may assume that for any σ, if $[\sigma] \cap P \neq \emptyset$, then $\mu([\sigma] \cap P) > 0$ (since we may assume that P only contains 1-x-random reals). Then we claim that there is some σ so that $P \cap [\sigma] \cap G = \emptyset$ but $P \cap [\sigma] \neq \emptyset$.

Suppose not. By Lemma 4.3 relativizing to x, there is a generalized x-Martin-Löf test $\{V_n\}_{n\in\omega}$ so that for any σ, if $[\sigma] \cap P$ is not empty, then $[\sigma] \cap P \cap (\bigcap_n V_n)$ is not empty. Then we build a sequence strings $\sigma_0 \prec \sigma_1 ...$ as follows.

Let $\sigma_0 = \emptyset$. Now suppose $[\sigma_i] \cap P \neq \emptyset$. Let $\tau \succ \sigma_i$ so that $[\tau] \cap P \neq \emptyset$ and $[\tau] \cap P \subseteq V_i$. By the property of $\{V_n\}_n$, there exists such a τ. Then by the assumption, let $\sigma_{i+1} \succ \tau$ so that $[\sigma_{i+1}] \cap P \subseteq U_i$.

Let $z = \bigcup_{i\in\omega} \sigma_i$. Then $z \in P \cap (\bigcap_{n\in\omega} U_n) \cap (\bigcap_{n\in\omega} V_n)$. Since $z \in \bigcap_{n\in\omega} V_n$, z is not weakly 2-x-random. But x is hyperarithmetic, z is not Δ^1_1-random, which contradicts to that G only contains Δ^1_1-random reals.

So there is some σ so that $P \cap [\sigma] \cap G = \emptyset$ but $P \cap [\sigma] \neq \emptyset$. Let $Q = P \cap [\sigma]$. Then $Q \in \boldsymbol{D}$ and $Q \leq P$. □

So by the same proof as in the previous sections, we have the following result.

Proposition 6.4. *The collection of Δ^1_1-random reals is not a $\mathbf{\Sigma}^0_3$-set.*

We give an application Proposition 6.4.

It was very difficult to separate Π^1_1-Martin-Löf randomness from Δ^1_1-random. The proof in [2] is pretty involved and only contains a sketch. Now we may apply the previous results to give a simpler proof (and even a stronger result).

An immediate conclusion of Proposition 6.4 is:

Corollary 6.5 (Chong, Nies and Yu [2])**.** *There is a Δ^1_1-random real z which is not Π^1_1-Martin-Löf random.*

By analyzing the proof of Proposition 6.4, we can have a total characterization whether they are different.

Theorem 6.6. *For each $x \geq_h \mathscr{O}$, there is a Δ^1_1-random real $z \equiv_h x$ which is not Π^1_1-Martin-Löf random.*

Proof. The the collection of Π^1_1-Martin-Löf random is a $\Sigma^0_2(\mathscr{O})$-set. Moreover, there is a $\mathscr{O}$-computably enumeration of the conditions in $\boldsymbol{D}$ (see Sacks [13]). Then hyperarithmetically in $\mathscr{O}$, by a finite extension argument, it is not difficult to construct a $\Delta^1_1(\mathscr{O})$-perfect tree T so that every infinite path in T is a Δ^1_1-random but not Π^1_1-Martin-Löf random. By Theorem 2.2, every real $x \in [T]$ is hyperaithmetically above $\mathscr{O}$. So for each $x \geq_h \mathscr{O}$, there is a Δ^1_1-random real $z \equiv_h x$ which is not Π^1_1-Martin-Löf random. □

We want to point out an observation there. In [13], Sacks does not use a forcing argument to study measure theoretic uniformity. In stead of that, he uses a model $\mathscr{M}(\omega_1^{\mathrm{CK}}, x)$. The advantage of his method is to show that $\mathscr{M}(\omega_1^{\mathrm{CK}}, x)$ satisfies $\Delta^1_1 - CA$ (and so $\omega_1^x = \omega_1^{\mathrm{CK}}$) for almost all reals x. Now the reason that a forcing argument is avoided seems clear since the forcing notion with Δ^1_1-sets with positive measures does not produce a generic real x with $\omega_1^x = \omega_1^{\mathrm{CK}}$.

7. Some remarks

We don't know what's the exact complexity of the collection of Π^1_1-random reals. We conjecture that it cannot be a $\boldsymbol{\Sigma}^0_{<\omega_1^{\mathrm{CK}}} = \bigcup_{\alpha<\omega_1^{\mathrm{CK}}} \boldsymbol{\Sigma}^0_\alpha$.

For any cardinal κ and number n, we use $\kappa - \boldsymbol{\Sigma}^0_{n+1}$ to denote the class of the sets which can be a union of less than κ-many $\boldsymbol{\Pi}^0_n$-sets. For example, $\aleph_1 - \boldsymbol{\Sigma}^0_{n+1}$-class is exactly same as $\boldsymbol{\Sigma}^0_{n+1}$-class. We also can define $\kappa - \boldsymbol{\Pi}^0_{n+1}$-class in the similar way. Then the following is true.

Theorem 7.1. *Assuming ZFC+Martin's axiom, then*

(1) *The collection of Kurtz random reals is not $2^{\aleph_0} - \boldsymbol{\Sigma}^0_2$;*
(2) *The collection of Schnorr random reals is not $2^{\aleph_0} - \boldsymbol{\Sigma}^0_3$;*
(3) *The collection of 1-random reals is not $2^{\aleph_0} - \boldsymbol{\Pi}^0_2$;*
(4) *The collection of weakly 2-random reals is not $2^{\aleph_0} - \boldsymbol{\Sigma}^0_3$;*
(5) *The collection of Δ^1_1-random reals is not $2^{\aleph_0} - \boldsymbol{\Sigma}^0_3$.*

Proof. All the negative results in the previous sections were proved by c.c.c. forcings except (1) and (3). But it is a theorem under ZFC+Martin's axiom that any set which is a union of less than $2^{\aleph_0}$ many meager sets is meager (see [7]). So under ZFC+Martin's axiom, (1)-(5) all are true. □

We don't know whether the conclusions of Theorem 7.1 can be proved under ZFC. We don't either know whether the following question is known.

Question 7.2. *Is it consistent with $ZFC + \neg CH$ that every $\boldsymbol{\Pi}^1_1$-set is a union of $\aleph_1$-many closed sets?*

References

[1] George Barmpalias, Joseph Miller, and Andre Nies. Randomness notions and partial relativization. 2010.
[2] Chi Tat Chong, André Nies, and Liang Yu. Lowness of higher randomness notions. *Israel journal of mathematics*, 66(1):39–60, 2008.
[3] R. Downey and D. Hirschfeldt. *Algorithmic randomness and complexity.* Springer-Verlag, Berlin. To appear.
[4] Rod Downey, Andre Nies, Rebecca Weber, and Liang Yu. Lowness and Π^0_2 nullsets. *J. Symbolic Logic*, 71(3):1044–1052, 2006.
[5] John M. Hitchcock, Jack H. Lutz, and Sebastiaan A. Terwijn. The arithmetical complexity of dimension and randomness. *ACM Trans. Comput. Log.*, 8(2):Art. 13, 22, 2007.

[6] G Hjorth and A Nies. Randomness in effective descriptive set theory. *J. London. Math. Soc.*, 75(2):495–508, 2007.
[7] Thomas Jech. *Set theory.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.
[8] Joseph S. Miller. Every 2-random real is Kolmogorov random. *J. Symbolic Logic*, 69(3):907–913, 2004.
[9] Joseph S. Miller. The K-degrees, low for K-degrees, and weakly low for K sets. *Notre Dame J. Form. Log.*, 50(4):381–391 (2010), 2009.
[10] Joseph S. Miller and Liang Yu. On initial segment complexity and degrees of randomness. *Trans. Amer. Math. Soc.*, 360(6):3193–3210, 2008.
[11] André Nies. *Computability and randomness*, volume 51 of *Oxford Logic Guides*. Oxford University Press, Oxford, 2009.
[12] André Nies, Frank Stephan, and Sebastiaan A. Terwijn. Randomness, relativization and Turing degrees. *J. Symbolic Logic*, 70(2):515–535, 2005.
[13] Gerald E. Sacks. *Higher recursion theory.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1990.

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