COFINAL MAXIMAL CHAINS IN THE TURING DEGREES

WEI WANG, LIUZHEN WU, AND LIANG YU

Abstract. Assuming ZFC, we prove that CH holds if and only if there exists a cofinal maximal chain of order type ω_1 in the Turing degrees. However it is consistent that $ZF+\neg CH+$"there exists a cofinal chain in the Turing degrees of order type ω_1".

1. Introduction

A *chain* in the Turing degrees is a set of degrees in which every two elements are Turing comparable. A chain is *maximal* if it cannot be properly extended, and *cofinal* if every degree is below one in the chain. The study of chains in the Turing degrees can be traced back to Sacks [8] and is related to the global theory of the Turing degrees. Obviously, assuming ZFC, the existence of a cofinal chain is equivalent to CH. An interesting question is how nice a maximal chain can be. In particular, does there exist a maximal cofinal chain of order type ω_1? Such a chain would provide a nice ranking for the Turing degrees. However, the existence of such a chain is not a simple question. Abraham and Shore [1] even constructed a maximal chain of order type ω_1 which is an initial segment of the Turing degrees. So the behavior of chains may be very abnormal. Hence it seems that there is no obvious way to construct such a chain. In this paper, we construct, by a very nonuniform method, such a chain. After we have done this, Shore pointed out that some technical results (e.g. Lemma 3.3) can be improved by some known results (see the remarks at the end). However, the proof of these results are difficult and some are unpublished. We give a self-contained and easy to reach proof here.

Since the construction of such a chain heavily depends on CH, we want to know whether there exists a more effective way. In other words, does the existence of such a chain imply CH? Until now, we don't know the answer. But we may prove that the existence of a cofinal chain of order type ω_1 does not imply CH.

We organize the paper as follows. In section 2, we briefly review some definitions and notation; In section 3, we prove the existence of a cofinal maximal chain of order type ω_1 under $ZFC+CH$; In section 4, we present two proofs that there is a model $\mathcal{N}$ of ZF in which there exists a cofinal chain of order type ω_1 but there is no well ordering of reals. Certainly the two proofs are forcing argument, and the second one is a little more general. However, we include the first proof, because it involves more recursion theory (of Gödel's L), thus may be more interesting for recursion theorists. In the last section, we give some remarks.

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2. Preliminary

We follow conventions in [7], [5], [6], and [9]. A *tree* T is a total function mapping $2^{<\omega}$ to $2^{<\omega}$ such that $T(\tau) \subset T(\sigma)$ if and only if $\tau \subset \sigma \in 2^{<\omega}$. Given trees T and T', T' is a *subtree* of T if and only if the range of T' is a subset of that of T. $[T]$ is the collection of infinite paths of a given tree T, i.e.

$$[T] = \{x \in 2^\omega : \exists y \in 2^\omega \forall n \in \omega (T(y \upharpoonright n) \subset x)\}.$$

$T' = Ext(T, \tau)$ is the subtree of T such that $T'(\sigma) = T(\tau\hat{\ }\sigma)$ for each σ.

We fix a recursive coding of $2^{<\omega}$ and identify finite strings with their codes. We say $\sigma < \tau$ if and only if $\ulcorner\sigma\urcorner < \ulcorner\tau\urcorner$ where $\ulcorner\xi\urcorner$ is the code of ξ, and we write $\langle 0^\sigma \rangle$ for $\langle 0^{\ulcorner\sigma\urcorner} \rangle$.

For some fix e, if for any τ and n there is a $\pi \supset \tau$ such that $\Phi_e(T(\pi); n) \downarrow$, then we define $T' = Tot(T, e)$ to be the subtree of T such that

(1) $T'(\emptyset) = T(\emptyset)$,
(2) if $T'(\tau) = T(\pi)$ then for $i < 2$ let ρ_i be the least extension of $\pi\hat{\ }\langle i \rangle$ with $\Phi_e(T(\rho_i)) \supset \Phi_e(T(\pi))$, define $T'(\tau\hat{\ }\langle i \rangle) = T(\rho_i)$.

For σ and ρ extending τ, we say that *σ and ρ e-split τ* if and only if $\Phi_e(\sigma)$ and $\Phi_e(\rho)$ are incompatible. Given τ and T, *there is an e-splitting of τ on T* if and only if there are extensions σ and ρ of τ such that $T(\sigma)$ and $T(\rho)$ e-split $T(\tau)$.

If there are e-splittings of all finite strings on T then we define $T' = Sp(T, e)$ to be the subtree of T such that

(1) $T'(\emptyset) = T(\emptyset)$,
(2) given $T'(\tau) = T(\pi)$, let ρ_0 and ρ_1 be the least pair such that $\pi \subseteq \rho_0 \cap \rho_1$, $\rho_0 < \rho_1$ and $T(\rho_0)$ and $T(\rho_1)$ e-split $T(\pi)$, define $T'(\tau\hat{\ }\langle i \rangle) = T(\rho_i)$ for $i < 2$.

Given a real x, ω_1^x is the least ordinal greater than ω such that

$$L_\alpha[x] \models \Sigma_1\text{-Replacement}.$$

$\mathcal{O}^x$ is Kleene's $\mathcal{O}$ relativized to x. For any $\alpha < \omega_1^x$, $x^{(\alpha)}$, the α-Turing jump, is well defined. Moreover, if $y \geq_T x$ then $y^{(\alpha)} \geq_T x^{(\alpha)}$. The following theorem, *Gandy's Basis Theorem*, will be used later.

Theorem 2.1 (Gandy, see [9]). *If A is a nonempty Σ^1_1 set of reals, then there must be some real $x \in A$ so that $x \leq_{\mathrm{T}} \mathcal{O}$.*

The domain of a binary relation R is the set

$$\operatorname{dom} R = \{x | \exists y((x, y) \in R \vee (y, x) \in R\}.$$

If $x \in \operatorname{dom} R$ then let

$$R \upharpoonright x = \{(y, z) \in R | (z, x) \in R\}.$$

If R well orders $\operatorname{dom} R$, then let $\operatorname{otp} R$ denote the order type of R.

3. The existence of a cofinal maximal chain of order type ω_1

In this section, we construct a cofinal maximal chain of order type ω_1. The construction is very nonuniform. The rough idea is that we first do a "jump" to compute a fixed real and then fill the "gap" between this jump and the degrees which we already constructed.

Chong and Yu [2] proved the following result.

Lemma 3.1 (ZF)**.** *For countable $A \subset 2^\omega$ and $x \in 2^\omega$ there is a minimal cover z of A such that $x \leq_{\mathrm{T}} z''$.*

Recall that z is *a minimal cover* of A, if and only if $x \leq_T z$ for every $x \in A$ and no $y <_T z$ can compute all $x \in A$.

With this and assuming CH one is able to construct a maximal chain in the Turing degrees such that every degree is bounded by a double jump of something in the chain. Actually we can improve this by eliminating double jumps. To this end we need the following technique lemma which should have been known for quite a long time (For example, see [7]). We include a proof here for the completeness.

Lemma 3.2 (Forklore)**.** *(ZF) There are reals x_0 and x_1 of minimal degrees such that $\emptyset'' \equiv_{\mathrm{T}} x_0 \vee x_1 \equiv_{\mathrm{T}} x_0'' \equiv_{\mathrm{T}} x_1''$.*

Proof. We will build stage by stage finite approximations $\sigma_0[s]$ and $\sigma_1[s] \in 2^{<\omega}$ to x_0 and x_1. As in typical minimal degrees constructions we will also build recursive trees $T_{0,s}$ and $T_{1,s}$ so that $\sigma_i[s] = T_{i,s}(\emptyset)$ for $i < 2$.

At the beginning let $\sigma_0[0] = \sigma_1[0] = \emptyset$ and $T_{0,0} = T_{1,0} = 2^\omega$. Suppose that $s = 5e$, $\sigma_0[s]$, $\sigma_1[s]$, $T_{0,s}$ and $T_{1,s}$ are defined.

At stage $s+1$, we make either $\Phi_e(x_0)$ is recursive or $\Phi_e(x_0)$ computes x_0.

Case 1. If for any $\tau \in 2^{<\omega}$ there is an e-splitting of τ on $T_{0,s}$, then let $T_{0,s+1} = Sp(T_{0,s}, e)$ and $T_{1,s+1} = Ext(T_{1,s}, \langle 0 \rangle)$.

Case 2. Otherwise fix any τ of which there is no e-splitting on $T_{0,s}$, let $T_{0,s+1} = Ext(T_{0,s}, \tau)$ and $T_{1,s+1} = Ext(T_{1,s}, \langle 10^\tau 1 \rangle)$.

In any case let $\sigma_0[s+1] = T_{0,s+1}(\emptyset)$ and $\sigma_1[s+1] = T_{1,s+1}(\emptyset)$.

At stage $s+2$, we make $\emptyset''$ know whether $\Phi_e(x_0)$ is total.

Case 1. If for all τ and n there is some $\pi \supset \tau$ with $\Phi_e(T_{0,s+1}(\pi); n) \downarrow$, then let $T_{0,s+2} = Tot(T_{0,s+1}, e)$ and $T_{1,s+2} = Ext(T_{1,s+1}, \langle 0 \rangle)$.

Case 2. Otherwise, fix $\tau \in 2^{<\omega}$ and $n \in \omega$ be the least pair such that $\Phi_e(T_{0,s}(\rho); n) \uparrow$ for all $\rho \supset \tau$, let $T_{0,s+2} = Ext(T_{0,s+1}, \tau)$ and $T_{1,s+2} = Ext(T_{1,s+1}, \langle 10^\tau 1 \rangle)$.

In any case let $\sigma_0[s+2] = T_{0,s+2}(\emptyset)$ and $\sigma_1[s+2] = T_{1,s+2}(\emptyset)$.

At stage $s+3$ and $s+4$, we repeat what we did at $s+1$ and $s+2$ respectively with the roles of 0 and 1 swaped.

At stage $s+5$, let $T_{0,s+5} = Ext(T_{0,s+4}, \langle i \rangle)$ and $T_{1,s+5} = T_{1,s+4}$ where $i = \emptyset''(e)$. In addition, let $\sigma_0[s+5] = T_{0,s+5}(\emptyset)$ and $\sigma_1[s+5] = T_{1,s+5}(\emptyset)$.

Finally let $x_0 = \bigcup_s \sigma_0[s]$ and $x_1 = \bigcup_s \sigma_1[s]$.

This completes the construction of x_0 and x_1.

It is easy to see that for $i < 2$, $\sigma_i[s] \subseteq \sigma_i[s+1]$ and $\sigma_i[s] \subset \sigma_i[s+1]$ when $s \equiv 3 - 2i \mod 5$, hence $x_i \in 2^\omega$ is well defined. It is also trivial that $x_0, x_1 \leq_{\mathrm{T}} \emptyset''$ and both are of minimal degrees.

Claim 1. *$x_0'' \leq_{\mathrm{T}} \emptyset''$ and $x_1'' \leq_{\mathrm{T}} \emptyset''$.*

To see $x_0'' \leq_{\mathrm{T}} \emptyset''$, note that the construction is recursive in $\emptyset''$, and at stage $5e+2$, $\Phi_e(x_0)$ is total if and only if Case 1 applies. Hence $x_0'' \leq_{\mathrm{T}} \emptyset''$. Similar argument shows that $x_1'' \leq_{\mathrm{T}} \emptyset''$.

Claim 2. $\emptyset'' \leq_{\mathrm{T}} x_0 \vee x_1$.

Let $s = 5e$ and $T_{0,s}$ and $T_{1,s}$ be given. x_1 can decide whether $T_{1,s}(\langle 0\rangle) \subseteq \sigma_{1,s+1}$ or $T_{1,s}(\langle 1\rangle) \subseteq \sigma_{1,s+1}$ and hence can compute the indices of $T_{0,s+1}$ and $T_{1,s+1}$. Similar computations at $s+2$, $s+3$ and $s+4$ using x_0 and x_1 produce the indices of $T_{0,s+4}$ and $T_{1,s+4}$ from $T_{0,s+1}$ and $T_{1,s+1}$. Finally $e \in \emptyset''$ if and only if $T_{0,s+5}(\langle 1\rangle) \subset x_0$, and $T_{0,s+5}$ and $T_{1,s+5}$ can be computed from $T_{0,s+4}$ and $T_{1,s+4}$ using x_0.

By induction $\emptyset'' \leq_{\mathrm{T}} x_0 \vee x_1$.

So the lemma is proved. □

Lemma 3.3. *(ZF) There is a maximal chain* $\mathbf{0} = \mathbf{c}_0 < \mathbf{c}_1 < \ldots < \mathbf{c}_\omega = \mathbf{0}''$ *in* $[\mathbf{0}, \mathbf{0}'']$.

Proof. Fix an enumeration $(\mathbf{d}_n : n \in \omega)$ of degrees below $\mathbf{0}''$. Let $\mathbf{c}_0 = \mathbf{0}$.

Suppose that $\mathbf{c}_n$ is defined with $\mathbf{c}_n'' = \mathbf{0}''$. Relativize Lemma 3.2 to get degrees $\mathbf{x}_0$ and $\mathbf{x}_1$ such that $\mathbf{x}_0 \vee \mathbf{x}_1 = \mathbf{c}'' = \mathbf{0}'' = \mathbf{x}_0'' = \mathbf{x}_1''$. If $\mathbf{d}_n < \mathbf{0}''$ then there must be some $i < 2$ with $\mathbf{x}_i \not\leq \mathbf{d}_n$, let $\mathbf{c}_{n+1} = \mathbf{x}_i$. If $\mathbf{d}_n = \mathbf{0}''$ then let $\mathbf{c}_{n+1} = \mathbf{x}_0$.

Finally let $\mathbf{c}_\omega = \mathbf{0}''$. □

One might ask whether the maximal chain in the lemma above could be cofinal in $[\mathbf{0}, \mathbf{0}'')$. The answer is negative. For assume that an r.e. degree $\mathbf{r} > \mathbf{0}$ were bounded by $\mathbf{c}_n$ but not by $\mathbf{c}_{n-1}$, then $n-1 > 0$ and $\mathbf{c}_n$ would have been r.e. in $\mathbf{c}_{n-1}$.

Theorem 3.4. *Assuming ZFC, CH holds if and only if there is a cofinal maximal chain of order type* ω_1 *in the Turing degrees.*

Proof. $(\Rightarrow)$ To show this direction, AC is not needed. Assuming CH, let $(\mathbf{d}_\alpha : \alpha \in \omega_1)$ be an enumeration of the Turing degrees. We will inductively construct a chain C_α for each $\alpha \in \omega_1$ such that

(1) for each $\beta < \alpha$ there is some $\mathbf{c} \in C_\alpha$ such that $\mathbf{c} \geq \mathbf{d}_\beta$,
(2) C_α is of order type $\alpha \times \omega + 1$ if α is a successor or 0, and
(3) C_α is of order type $\alpha \times \omega$ if α is a limit ordinal.

Moreover we will make $C_\beta \subset C_\alpha$ whenever $\beta < \alpha < \omega_1$.

At the beginning let $C_0 = \{\mathbf{0}\}$.

Suppose that $\alpha = \beta + 1$ and C_β is defined. Given $\mathbf{d}_\beta$, we apply Lemma 3.1 [1] to get a minimal cover $\mathbf{a}$ of C_β with $\mathbf{a}'' \geq \mathbf{d}_\beta$. Then we relativize Lemma 3.3 to get a maximal chain $\mathbf{a} = \mathbf{a}_0 < \mathbf{a}_1 < \ldots < \mathbf{a}_\omega = \mathbf{a}''$ in $[\mathbf{a}, \mathbf{a}'']$. Let $C_\alpha = C_\beta \cup \{\mathbf{a}_\gamma : \gamma \leq \omega\}$.

For α a limit, if C_γ's are defined for $\gamma < \alpha$, let $C_\alpha = \bigcup_{\gamma<\alpha} C_\gamma$.

Finally let $C = \bigcup_{\alpha<\omega_1} C_\alpha$. It is easy to see that the order type of C is ω_1, C is a maximal chain and for each $\alpha < \omega_1$ the $(\alpha \times \omega + \omega + 1)$-th element of C bounds $\mathbf{d}_\alpha$.

$(\Leftarrow)$ To show this direction, we need $\aleph_1 - AC$. Since every degree bounds at most countably many degrees, CH follows immediately if there is a maximal chain as described. □

Note that it is not difficult to see, by a usual fusion argument, that assuming $ZF + CH$, there are $2^{\aleph_1}$ many such chains.

[1]As promised in the introduction that our proof is self-contained, we explain how to eliminate the use of Lemma 3.1. By a slight modification of the proof, one can prove Theorem 3.4 only using the result that the double jump of minimal degrees can be arbitrarily high (and this is pretty easy to show, see [7]). The reason that we apply Lemma 3.1 here is just to make the proof uniform.

4. Cofinal chains v.s. CH

In this section, we construct a model $\mathcal{N}$ such that

$\mathcal{N} \models ZF +$ there is a cofinal chain in the Turing degrees of order type $\omega_1 + \neg CH$.

The general idea is to start with a ground model of $\mathcal{M}$ satisfying $ZFC + CH + \omega_1 = \omega_1^L$ and construct a HOD model $\mathcal{N}$ in a generic extension so that $\mathcal{N}$ models $ZF+$"there is no a well ordering of reals".

The following result guarantees the failure of CH in $\mathcal{N}$.

Proposition 4.1. *For any model $\mathcal{M} \models ZFC + CH$. If $\mathcal{N}$ is an extension of $\mathcal{M}$ preserving $(\omega_1)^{\mathcal{M}}$ so that $\mathcal{N} \models ZF+$"there is a cofinal chain of Turing degrees", then $\mathcal{N} \models \neg CH$ if and only if $\mathcal{N} \models$ "there is no well ordering of reals".*

Proof. Suppose that $\mathcal{M}$ satisfies the assumption.

If $\mathcal{N} \models$ "there is a well ordering of reals", then, since $\mathcal{N} \models ZF+$"there is a cofinal chain of Turing degrees", we have that $\mathcal{N} \models CH$.

Now suppose that $\mathcal{N} \models$ "there is a no well ordering of reals". Take $C \in \mathcal{M}$ to be a subset of $(2^\omega)^{\mathcal{M}}$ so that $<_{\mathrm{T}}$ is a well ordering of C of which the order type is $(\omega_1)^{\mathcal{M}}$. Since $\mathcal{N}$ is an extension of $\mathcal{M}$ preserving $(\omega_1)^{\mathcal{M}}$, we have that $\mathcal{N} \models |\omega| < |C|$. Since there is no well ordering of 2^ω in $\mathcal{N}$, we have that $\mathcal{N} \models |C| < |2^\omega|$. So $\mathcal{N} \models \neg CH$. $\square$

We include two proofs of the existence of models $\mathcal{N}$. The first one is a forcing argument over L and involves some higher recursion theory. So, we guess that it is more interesting for recursion theorists. However, the second proof is a forcing argument over any model of $ZFC + CH + \omega_1 = \omega_1^L$, which is a little more general.

4.1. Forcing over L.

The ground model is L. Let $\mathbb{P}$ be Cohen forcing. In other words, $\mathbb{P} = (2^{<\omega}, \leq)$ where $\sigma \leq \tau$ if and only if $\sigma \succeq \tau$. $\mathbb{P}_{\aleph_1}$ is the finite support product of Cohen forcings of length $\aleph_1$. In other words, $\mathbb{P}_{\aleph_1} = (\mathbf{P}_{\aleph_1}, \leq)$ where $p \in \mathbf{P}_{\aleph_1}$ if and only if $\operatorname{dom} p \subset \omega_1$ is finite and $p(\alpha) \in 2^{<\omega}$ for each $\alpha \in \operatorname{dom} p$. Moreover $p \leq q$ if and only if for every $\alpha \in \operatorname{dom} q$, $\alpha \in \operatorname{dom} p$ and $p(\alpha) \succeq q(\alpha)$.

For every $\alpha < \omega_1$, let $\mathbb{P}_{<\alpha} = (\mathbf{P}_{<\alpha}, \leq)$ where $p \in \mathbf{P}_{<\alpha}$ if and only if $p \in \mathbf{P}_{\aleph_1}$ and $\operatorname{dom} p \subseteq \alpha$. The partial order in $\mathbb{P}_{<\alpha}$ is the restriction of that in $\mathbb{P}_{\aleph_1}$ to $\mathbf{P}_{<\alpha}$. So $\mathbb{P}_{\aleph_1} = \mathbb{P}_{<\aleph_1}$.

Let

$$E = \{\alpha < \omega_1 \mid \alpha \text{ is limit}\}.$$

For every $\alpha < \omega_1$, $\mathbb{P}_{<\alpha} \in L_{\omega_1^\alpha}$ where ω_1^α is the least countable ordinal $\gamma > \alpha$ so that $L_\gamma \models \Sigma_1$-Replacement.

Lemma 4.2. *There exist an uncountable set $E_1 \subseteq E$ in L and binary relations $\{R_\alpha\}_{\alpha \in E_1} \in L$ so that for each $\alpha \in E_1$*

- *$R_\alpha \in L$ is a well ordering of order type α binary relation over $\{2^n \mid n \in \omega\}$; and*
- *For each $\beta \leq \alpha$, there is a function $f \leq_{\mathrm{T}} R_\alpha$ which is an isomorphism from R_β to an initial segment of R_α.*

Proof. We build E_1 and $\{R_\alpha\}_{\alpha \in E_1}$ by induction. Let $E_{1,0} = \emptyset$.

At stage $\gamma < \omega_1$, suppose that we already have $E_{1,\gamma}$ and $\{R_\alpha\}_{\alpha \in E_{1,\gamma}}$. Let $R \in L$ be a well ordering over $\{2^{2n} | n \in \omega\}$ such that the order type of R is a limit ordinal greater than that of R_α for every $\alpha \in E_{1,\gamma}$. So for every $\alpha \in E_{1,\gamma}$, there is a unique function f_α which maps R_α isomorphically to an initial segment of R. Let $x \in L$ be a real that computes all the f_α's. Then let $E_{1,\gamma+1} = E_{1,\gamma} \cup \{\alpha_1\}$ where $\alpha_1 = \beta + \omega + \omega$. Let R^1 be a binary relation over $\{2^{2n+1} | n \in \omega\}$ so that for any $n \neq m \in \omega$, $(2^{2n+1}, 2^{2m+1}) \in R^1$ if and only if either $n \in x$ and $m \notin x$, or $n \in x \leftrightarrow m \in x$ and $n < m$. So $R^1 \geq_T x$ in L and is a well ordering of order type $\omega + \omega$ over $\{2^{2n+1} | n \in \omega\}$. Let

$$R_{\alpha_1} = R \cup R^1 \cup \{(2^{2n}, 2^{2m+1}) | n, m \in \omega\}.$$

This finished the construction at stage γ.

Let $E_1 = \bigcup_{\gamma < \omega_1} E_{1,\gamma}$.
Then E_1 and $\{R_\alpha\}_{\alpha \in E_1}$ are as required. $\square$

From now on, we fix such a set E_1 and sequence $\{R_\alpha\}_{\alpha \in E_1}$ in L.

Let G be a $\mathbb{P}_{\aleph_1}$-generic set over L. Then G can be viewed as an ω_1-sequence $\{G_\alpha\}_{\alpha < \omega_1}$ so that for every $\alpha < \omega_1$, $G_\alpha = \bigcup_{p \in G} p(\alpha)$ is a real. Let $G_{<\alpha} = \{(\beta, G_\beta) \mid \beta < \alpha\}$.

Definition 4.3. For any $\alpha \leq \gamma \in E_1$ and a sequence real $\{z_\beta\}_{\beta < \alpha}$, define *the join of* $\{z_\beta\}_{\beta < \alpha}$ *along* R_γ by induction on α, to be the set

$$\bigoplus_{R_\gamma} \{z_\beta\}_{\beta < \alpha} = R_\gamma \oplus \{2^n \cdot 3^j \mid \exists \beta < \alpha (j \in z_\beta \wedge \beta = \mathrm{otp}(R_\gamma \restriction n)\}.$$

The follow lemma says that the join operator is order preserved.

Lemma 4.4. *For any* $\beta_1 \leq \beta_2 \leq \alpha_1 \leq \alpha_2 \in E_1$,

$$\bigoplus_{R_{\beta_2}} \{G_\gamma\}_{\gamma < \beta_1} \leq_{\mathrm{T}} \bigoplus_{R_{\alpha_2}} \{G_\gamma\}_{\gamma < \alpha_1}.$$

Proof. Suppose that $\beta_1 \leq \beta_2 \leq \alpha_1 \leq \alpha_2 \in E_1$. Obviously $R_{\beta_2} \leq_{\mathrm{T}} R_{\alpha_2}$.

Fix an R_{α_2}-recursive function f which is an isomorphism from R_{β_2} to an initial segment of R_{α_2}. Then for any n, j,

$$2^n \cdot 3^j \in \bigoplus_{R_{\beta_2}} \{G_\gamma\}_{\gamma < \beta_1} \text{ if and only if } 2^{f(n)} \cdot 3^j \in \bigoplus_{R_{\alpha_2}} \{G_\gamma\}_{\gamma < \alpha_1}.$$

So $\bigoplus_{R_{\beta_2}} \{G_\gamma\}_{\gamma < \beta_1} \leq_{\mathrm{T}} \bigoplus_{R_{\alpha_2}} \{G_\gamma\}_{\gamma < \alpha_1}$. $\square$

The following lemma helps us to build a cofinal chain.

Lemma 4.5. *For any real x and ordinal $\alpha < E_1$, if $x \in L[G_{<\alpha}]$ then there is some countable $\beta \geq \alpha$ in E_1 so that* $x \leq_{\mathrm{T}} (\bigoplus_{R_\beta} \{G_\gamma\}_{\gamma < \beta})^{(\beta)}$.

Proof. Suppose that $x \in L[G_{<\alpha}]$. Note that for any $\alpha \leq \beta < \omega_1$, $L[G_{<\alpha}] = L[\bigoplus_{R_\alpha} \{G_\gamma\}_{\gamma < \alpha}]$ since $R_\alpha \in L$. So there must be some $\beta_1 \geq \alpha$ such that $x \in L_{\beta_1}[\bigoplus_{R_\alpha} \{G_\gamma\}_{\gamma < \alpha}]$.

Fix a countable ordinal β_1 so that $x \in L_{\beta_1}[\bigoplus_{R_\alpha} \{G_\gamma\}_{\gamma < \alpha}]$. For $i < \omega$, let n_i be such that $\mathrm{otp}(R_{\beta_1} \restriction 2^{n_i}) = i$. For $S \subseteq \omega \times \omega$, tentatively we say that S is *good*, if and only if

- $\forall m, n \in \omega ((2^m, 2^n) \in R_{\beta_1} \leftrightarrow (2^m, 2^n) \in S)$, and
- $\forall n, i \in \omega ((i, 2^n) \in S \rightarrow \exists m \in \omega (i = 2^m))$.

If S is good and in addition there exists n_S such that

$$\{k|(k,n_S)\in S\}=\{2^{n_i}|i\in\bigoplus_{R_\alpha}\{G_\gamma\}_{\gamma<\alpha}\},$$

then S is *very good.*

Let $\mathscr{X}$ be the following set of pairs

$$\{(\omega,S)\mid S\subseteq\omega\times\omega \text{ is very good } \wedge(\omega,S)\models(KP+V=L[n_S])\}.$$

Then for any $(\omega,S)\in\mathscr{X}$, it codes an end extension of $L_{\omega_1^{\beta_1}}[\bigoplus_{R_\alpha}\{G_\gamma\}_{\gamma<\alpha}]$. So $x\leq_{\mathrm{T}} S$. Moreover, $\mathscr{X}$ is Δ^1_1 in $R_{\beta_1}\oplus(\bigoplus_{R_\alpha}\{G_\gamma\}_{\gamma<\alpha})$. By Lemma 4.4, $\bigoplus_{R_\alpha}\{G_\gamma\}_{\gamma<\alpha}\leq_{\mathrm{T}}\bigoplus_{R_{\beta_1}}\{G_\gamma\}_{\gamma<\alpha}$. Thus $\mathscr{X}$ is Δ^1_1 in $\bigoplus_{R_{\beta_1}}\{G_\gamma\}_{\gamma<\alpha}$. By the Gandy's Basis Theorem, there must be some S so that $(\omega,S)\in\mathscr{X}$ and $S\leq_{\mathrm{T}}\mathcal{O}^{\bigoplus_{R_{\beta_1}}\{G_\gamma\}_{\gamma<\alpha}}$.

Let $\beta_2\geq\omega_1^{\bigoplus_{R_{\beta_1}}\{G_\gamma\}_{\gamma<\alpha}}$. Then $\mathcal{O}^{\bigoplus_{R_{\beta_1}}\{G_\gamma\}_{\gamma<\alpha}}$ is Δ^1_1 in $\bigoplus_{R_{\beta_2}}\{G_\gamma\}_{\gamma<\alpha}$. So there must be some $\beta_3<\omega_1^{\bigoplus_{R_{\beta_2}}\{G_\gamma\}_{\gamma<\alpha}}$ in E_1 such that $\mathcal{O}^{\bigoplus_{R_{\beta_1}}\{G_\gamma\}_{\gamma<\alpha}}\leq_{\mathrm{T}}(\bigoplus_{R_{\beta_2}}\{G_\gamma\}_{\gamma<\alpha})^{(\beta_3)}$. By Lemma 4.4, we have that

$$\mathcal{O}^{\bigoplus_{R_{\beta_1}}\{G_\gamma\}_{\gamma<\alpha}}\leq_{\mathrm{T}}(\bigoplus_{R_{\beta_2}}\{G_\gamma\}_{\gamma<\alpha})^{(\beta_3)}\leq_{\mathrm{T}}(\bigoplus_{R_{\beta_3}}\{G_\gamma\}_{\gamma<\beta_3})^{(\beta_3)}.$$

Let β be β_3. □

In $L[G]$, let $x^g_\alpha=(\oplus_{R_\alpha}\{G_\gamma\}_{\gamma<\alpha})^{(\alpha)}$ for every $\alpha\in E_1$, and let τ_α be the canonical $\mathbb{P}_{\aleph_1}$-name of x^g_α. Let

$$A=\{z\mid\exists\alpha\in E_1(z\equiv_{\mathrm{T}}x^g_\alpha)\}.$$

Obviously A is a chain of Turing degrees in $L[G]$.

Set

$$\mathcal{N}=(HOD(trcl(\{A\}))^{L[G]},$$

where $trcl(\{A\})\in L[G]$ is the transitive closure of $\{A\}$.

We have the following facts about the forcing and $\mathcal{N}$.

Lemma 4.6. *For any $\alpha<\omega_1$,*

(1) *$\mathbb{P}_{<\alpha}$ preserves cardinals;*
(2) *$G_{<\alpha}$ is a $\mathbb{P}_{<\alpha}$-generic set over L;*
(3) *$A\in\mathcal{N}$ is a cofinal chain of order type ω_1 in the Turing degrees;*
(4) *$(2^\omega)^{L[G]}=(2^\omega)^{\mathcal{N}}$.*

Proof. (1) and (2) are obvious and (4) follows from (3) immediately. We only prove (3).

For any real $x\in L[G]$, the must be some ordinal $\alpha<\omega_1$ so that $x\in L[G_{<\alpha}]$. So there is a countable ordinal $\alpha_1\geq\alpha$ so that $x\in L_{\alpha_1}[G_{<\alpha}]$. By Lemma 4.5, for any countable ordinal $\beta\geq\alpha_1$ in E_1, $x\leq_{\mathrm{T}}(\oplus_{R_\beta}\{G_\gamma\}_{\gamma<\beta})^{(\beta)}$. So $x\leq_{\mathrm{T}}x^g_\beta$. □

So $\mathcal{N}$ is a cardinal preserved extension of L satisfying $ZF+$"there exists a cofinal chain of Turing degrees".

If we take $\mathcal{N}'=HOD(\{G_\alpha|\alpha<\omega_1\})$, then in $\mathcal{N}'$ there is no well ordering of 2^ω by Cohen's proof (see [6]). But to follow Cohen's argument for $\mathcal{N}$, we need some additional care to preserve a name of A. So, we work with a subset of permutations of ω_1.

If π is a permutation of ω_1 then let $\operatorname{spt}\pi = \{\alpha < \omega_1 | \pi(\alpha) \neq \alpha\}$. Let H_0 be the set of all permutations π of ω_1 with $|\operatorname{spt}\pi| < \omega$. Let

$$H = \{\pi \in H_0 | \forall \alpha < \omega_1 (\pi(\alpha) + \omega = \alpha + \omega)\}.$$

For each $\pi \in H$, let $\bar{\pi}$ be the permutation of $L^{\mathbf{P}_{\aleph_1}}$ induced by π.

Let B be the set of $\mathbb{P}_{\aleph_1}$-names of reals, and define a $\mathbb{P}_{\aleph_1}$-name of A as follows

$$\dot{A} = \{(\tau, p) \in B \times \mathbf{P}_{\aleph_1} | \exists \alpha \in E_1 (p \Vdash (\tau \equiv_{\mathrm{T}} \tau_\alpha))\}.$$

Lemma 4.7. *If $\pi \in H$ and $\gamma \in E_1$ then $\Vdash \tau_\gamma \equiv_{\mathrm{T}} \bar{\pi}(\tau_\gamma)$.*

Proof. Note that the canonical name $\check{\gamma}$ of γ is invariant under π and R_γ is definable in γ as L is the least inner model. So τ_γ^G and $\bar{\pi}(\tau_\gamma)^G$ only differ from each other on finitely many columns, for any $\mathbb{P}_{\aleph_1}$-generic G. The lemma follows immediately. $\square$

Thus we have the following conclusion

Corollary 4.8. *If $\pi \in H$ then $\dot{A} = \bar{\pi}(\dot{A})$.*

So, the permutations in H are sufficient for showing

$$\mathcal{N} \models \text{ There is no well ordering of reals.}$$

4.2. **Forcing over models of CH.** Suppose that the ground model

$$\mathcal{M} \models ZFC + CH + \omega_1 = \omega_1^L.$$

We fix a cofinal chain in the Turing degrees $(x_\alpha : \alpha < \omega_1)$ in $\mathcal{M}$ such that $x_\alpha <_T x_\beta$ for $\alpha < \beta < \omega_1$.

We define $\mathbb{P}, \mathbb{P}_{<\alpha}, \mathbb{P}_{\aleph_1}, E$ and H as in the last subsection. For $x \subseteq \omega$ and $i \in \omega$, let $(x)_i = \{j \in \omega | \langle i, j\rangle \in x\}$.

For each α in E, fix a bijection $f_\alpha : \omega \to \alpha$. We define $(y_\alpha : \alpha \in E)$ by induction:

(1) $y_\omega = \emptyset$.
(2) Suppose that y_β is defined for each $\beta \in E \cap \alpha$. Let y_α be such that

$$(y_\alpha)_{2n} = y_\beta \text{ and } (y_\alpha)_{2n+1} = \{\langle i, j\rangle | f_\beta(i) = f_\alpha(j)\}$$

for each $\beta \in E \cap \alpha$ and $n = f_\alpha^{-1}(\beta)$.

For $\alpha \in \aleph_1$, let

$$\Gamma^*_\alpha = \{\langle \check{k}, p\rangle | p(\alpha)(k) = 1\},$$

and for $\alpha \in E$ let

$$\Gamma^*_{<\alpha} = \{\langle \check{\langle i, k\rangle}, p\rangle | p(f_\alpha(i))(k) = 1, p \in \mathbb{P}_{<\alpha}\}.$$

Clearly, Γ^*_α and $\Gamma^*_{<\alpha}$ are names, and if G is a $\mathbb{P}_{\aleph_1}$-generic then

$$G^*_\alpha = \{k | \exists p \in G(p(\alpha)(k) = 1)\} = (\Gamma^*_\alpha)_G$$

and

$$G^*_{<\alpha} = \{\langle i, k\rangle | \exists p \in G(p(f_\alpha(i))(k) = 1)\} = (\Gamma^*_{<\alpha})_G.$$

Lemma 4.9. $\Vdash (\check{x}_\alpha \oplus \check{y}_\alpha \oplus \Gamma^*_{<\alpha} | \alpha \in \check{E})$ *is a cofinal chain in the Turing degrees.*

Proof. Let G be a $\mathbb{P}_{\aleph_1}$-generic over $\mathcal{M}$. We work in $\mathcal{M}[G]$.

Firstly, we show that $x_\beta \oplus y_\beta \oplus G^*_{<\beta} \leq_T x_\alpha \oplus y_\alpha \oplus G^*_{<\alpha}$ for $\beta, \alpha \in E$ with $\beta < \alpha$. Clearly, $x_\beta \oplus y_\beta \leq_T x_\alpha \oplus y_\alpha$. It suffices to show $G^*_{<\beta} \leq_T y_\alpha \oplus G^*_{<\alpha}$. Let $n = f_\alpha^{-1}(\beta)$. For $\langle i, k\rangle$, find some j with $\langle i, j\rangle \in (y_\alpha)_{2n+1}$, then

$$\langle i, k\rangle \in G^*_{<\beta} \leftrightarrow \langle j, k\rangle \in G^*_{<\alpha}.$$

Secondly, we show that the chain is cofinal in $\mathcal{M}[G]$. For $x \in 2^\omega \cap \mathcal{M}[G]$, let τ be a name for x, and let $\beta \in E$ be such that for each $k < \omega$, there exists p with $p \in \mathbb{P}_{<\beta} \cap G$ deciding $\tau_G(k)$. For each $p \in \mathbb{P}_{<\beta}$, let

$$p^*_\beta = \{\langle i, k, b\rangle | b < 2, p(f_\beta(i))(k) = b\}.$$

p^*_β is a finite subset of ω, and can be identified with a number under some recursive coding of finite subsets of ω. The set of all p^*_β's is obviously recursive. Let $z : \omega \to \omega$ be such that $z(x) = c < 2$ if $x = \langle p^*_\beta, m\rangle$ and $p \Vdash \tau(\check{m}) = \check{c}$, or $z(x) = 2$ if otherwise. To decide $x(m)$, find (p^*, m, c) such that $z(\langle p^*, m\rangle) = c < 2$ and

$$\forall \langle i, k, b\rangle \in p^*(b = G^*_{<\beta}(\langle i, k\rangle)).$$

Then $x(m) = c$. So $x \leq_T z \oplus G^*_{<\beta}$. Now fix $\alpha \in E - \beta$ such that $z \leq_T x_\alpha$. Then

$$x \leq_T z \oplus G^*_{<\beta} \leq_T x_\beta \oplus G^*_{<\alpha} \leq_T x_\alpha \oplus y_\alpha \oplus G^*_{<\alpha}.$$

Hence, the chain is cofinal. □

Let

$$\Lambda_\alpha = \{(\bar{\pi}(\Gamma^*_{<\alpha}), \emptyset) | \pi \in H, \forall \gamma \geq \alpha(\pi(\gamma) = \gamma)\}.$$

For $\pi \in H$, let $\tilde{\pi}$ be the automorphism of $\mathbb{P}_{\aleph_1}$ induced by π. Then $\tilde{\pi} \upharpoonright \mathbb{P}_{<\alpha}$ is an automorphism of $\mathbb{P}_{<\alpha}$ for $\alpha \in E$. As only $p \in \mathbb{P}_{<\alpha}$ are involved in $\Gamma^*_{<\alpha}$, Λ^*_α depends only on $\{\tilde{\pi} \upharpoonright \mathbb{P}_{<\alpha} : \pi \in H\}$. Thus, Λ_α is fixed by all $\bar{\pi}$ for $\pi \in H$. Moreover,

$$\Vdash \Lambda_\alpha \text{ is a subset of the Turing degree of } \Gamma^*_{<\alpha}.$$

Similar to the last subsection, let

$$\mathcal{N} = HOD(\{\langle x_\alpha, y_\alpha, (\Lambda_\alpha)_G | \alpha \in E\rangle\})^{\mathcal{M}[G]}.$$

It follows that

$$\Vdash (HOD(trcl(\{\langle \check{x}_\alpha, \check{y}_\alpha, \Lambda_\alpha | \alpha \in \check{E}\rangle\})) \models \text{ There exists a cofinal chain}).$$

It follows by the homogeneity of $\mathbb{P}_{\aleph_1}$ and the above remark that

$$\mathcal{N} \models \text{There is no well ordering of reals.}$$

As $(\omega_1)^{\mathcal{N}} = \omega_1^L$, by Proposition 4.1, $\mathcal{N} \models \neg CH$.

5. Some comments

Finally we give some further comments.

(1). One also can construct a cofinal maximal chain of order type ω_1 in the hyperdegrees by the same method as above based on the results in [10].

(2). One may want to "localize" our result. Such as, by Lemma 3.3, there exists a cofinal maximal chain of order type ω^2 in arithmetical sets. However it is not difficult to see that there exists no a cofinal chain in $[\mathbf{a}, \mathbf{a}^n)$ for any degree $\mathbf{a}$ and number $n > 0$. Abraham and Shore's result [1] shows that the order types of cofinal maximal chains in $[\mathbf{a}, \mathbf{b})$ range over all of the ordinals below ω_1.

(3). Obviously, every minimal degree can be extended to be a cofinal maximal chain of order type ω_1. By Lemma 3.3, every Turing degree greater or equal to $\mathbf{0}''$ can be extended to be a cofinal maximal chain of order type ω_1.[2]

(4). Not every Turing degree can be extended to be a cofinal maximal chain of order type ω_1. Such as Δ^0_2 generic degrees, 2-generic degrees, 2-random degrees and hyperimmune-free random degrees, since they do not bound minimal degrees.

(5). It is open how to characterize the Turing degrees which can be extended to be a cofinal maximal chain of order type ω_1. Shore told us that such degrees can be defined based on Slaman and Woodin's results. But we don't know whether such degrees can be "naturally defined" in $(\mathcal{D}, \leq)$.

(6). We don't know whether ZF + "there exists a cofinal maximal chain of Turing degrees of order type ω_1" implies CH.

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Institute of Logic and Cognition, Sun Yat-sen University, 135 Xingang Xi Road, Guangzhou 510275, P.R. China
E-mail address: wwang.cn@gmail.com

Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger Strasse 25, A-1090 Vienna, Austria
E-mail address: liu.zhen.wu@univie.ac.at

Department of Mathematics, Nanjing University, 22 Hankou Road, Nanjing 210093, P.R. China
E-mail address: yuliang.nju@gmail.com

[2]Shore pointed out that $\mathbf{0}''$ can be replaced with $\mathbf{0}'$ in Lemma 3.3 since every $\mathbf{GH}_1$ degree bounds a minimal degree avoiding certain types of cones and every minimal degree is $\mathbf{GL}_2$ (these facts can be found in [7]). Together with Cooper's theorem on jumps of minimal degrees [4], one can show that every Turing degree greater or equal to $\mathbf{0}'$ can be extended to be a cofinal maximal chain of order type ω_1.