BOUNDING NON-$\mathbf{GL}_2$ AND R.E.A.

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ABSTRACT. We prove that every degree $\mathbf{a}$ bounding some non-$\mathbf{GL}_2$ degree is recursively enumerable in and above (r.e.a.) some 1-generic degree.

1. INTRODUCTION

A way to classify Turing degrees according to complexity is to calculate their iterated Turing jumps.

Definition 1.1. (1) *For* $n \geq 0$, $\mathbf{L}_n = \{\mathbf{a} \leq \mathbf{0}' | \mathbf{a}^{(n)} = \mathbf{0}^{(n)}\}$ *(where* $\mathbf{x}^{(0)} = \mathbf{x}$*) is the class of* low$_n$ degrees *and* $\mathbf{H}_n = \{\mathbf{a} \leq \mathbf{0}' | \mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}\}$ *is the class of* high$_n$ degrees.

(2) *Similarly,* $\mathbf{GL}_n = \{\mathbf{a} | \mathbf{a}^{(n)} = (\mathbf{a} \vee \mathbf{0}')^{(n-1)}\}$ *is the class of* generalized low$_n$ degrees *and* $\mathbf{GH}_n = \{\mathbf{a} | \mathbf{a}^{(n)} = (\mathbf{a} \vee \mathbf{0}')^{(n)}\}$ *is the class of* generalized high$_n$ degrees

Intuitively a degree in $\mathbf{L}_n$ or $\mathbf{GL}_n$ realizes the least n-th jump and thus is supposed to be less complicated, while a degree in $\mathbf{H}_n$ or $\mathbf{GH}_n$ realizes the greatest n-th jump and thus is considered to be more complicated. A class of special interests in the generalized high/low hierarchies is the class $\mathbf{GH}_1$ of the generalized high$_1$ degrees. Martin's characterization of sets of degrees in $\mathbf{H}_1$ provides great technical convenience for working with $\mathbf{GH}_1$.

Definition 1.2. *Given two functions* $f, g : \omega \to \omega$, f dominates g *if there exists some* x *such that* $f(y) > g(y)$ *for all* $y > x$.

Theorem 1.3 (Martin [8])**.** *If* $Y \leq_{\mathrm{T}} X$*, then* $Y'' \leq_{\mathrm{T}} X'$ *if and only if there exists a function* $f \leq_{\mathrm{T}} X$ *dominating all functions recursive in* Y.

As for any $\mathbf{a}$,

$$\mathbf{a} \notin \mathbf{GL}_2 \Leftrightarrow \mathbf{a}'' > (\mathbf{a} \vee \mathbf{0}')' \Leftrightarrow \mathbf{a} \vee \mathbf{0}' \notin \mathbf{GH}_1(\mathbf{a}),$$

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the following non-domination property of non-generalized-low$_2$ degrees follows easily.

Corollary 1.4. *If $A \in \mathbf{a} \notin \mathbf{GL}_2$, then for any function $f \leq_{\mathrm{T}} A \oplus \emptyset'$ there exists a function $g \leq_{\mathrm{T}} A$ such that $\exists^\infty x(g(x) > f(x))$.*

We shall refer to both Theorem 1.3 and Corollary 1.4 as *Martin's Domination Theorem*. Corollary 1.4 turns out to be very useful in studying non-$\mathbf{GL}_2$ degrees, especially for properties that involve Δ_2 predicates. For instance some strong results on the structure of the principal ideals $\mathcal{D}(\mathbf{a})$ of the non-$\mathbf{GL}_2$ degrees $\mathbf{a}$ have been obtained this way. In particular, Jockusch and Posner [4] have shown that every non-$\mathbf{GL}_2$ degree $\mathbf{a}$ bounds a 1-generic degree (in fact can be split into two 1-generic degrees). For more details, see e.g. Chapter IV of Lerman's monograph [7] where also a proof of Jockusch and Posner's result can be found (see Theorem IV.3.5).

In this article, we shall prove the following extension of Jockusch and Posner's theorem.

Theorem 1.5 (Main Theorem)**.** *If $\mathbf{a} \geq \mathbf{b} \notin \mathbf{GL}_2$ then $\mathbf{a}$ is recursively enumerable in and above (r.e.a.) some 1-generic degree.*

This may be regarded as a degree theoretic summarization of many interesting phenomena of non-$\mathbf{GL}_2$ degrees (e.g. poset or lattice embeddings below a non-$\mathbf{GL}_2$ degree) which were previously proved from Martin's Domination Theorem.

Also note that neither being non-$\mathbf{GL}_2$ nor beeing r.e.a. is upward closed (see Lerman [7], Theorem IV.1.11 and Kumabe [6], respectively). So far we did not know any upper cone with a base incomparable with $\mathbf{0}'$ consisting of r.e.a. degrees. Our Main Theorem gives us as many as possible upper cones of r.e.a. degrees.

We organize this paper as follows. We conclude this section by introducing some notation and the concepts used in our Main Theorem. In Section 2 we shall apply a result on recursive linear orderings to prove that each non-$\mathbf{L}_2$ degree below $\mathbf{0}'$ is r.e.a. some 1-generic degree. In Section 3 we shall first prove a jump inversion theorem for $\mathbf{a} \vee \mathbf{0}'$ with $\mathbf{a} \notin \mathbf{GL}_2$. With this and the main result in Section 2 we will deduce that non-$\mathbf{GL}_2$ degrees are r.e.a. 1-generics. Then we shall prove the Main Theorem, applying a join theorem by Jockusch and Posner. Finally, in Section 4 we will close this paper with some open questions.

Our notation is standard. For unexplained notation see Lerman [7]. As usual we identify a set of natural numbers, $A \subseteq \omega$, with its characteristic sequence. Finite binary sequences are denoted by lower case Greek letters; $|\sigma|$ denotes the length of σ, $\sigma \prec \tau$ denotes that τ properly extends σ; and, similarly, $\sigma \prec A$ denotes that σ is an initial segment of (the characteristic sequence of) A. If σ is an initial segment of a subset of A then we write $\sigma \subset A$.

We write $\Phi_x(\sigma; y)\downarrow$ if $x, y < |\sigma|$ and $\Phi_x(\sigma; y)$ is defined in $\leq |\sigma|$ many steps; and we write $\Phi_x(\sigma; y)\uparrow$ otherwise. Note that, for given x, y and σ, it is decidable whether $\Phi_x(\sigma; y)\downarrow$ holds, and if so, $\Phi_x(A; y) = \Phi_x(\sigma; y)$ for any set A with $\sigma \prec A$.

We introduce the 1-generic sets as those sets which force their jump. (For the equivalence with the original definition, see e.g. Lerman [7], Lemma IV.2.2.)

Definition 1.6. (1) *Let $\sigma \in 2^{<\omega}$ and $x, y < \omega$. Then $\sigma \Vdash \Phi_x(G; y)\downarrow$ if $\Phi_x(\sigma; y)\downarrow$ and $\sigma \Vdash \Phi_x(G; y)\uparrow$ if $\forall \tau \succ \sigma(\Phi_x(\tau; y)\uparrow)$. Correspondingly, $\sigma \Vdash e \in G'$ if $\sigma \Vdash \Phi_e(G; e)\downarrow$, and $\sigma \Vdash e \notin G'$ if $\sigma \Vdash \Phi_e(G; e)\uparrow$.*

(2) *G is* 1-generic *if, for each e,*

$$\exists \sigma \prec G(\sigma \Vdash e \in G' \text{ or } \sigma \Vdash e \notin G').$$

(3) *A degree $\mathbf{a}$ is* 1-generic *if it contains a 1-generic set.*

Note that $\sigma \Vdash e \in G'$ implies that $e \in X'$ for any set X such that $\sigma \prec X$. Similarly, $\sigma \Vdash e \notin G'$ implies that $e \notin X'$ for any set X such that $\sigma \prec X$.

Finally, we review the concept of recursive enumerability in and above.

Definition 1.7. (1) *A degree $\mathbf{a}$ is* recursively enumerable in $\mathbf{b}$ *if there are sets $A \in \mathbf{a}$ and $B \in \mathbf{b}$ such that $A = W^B$ for some recursively enumerable operator W.*

(2) *A degree $\mathbf{a}$ is* recursively enumerable in and above *(or* r.e.a. *for short) $\mathbf{b}$ if $\mathbf{a} > \mathbf{b}$ and $\mathbf{a}$ is recursively enumerable in $\mathbf{b}$.*

(3) *A degree $\mathbf{a}$ is* r.e.a. *if $\mathbf{a}$ is r.e.a. some degree.*

2. Non-$\mathbf{L}_2$ Degrees

In this section we prove the special case of our main theorem for degrees below $\mathbf{0}'$.

Theorem 2.1. *Every non-$\mathbf{L}_2$ degree $\mathbf{a}$ below $\mathbf{0}'$ is r.e.a. some 1-generic degree.*

We first give the main ideas of the proof. Let $\mathbf{a} \leq \mathbf{0}'$ be a non-$\mathbf{L}_2$ degree. We have to show that there is a degree $\mathbf{g}$ such that

(i) $\mathbf{g}$ is 1-generic,
(ii) $\mathbf{g} < \mathbf{a}$, and
(iii) $\mathbf{a}$ is r.e. in $\mathbf{g}$.

In fact, since $\mathbf{a}$ is non-$\mathbf{L}_2$ by assumption and since any 1-generic degree is $\mathbf{GL}_1$ (hence any 1-generic degree below $\mathbf{0}'$ is low$_1$; see e.g. Lerman [7], Lemma IV.2.3), (i) will imply that $\mathbf{a} \neq \mathbf{g}$. So we may replace (ii) by the weaker requirement

(ii') $\mathbf{g} \leq \mathbf{a}$.

Moreover, if $\mathbf{a}$ is recursively enumerable then $\mathbf{a}$ is recursively enumerable in all degrees $\mathbf{b} < \mathbf{a}$ whence in this case Theorem 2.1 follows from Jockusch and Posner's result that any non-$\mathbf{GL}_2$ degree bounds a 1-generic degree. So in the following, w.l.o.g. we may assume that **a** does not contain any r.e. set.

Now in order to get the desired 1-generic degree **g** we exploit some results on recursive linear orderings. A linear ordering $<_L$ is *recursive* if ω is the domain of $<_L$ and the relation $x <_L y$ is recursive. Here we will consider orderings $<_L$ of order type $\omega+\omega^*$ where ω^* is the order type of the negative integers, i.e., $\omega + \omega^* = 0, 1, 2, \dots, -3, -2, -1$. In the following we let U_L denote the ω-part of $<_L$. Correspondingly, $\overline{U_L} = \omega - U_L$ will be the ω^* part of $<_L$.

Note that for a recursive linear ordering $<_L$ of order type $\omega + \omega^*$, any infinite subset of the ω-part U_L is cofinal in U_L (with respect to $<_L$). This implies that U_L is recursively enumerable in any infinite $S \subseteq U_L$ since

$$x \in U_L \Leftrightarrow \exists\, y \in S\ (x <_L y).$$

So, for our proof, it suffices to give a recursive linear ordering $<_L$ of order type $\omega + \omega^*$ and a 1-generic set G such that $U_L \in \mathbf{a}$, $G \subseteq U_L$ and $G \leq_{\mathrm{T}} U_L$. Then, for $\mathbf{g} = deg(G)$, (i) holds by 1-genericity of G, (ii') holds by $G \leq_{\mathrm{T}} U_L \in \mathbf{a}$, and (iii) holds by $G \subseteq U_L$.

In order to obtain the required sets U_L and G we use the following two lemmas. The former lemma guarantees that there is some recursive linear ordering $<_L$ of order type $\omega + \omega^*$ such that the ω-part U_L has degree **a**. The second lemma implies that the ω-part U_L of any such ordering contains a 1-generic set; in fact it allows the construction of 1-generic subsets G of U_L by any finite extension argument. In the actual proof we give such a construction which in addition will ensure that the constructed 1-generic set G is recursive in **a**.

Lemma 2.2 (Harizanov)**.** *Let* $\mathbf{a} \leq \mathbf{0}'$*. There is a recursive linear ordering* $<_L$ *of order type* $\omega + \omega^*$ *such that the* ω*-part* U_L *has degree* **a***.*

For a proof see Harizanov [1, Proposition 3.1].

Lemma 2.3 (Hirschfeldt and Shore)**.** *Suppose* $<_L$ *is a recursive linear ordering of order type* $\omega+\omega^*$ *such that there is no recursive infinite descending sequence of* $<_L$*. Then, for any string* $\sigma \subset U_L$ *and for any number* $e < \omega$*, there is a string* τ *such that* $\sigma \prec \tau \subset U_L$ *and either* $\tau \Vdash e \in G'$ *or* $\tau \Vdash e \notin G'$*.*

Proof. We follow the last paragraph in the proof of [2, Theorem 3.4].

For a contradiction assume that the claim fails. Then we may fix $\sigma \subset U_L$ and $e \geq 0$ such that

$$\forall\, \tau\ (\sigma \prec \tau \subset U_L \Rightarrow [\tau \nVdash e \in G' \ \&\ \tau \nVdash e \notin G'],$$

i.e.,

$$\forall\, \tau\ (\sigma \prec \tau \subset U_L \Rightarrow [\Phi_e(\tau; e)\uparrow\ \ \&\ \exists\, \tau' \succ \tau\ (\Phi_e(\tau'; e)\downarrow)]). \tag{1}$$

Note that, for τ and τ' as in (1), $\tau' \not\subset U_L$.

Now, contrary to assumption, we will inductively define a recursive descending sequence $\langle x_n : n \geq 0\rangle$ of $<_L$, $x_0 >_L x_1 >_L x_2 \dots$, where in addition we will ensure that $x_0 < x_1 < x_2 < \dots$ and that x_n is in the ω^* part $\overline{U_L}$ of $>_L$ (for all $n \geq 0$).

Let x_0 be any element of $\overline{U_L}$ such that $x_0 > |\sigma|$. For the inductive step, given numbers $x_0, x_1, \dots x_n$ in $\overline{U_L}$ such that $|\sigma| < x_0 < x_1 < \cdots < x_n$ and $x_n <_L x_{n-1} <_L \cdots <_L x_0$, it suffices to effectively find $x_{n+1} \in \overline{U_L}$ such that $x_n < x_{n+1}$ and $x_{n+1} <_L x_n$.

This is achieved as follows. For $k \geq 0$ let τ_k be the extension $\sigma 0^k$ of σ. Note that, by $\sigma \subset U_L$, $\tau_k \subset U_L$ too. So, by (1), there will be an extension τ'_k of τ_k such that $\Phi_e(\tau'_k; e) \downarrow$. Moreover, we can find the least such extension τ'_k effectively, and, as observed above, $\tau'_k \not\subset U_L$. So there is a number $y_k \geq |\tau_k| \geq k$ such that $\tau'_k(y_k) = 1$ and $y_k \in \overline{U_L}$. Moreover, since the elements of U_L $<_L$-precede the elements of $\overline{U_L}$, we can effectively find such a number y_k by letting y_k be the $<_L$-greatest number y such that $|\tau_k| \leq y < |\tau'_k|$ and $\tau'_k(y) = 1$. Hence we can effectively enumerate the set $Y = \{y_k : k \geq 0\}$ in order and, by $y_k \geq k$ and $y_k \in \overline{U_L}$, Y is an infinite subset of $\overline{U_L}$. So, since there are only finitely many numbers y such that $x_n \leq_L y$ (since x_n is in the ω^*-part of $<_L$) or $y \leq x_n$, we obtain the desired number x_{n+1} by letting $x_{n+1} = y_k$ for the least k such that $y_k <_L x_n$ and $y_k > x_n$. □

Note that for a recursive linear ordering $<_L$ of order type $\omega + \omega^*$ such that the degree of U_L is not r.e., the hypothesis of Lemma 2.3 is satisfied. Namely, for any infinite $<_L$-descending sequence $x_0 >_L x_1 >_L x_2 \dots$, all members of the sequence have to be in the ω^*-part and the sequence is cofinal in this part, whence

$$x \in \overline{U_L} \Leftrightarrow \exists\, n\ (x_n <_L x).$$

So if this sequence were recursive then $\overline{U_L}$ were r.e., hence $deg(U_L) = deg(\overline{U_L})$ were r.e. too.

Having explained the basic format of the proof of Theorem 2.1, we now turn to the actual construction.

Proof. Let $\mathbf{a} \leq \mathbf{0}'$ be non-$\mathbf{L}_2$ where w.l.o.g. $\mathbf{a}$ is not r.e. By Lemma 2.2 fix a recursive linear ordering $<_L$ of order type $\omega + \omega^*$ such that the ω-part U_L has degree $\mathbf{a}$. As pointed out above, it suffices to define a 1-generic set G such that $G \leq_{\mathrm{T}} U_L$ and $G \subseteq U_L$.

We will define G by a finite extension argument, where the initial segment γ_s of G of length s is defined at stage s of the construction. Then $G \subseteq U_L$ is ensured by guaranteeing $\gamma_s \subset U_L$ and $G \leq_{\mathrm{T}} U_L$ is ensured by making the construction effective in U_L (i.e., in $\mathbf{a}$). Finally, 1-genericity of G is obtained by meeting the requirements

$$R_e : \exists s\ (\gamma_s \Vdash e \in G' \text{ or } \gamma_s \Vdash e \notin G')$$

for all numbers $e \geq 0$.

The strategy for meeting the genericity requirements is based on Lemma 2.3. As pointed out above, $\mathbf{a}$ not being r.e. implies that the hypothesis of the lemma is satisfied. So, for $P = \{\sigma : \sigma \subset U_L\}$ and for any $\sigma \in P$ and $e \geq 0$,

$$\exists\, \tau \in P\, [\sigma \prec \tau\ \&\ (\tau \Vdash e \in G' \text{ or } \tau \Vdash e \notin G')]. \tag{2}$$

So, if we let $\tau(\sigma, e)$ be the least (i.e. shortest and leftmost) string τ witnessing (2), i.e.,

$$\tau(\sigma, e) = \mu\tau\ [\sigma \prec \tau \in P\ \&\ (\tau \Vdash e \in G' \text{ or } \tau \Vdash e \notin G')],$$

then the function $\tau : P \times \omega \to P$ is total. Moreover, $\tau \leq_{\mathrm{T}} \emptyset'$ since $P \leq_{\mathrm{T}} \emptyset'$ (note that $P \leq_{\mathrm{T}} U_L$ and $U_L \in \mathbf{a} \leq \mathbf{0}'$) and since the questions whether $\tau \Vdash e \in G'$ (i.e., $\Phi_e(\tau; e)\downarrow$) and $\tau \Vdash e \notin G'$ (i.e., $\Phi_e(\tau'; e)\uparrow$ for all $\tau' \succ \tau$) hold can be respectively answered effectively relative to $\emptyset'$.

Now if - in place of $G \leq_{\mathrm{T}} U_L$ - we only had to ensure $G \leq_{\mathrm{T}} \emptyset'$ then - based on the function τ - we could easily define the desired 1-generic set $G \subseteq U_L$ by a straightforward finite extension argument as follows. Given a finite initial segment γ_s of G and the first requirement R_e not yet met we meet R_e by extending γ_s to $\tau(\gamma_s, e)$, i.e., by letting $\gamma_{s'} = \tau(\gamma_s, e)$ for $s' = |\tau(\gamma_s, e)|$.

In the actual construction, using $U_L \in \mathbf{a}$ (in place of $\emptyset'$) as an oracle we will work with some $\mathbf{a}$-approximations $\tau(\sigma, e, s)$ of $\tau(\sigma, e)$, called e-targets below, and we will argue that, by Martin's Domination Theorem, we will eventually get some correct e-target. (The argument resembles the proof of Jockusch and Posner's Theorem on bounding 1-generic degrees given in Lerman [7, Theorem IV.3.5]. Our argument is somewhat more subtle, however, since here we also have to approximate the Π^0_1-property $\tau \Vdash e \notin G'$ while there it sufficed to deal with the Σ^0_1-property $\tau \Vdash e \in G'$.)

We first define functions f and m where $f(s)$ bounds the size of the strings $\tau(\sigma, e)$ for strings $\sigma \in P$ of length $\leq s$ and for $e \leq s$, while $m(s)$ bounds, for any string τ of length $\leq f(s)$ and for any number $e \leq s$ such that $\tau \not\Vdash e \notin G'$, the length of the least string ρ witnessing this fact.

$$f(s) = \max\{|\tau(\sigma, e)| : \sigma \in P \cap 2^{\leq s}\ \&\ e \leq s\}$$

$$m(s) = \max\{|\rho(\tau, e)| : \tau \in 2^{\leq f(s)}\ \&\ e \leq s\}$$

where

$$\rho(\tau, e) = \begin{cases} \mu\rho(\tau \prec \rho \Vdash e \in G') & \text{if } \tau \not\Vdash e \notin G' \\ \tau & \text{otherwise.} \end{cases}$$

Note that f and m are total, nondecreasing, and recursive in $\emptyset'$. Moreover, $s \leq f(s) \leq m(s)$ and, for any $\tau \in 2^{\leq f(s)}$,

$$\tau \Vdash e \notin G' \Leftrightarrow \neg\exists \tau' \succ \tau\ (|\tau'| \leq m(s)\ \&\ \tau' \Vdash e \in G'). \tag{3}$$

So $\tau \Vdash e \notin G'$ can be decided recursively in m.

Finally, by Martin's Domination Theorem, fix a strictly increasing function $g \leq_{\mathrm{T}} U_L$ such that $\exists^\infty s(g(s) > m(s))$.

Before proceeding to the formal construction we give some more intuition by explaining the strategy for meeting a single requirement R_e. At stage

$s+1$, given the previously defined finite initial segment $\gamma_s \in P \cap 2^s$ of G, we search in $2^{<g(s)}$ for a string τ with $\gamma_s \prec \tau \in P$ such that

$$\text{either } \tau \Vdash e \in G' \text{ or } \forall \rho \in 2^{<g(s)} (\tau \preceq \rho \Rightarrow \rho \nVdash e \in G')$$

holds. We let $\tau(\gamma_s, e, s)$ be the least such string τ (if there is no such τ then we let $\tau(\gamma_s, e, s) = \gamma_s$) and we call $\tau(\gamma_s, e, s)$ the *e-target* at stage s. Note that $|\tau(\gamma_s, e, s)| \leq f(s)$ as $\tau(\gamma_s, e) \in 2^{\leq f(s)}$ has the described property with the possible exception of $\tau(\gamma_s, e) \notin 2^{\leq g(s)}$. Moreover, $\tau(\gamma_s, e, s)$ can be computed relative to $\mathbf{a}$.

Now if $\tau(\gamma_s, e, s) \Vdash e \in G'$ then we will meet requirement R_e by extending G along $\tau(\gamma_s, e, s)$, i.e., by letting $\gamma_t = \tau(\gamma_s, e, s) \restriction t+1$ for $s < t \leq |\tau(\gamma_s, e, s)|$. If $\tau(\gamma_s, e, s) \nVdash e \in G'$ then we *guess* that $\tau(\gamma_s, e, s) \Vdash e \notin G'$. Again, we extend G along $\tau(\gamma_s, e, s)$. If our guess is correct, this will ensure that R_e is met. If our guess is not correct then there will be a stage $t > s$ at which this becomes apparent by a string τ' of length $\leq g(t)$ such that $\tau(\gamma_s, e, s) \prec \tau'$ and $\tau' \Vdash e \in G'$. At the first such stage t the target $\tau(\gamma_s, e, s)$ is called *t-incorrect for e* and a new target is appointed at stage t.

It is crucial that this process has to be repeated at most finitely often. Namely, by choice of g, there will be a least stage $t \geq s$ such that $m(t) < g(t)$ (hence $m(s) < g(t)$). Now, if $\tau(\gamma_s, e, s)$ is t-correct for e, then, by (3), $\tau(\gamma_s, e, s) \Vdash e \notin G'$ and $\tau(\gamma_s, e, s)$ is correct for e at all stages $\geq s$. So R_e is met and no new e-target will be considered. On the other hand, if $\tau(\gamma_s, e, s)$ is t-incorrect for e then at stage t the e-target is replaced by $\tau(\gamma_t, e, t)$ and, by $m(t) < g(t)$ and (3), this target will never become incorrect for e. So again, R_e is met and the action for R_e is finite.

In the actual construction, there might be conflicts among the requirements which are resolved by giving priority to requirements with lesser index. If a requirement is injured its target is cancelled and a new attempt for meeting the requirement based on a new target will be started later.

Now we describe the construction formally. For $s = 0$ we let γ_0 be the empty string.

At stage $s+1$, we let $t(e, s)$ be the least stage $t \leq s$ such that $\tau(\gamma_t, e, t)$ is s-correct for e and $\tau(\gamma_t, e, t)$ is comparable with γ_s; and we let $t(e, s) = s+1$ if no such stage $t \leq s$ exists. We say that R_e *requires attention* if $e < s+1$ and

(a) $t(e, s) \leq s$ and $\gamma_s \prec \tau(\gamma_{t(e,s)}, e, t(e, s))$ or
(b) $t(e, s) = s+1$.

If no requirement R_e requires attention then let $\gamma_{s+1} = \gamma_s 0$. Otherwise, fix e minimal such that R_e requires attention and let $\gamma_{s+1} = \tau(\gamma_{t(s,e)}, e, t(s, e)) \restriction s+1$. This completes the construction. Note that the construction is recursive in $\mathbf{a}$ and $\gamma_s \in P$ for all $s \geq 0$. So $G \leq_{\mathrm{T}} U_L$ and $G \subseteq U_L$. Finally, 1-genericity of G follows from the following claim.

Claim. Every requirement R_e is met and requires attention at most finitely often.

The Claim is established by induction on e. Fix e and, by inductive hypothesis, assume that no requirement $R_{e'}$, $e' < e$ requires attention after stage s_e. Moreover, choose $s+1 > e, s_e$ such that $g(s) > m(s)$. Then, for $t = t(e,s)$ as in the construction, $\tau(\gamma_t, e, t)$ is comparable with γ_s. Moreover, by $g(s) > m(s)$ and by (3), $\tau(\gamma_t, e, t) \Vdash e \in G'$ or $\tau(\gamma_t, e, t) \Vdash e \notin G'$, and, by $s > s_e, e$, G is extended along $\tau(\gamma_t, e, t)$. So R_e is met and R_e will not require attention after stage $|\tau(\gamma_t, e, t)|$.

This completes the proof. □

The proof of Theorem 2.1 gives the following improvement of Hirschfeldt and Shore [2, Theorem 2.11] when U_L is non-$\mathbf{L}_2$.

Corollary 2.4. *Every X-recursive linear order $<_L$ of type $\omega + \omega^*$ with the ω part U_L non-$\mathbf{L}_2$ in X has an infinite ascending or descending sequence A which is recursive in U_L and low in X.*

Proof. Assume that there is no X-recursive infinite descending sequence of the order $<_L$. Then, by relativizing the proof of Theorem 2.1, we get $G \subseteq U_L$ 1-generic in X and recursive in U_L. From $G \oplus X$ we compute an infinite ascending sequence A in U_L. A is low in X as so is $G \oplus X$. □

Readers may also note that an improvement of Jockusch's observation [2, Corollary 2.14] follows easily from Hirschfeldt-Shore's density lemma 2.3.

Corollary 2.5. *Every $\mathbf{a} \leq \mathbf{0}'$ is r.e. in a 1-generic degree $\mathbf{g} < \mathbf{0}'$.*

3. Bounding Non-$\mathbf{GL}_2$ Degrees

Posner proved (see Lerman [7, Theorem IV.4.8]) that every $\mathbf{GH}_1$ degree is r.e.a. some degree. Readers familiar with the following theorem of Jockusch and Posner may find that Posner's result can be extended to non-$\mathbf{GL}_2$ degrees with the help of Theorem 2.1.

Theorem 3.1 (Jockusch and Posner [4], Theorem 2)**.** *If $\mathbf{a} \notin \mathbf{GL}_2$, $\mathbf{c} \geq \mathbf{a} \vee \mathbf{0}'$ and $\mathbf{c}$ is r.e. in $\mathbf{a}$, then there is $\mathbf{b} \leq \mathbf{a}$ such that $\mathbf{b}' = \mathbf{c}$.*

Then the improvement of Posner's result follows.

Proposition 3.2. *For every non-$\mathbf{GL}_2$ degree $\mathbf{a}$ there is a degree $\mathbf{d} < \mathbf{a}$ such that $\mathbf{a}$ is r.e. and non-$\mathbf{L}_2$ over $\mathbf{d}$.*

Proof. Let $\mathbf{a}$ be a non-$\mathbf{GL}_2$ degree. By Theorem 3.1, there is $\mathbf{b} \leq \mathbf{a}$ such that $\mathbf{b}' = \mathbf{a} \vee \mathbf{0}'$. Then as $\mathbf{a}$ is non-$\mathbf{GL}_2$ we have $\mathbf{a}'' > (\mathbf{a} \vee \mathbf{0}')' = \mathbf{b}''$, i.e. $\mathbf{a}$ is non-$\mathbf{L}_2$ over $\mathbf{b}$. Now relativize Theorem 2.1 to produce $\mathbf{d}$ between $\mathbf{b}$ and $\mathbf{a}$ such that $\mathbf{a}$ is r.e. over $\mathbf{d}$ and $\mathbf{d}$ is low over $\mathbf{b}$. Hence $\mathbf{a}$ is also non-$\mathbf{L}_2$ over $\mathbf{d}$. □

Actually we can make $\mathbf{d}$ above to be 1-generic, like in Theorem 2.1. To this end, we first improve Theorem 3.1 when $\mathbf{c} = \mathbf{a} \vee \mathbf{0}'$ to yield a 1-generic $\mathbf{b}$.

Proposition 3.3. *If* $\mathbf{a} \notin \mathbf{GL}_2$ *then there is a 1-generic degree* $\mathbf{g} \leq \mathbf{a}$ *such that* $\mathbf{g}' = \mathbf{a} \vee \mathbf{0}'$.

Proof. Just as the proof of Theorem 2.1, the proof is based on the proof of Jockusch and Posner's Theorem on bounding 1-generic degrees given in Lerman [7, Theorem IV.3.5].

Fix $A \in \mathbf{a}$. Since 1-generic sets are $\mathbf{GL}_1$, it suffices to construct a 1-generic set G such that $G \leq_{\mathrm{T}} A$ and $A \leq_{\mathrm{T}} G'$.

We will define G by a finite extension argument, where the initial segment γ_s of G of length s is defined at stage s of the construction. 1-genericity of G is guaranteed by meeting the requirements

$$R_e : \exists s\ (\gamma_s \Vdash e \in G' \text{ or } \gamma_s \Vdash e \notin G')$$

for all numbers $e \geq 0$. $G \leq_{\mathrm{T}} A$ is ensured by making the construction effective in A. Finally, $A \leq_{\mathrm{T}} G'$ is ensured by coding A into G in a way such that G' can detect the coding locations. I.e., for any number n there will be a stage $s(n)$ such that $\gamma_{s(n)+1} = \gamma_{s(n)} A(n)$ and $s(n)$ can be computed from G'.

The strategy for meeting the 1-genericity requirements is based on the following functions. Define

$$f(s) = \mu t \geq s[\forall \sigma \in 2^s \forall e < s(\sigma \Vdash e \notin G' \text{ or } \exists \tau \in 2^{\leq t}(\sigma \prec \tau \Vdash e \in G'))].$$

Clearly f is total and recursive in $\emptyset'$. Hence by Martin's Domination Theorem there is a total function $g \leq_{\mathrm{T}} A$ such that $\exists^\infty s(g(s) > f(s))$ where w.l.o.g. $g(s) > s$ and g is strictly increasing.

Then requirement R_e can be met as follows. Given a stage s at which R_e is not yet satisfied we search for a string τ of length $\leq g(s)$ which extends γ_s and forces $e \in G'$. If there is such a string then we pick the least such string τ and extend G along τ thereby ensuring that $\gamma_t \Vdash e \in G'$ for $t = |\tau|$. (In the actual construction, while extending G along τ, τ will be called the e-target at stage s' and will be denoted by $\delta_{s'}$, $s \leq s' \leq |\tau|$.) If we do not find such a string then we repeat our search at the next stage $s+1$ (and so on). Now if we never find an extension τ as above then we may argue that $\gamma_{s'} \Vdash e \notin G'$ for some $s' \geq s$. Namely, by choice of g there will be $s' \geq s$ with $g(s') \geq f(s')$ and, by definition of f and the failure of the existence of a string τ as above, $\gamma_{s'} \Vdash e \notin G'$.

The coding of A into G will be as follows. Let $e_0 < e_1 < e_2 \ldots$ be the sequence of all numbers e such that $e \in G'$. Note that, by 1-genericity of G,

$$\forall\, n \geq 0\ \forall^\infty\, s\ (\gamma_s \Vdash e_n \in G').$$

So we may inductively define $s(0) < s(1) < s(2) \ldots$ by

$$s(0) = \mu s(s > e_0\ \&\ \gamma_s \Vdash e_0 \in G')$$

and

$$s(n+1) = \mu s(s > \max(e_{n+1}, s(n))\ \&\ \gamma_s \Vdash e_{n+1} \in G').$$

Clearly, e_n and $s(n)$ are computable from G and G' (hence from G'). So we achieve $A \leq_{\mathrm{T}} G'$ by ensuring

$$\gamma_{s(n)+1} = \gamma_{s(n)} A(n).$$

In the construction we will guess whether $s = s(n)$ for some n and, if so, set $\gamma_{s+1} = \gamma_s A(n)$. Roughly speaking, this is done as follows. If requirement R_e becomes satisfied at stage s then we guess that $e = e_n$ and $s = s_n$ if the requirements $R_{e_0}, \ldots, R_{e_{n-1}}$, $e_0 < e_1 < \ldots e_{n-1} < e$ have been previously satisfied. Correspondingly we set $G(s) = A(n)$. Now this guess may turn out to be incorrect since for some $e' < e$ the fact that $e' \in G'$ can be forced only by some $\gamma_{s'}$ with $s' > s$. In this case we say that R_e is injured at stage s' and we make a new attempt for meeting the requirement later. Then we will argue that for the final attempt the guess will be correct. We let $c_e(s)$ denote the least number n which has not been previously coded by satisfying a requirement $R_{e'}$ where $e' < e$ and the coding had not been injured by the end of stage s.

We next formally describe the construction.

At stage $s = 0$ we let $\gamma_0 = \emptyset$. No requirement is active at stage 0, δ_0 is not defined, and $c_e(0) = 0$ for all $e \geq 0$.

Stage $s + 1$ of the construction is as follows.

Requirement R_e *requires attention at stage* $s + 1$ if $e \leq s$ and R_e is not satisfied at stage s and

(i) δ_s is an e-target or
(ii) there is a string $\tau \in 2^{\leq g(s)}$ such that $\gamma_s \preceq \tau \Vdash e \in G'$.

If no requirement requires attention then let $\gamma_{s+1} = \gamma_s 0$. Otherwise, fix e minimal such that R_e requires attention. Declare that R_e is *active at stage* $s + 1$ and do the following.

- If (i) holds let $\eta = \delta_s$. Otherwise let η be the least string τ as in (ii). Distinguish the following two cases.
- If $\gamma_s \prec \eta$ then let $\gamma_{s+1} = \eta \upharpoonright s+1$, appoint $\delta_{s+1} = \eta$ as e-target, and let $c_{e'}(s + 1) = c_{e'}(s)$ for all $e' \geq 0$.
- If $\gamma_s = \eta$ then let $\gamma_{s+1} = \gamma_s A(c_e(s))$, declare R_e to *become satisfied* and all requirements $R_{e'}$ with $e' > e$ to be *injured* at stage $s + 1$, and let $c_{e'}(s + 1) = c_{e'}(s)$ for $e' \leq e$ and $c_{e'}(s + 1) = c_e(s) + 1$ for $e' > e$. (In this case δ_{s+1} is not defined.)

We call $R_{e'}$ *satisfied* at stage $s + 1$ if $R_{e'}$ became satisfied at a stage $t \leq s$ and $R_{e'}$ was not injured at any stage t' with $t < t' \leq s + 1$.

This completes the construction.

Note that the construction is effective in A, hence $G \leq_{\mathrm{T}} A$. So the correctness of the construction can be established by the following two claims.

Claim 1. Every requirement R_e is met and requires attention at most finitely often.

The proof of Claim 1 is by induction on e. Given e, by inductive hypothesis, fix a stage $t_e \geq e$ such that no requirement $R_{e'}$ with $e' < e$ requires attention at any stage $t \geq t_e$. Then R_e will not be injured after stage t_e. So if R_e is satisfied at some stage $s+1 > t_e$ then R_e will not require attention after stage $s+1$ and R_e will be met since $\gamma_{s+1} \Vdash e \in G'$ by construction.

So, for the remainder of the argument, we may assume that R_e is never satisfied after stage t_e.

First we show that R_e does not require attention after stage t_e. For a contradiction assume that R_e requires attention at stage $s+1 > t_e$. Then either R_e is satisfied at stage $s+1$ or an e-target δ_{s+1} will be appointed at stage $s+1$, R_e will continue to be active at the successive stages, and eventually be statisfied at stage $s'+1$ for $s' = |\delta_{s+1}|$. In either case this contradicts our assumption that R_e will not be satisfied after stage t_e.

Finally, in order to show that R_e is met, we argue that, for some $s > t_e$, $\gamma_s \Vdash e \notin G'$. Fix $s > t_e$ such that $g(s) \geq f(s)$. Then, by definition of f, either $\gamma_s \Vdash e \notin G'$ or there is a string $\tau \in 2^{\leq f(s)}$ such that $\gamma_s \preceq \tau \Vdash e \in G'$. The latter, however, cannot happen since otherwise R_e will require attention at stage $s \geq t_e$.

Claim 2. For $n \geq 0$, $\gamma_{s(n)+1} = \gamma_{s(n)} A(n)$.

Note that, for a requirement R_e which becomes satisfied at some stage $s+1$, $\gamma_s \Vdash e \in G'$ hence $e \in G'$. Conversely, if $e \in G'$ then, by R_e being met, $\gamma_s \Vdash e \in G'$ for all sufficiently large s. So, by Claim 1, R_e will eventually become satisfied. In fact, there will be a last stage t such that R_e becomes satisfied at stage $t+1$ and R_e will not be injured later. So, in particular, for $n \geq 0$, we may let $t(n)+1$ be the last stage at which R_{e_n} becomes satisfied. We will show that

$$t(n) = s(n) \;\&\; \forall t \geq t(n) \; (c_{e_n}(t) = n). \tag{4}$$

This will imply Claim 2, since by (4) and by construction,

$$\gamma_{s(n)+1} = \gamma_{t(n)+1} = \gamma_{t(n)} A(c_{e_n}(t(n))) = \gamma_{s(n)} A(n).$$

The proof of (4) is by induction on n.

Let $n = 0$. Since no R_e with $e < e_0$ will ever become satisfied, none of these requirements will ever become active. So R_{e_0} will become active whenever R_{e_0} requires attention, R_{e_0} is never injured, and $c_{e_0}(s) = 0$ for all stages $s \geq 0$. So $t(0)+1$ is the unique stage at which R_{e_0} becomes satisfied and $c_{e_0}(t) = 0$ for all stage $t \geq t(0)$. To show that $t(0) = s(0)$, note that by definition of $s(0)$ and by construction, R_{e_0} will not be satisfied by stage $s(0)$ and R_{e_0} will become active at stage $s(0)+1$ since $\gamma_{s(0)} \Vdash e_0 \in G'$. So, for η as described in the construction, $\eta = \gamma_{s(0)}$ or $\eta = \delta_{s(0)}$ where $\delta_{s(0)}$ is appointed as e_0-target at stage s_0. Since, as one can easily check, an e-target δ_s is the least string τ with $\gamma_s \preceq \tau \Vdash e \in G'$, in either case $\eta = \gamma_{s(0)}$ and R_{e_0} becomes satisfied at stage $s(0)+1$.

For the inductiv step fix n and assume that (4) holds. We have to show that $t(n+1) = s(n+1)$ and $c_{e_{n+1}}(t) = n+1$ for all $t \geq t(n+1)$. Note that by choice of $t(n)$ and by (4), $A(n)$ is coded into G at stage $t(n)+1$ by R_{e_n} becoming satisfied. So, since R_{e_n} will not be injured at any later stage, $c_{e_{n+1}}(t) = n+1$ holds for all $t > t(n)$ hence for all $t \geq t(n+1)$. Finally, the argument that $t(n+1) = s(n+1)$ resembles the proof of the corresponding claim for 0 in place of $n+1$. Since $R_{e_{n+1}}$ is injured at stage $t(n)+1$ it easily follows from the inductive hypothesis that, after stage $t(n)+1 = s(n)+1$, $R_{e_{n+1}}$ will become active whenever it requires attention and it will not be injured anymore. So $t(n+1)$ will be the first stage $> s(n)+1$ at which $R_{e_{n+1}}$ becomes satisfied. But then, as in case of $n = 0$, we can easily argue that $t(n+1) = s(n+1)$.

This completes the proof of Claim 2 and of Proposition 3.3. □

The following proposition allows us to relativize Theorem 2.1.

Proposition 3.4 (Yu [12], Proposition 2.2)**.** *$G_0 \oplus G_1$ is 1-generic if and only if G_0 is 1-generic and G_1 is 1-generic in G_0.*

Theorem 3.5. *For every non-$\mathbf{GL}_2$ degree $\mathbf{a}$ there is a 1-generic degree $\mathbf{g} < \mathbf{a}$ such that $\mathbf{a}$ is r.e.a. $\mathbf{g}$ and $\mathbf{g}' = \mathbf{a} \vee \mathbf{0}'$. Hence $\mathbf{a} \notin \mathbf{L}_2(\mathbf{g})$.*

Proof. By Proposition 3.3, choose a 1-generic $\mathbf{g}_0 \leq \mathbf{a}$ such that $\mathbf{g}_0' = \mathbf{a} \vee \mathbf{0}'$. Then $\mathbf{a}$ is non-$\mathbf{L}_2$ above $\mathbf{g}_0$. Now relativize Theorem 2.1 to get $\mathbf{g}_1 < \mathbf{a}$ such that $\mathbf{a}$ is r.e.a. $\mathbf{g} = \mathbf{g}_0 \vee \mathbf{g}_1$ and $\mathbf{g}_1$ is 1-generic in $\mathbf{g}_0$. By Proposition 3.4, $\mathbf{g}$ is also 1-generic.

As 1-generic degrees are $\mathbf{GL}_1$,

$$\mathbf{g}' = \mathbf{g} \vee \mathbf{0}' = \mathbf{g}_0 \vee \mathbf{g}_1 \vee \mathbf{0}' = \mathbf{g}_1 \vee \mathbf{g}_0' = \mathbf{g}_1 \vee \mathbf{a} \vee \mathbf{0}' = \mathbf{a} \vee \mathbf{0}'.$$

Hence $\mathbf{a}'' > (\mathbf{a} \vee \mathbf{0}')' = \mathbf{g}''$ and $\mathbf{a} \notin \mathbf{L}_2(\mathbf{g})$. □

As Sasso [10] proved the existence of proper $\mathbf{GL}_2$ minimal degrees, Theorem 3.5 is optimal in terms of the generalized high/low hierarchies.

With Theorem 3.5 we can improve Theorem 3.1.

Corollary 3.6. *If $\mathbf{a} \notin \mathbf{GL}_2$ and $\mathbf{c} \geq \mathbf{a} \vee \mathbf{0}'$ is r.e. in $\mathbf{a}$, then there is an* r.e.a. *degree $\mathbf{b} < \mathbf{a}$ with $\mathbf{b}' = \mathbf{c}$.*

Proof. By Theorem 3.5, fix $\mathbf{g} < \mathbf{a}$ such that $\mathbf{a}$ is r.e.a. $\mathbf{g}$ and $\mathbf{g}' = \mathbf{a} \vee \mathbf{0}'$. Now the desired $\mathbf{b}$ is given by Robinson's Jump Interpolation Theorem ([9, Theorem 2]). □

Jockusch and Posner proved the following result.

Theorem 3.7 (Jockusch and Posner [4], Corollary 7)**.** *If $\mathbf{a} \geq \mathbf{b} \notin \mathbf{GL}_2$ then there is a 1-generic $\mathbf{g} < \mathbf{a}$ such that $\mathbf{a} = \mathbf{b} \vee \mathbf{g}$.*

Now we can prove the Main Theorem with the help of their result.

Proof of the Main Theorem. By Theorem 3.5, choose a 1-generic $\mathbf{g}_0 < \mathbf{b}$ such that $\mathbf{b}$ is r.e.a. $\mathbf{g}_0$ and $\mathbf{b} \notin \mathbf{L}_2(\mathbf{g})$. Then relativize Theorem 3.7 to yield $\mathbf{g}_1$ such that $\mathbf{g}_1$ is 1-generic in $\mathbf{g}_0$, $\mathbf{g}_0 \vee \mathbf{g}_1 < \mathbf{a}$ and $\mathbf{a} = \mathbf{b} \vee \mathbf{g}_0 \vee \mathbf{g}_1$. Hence $\mathbf{a}$ is r.e.a. $\mathbf{g} = \mathbf{g}_0 \vee \mathbf{g}_1$ and $\mathbf{g}$ is 1-generic by Proposition 3.4 again. □

Corollary 3.8. *There are continuum many mutually incomparable upper cones which are incomparable with* $[\mathbf{0}', \infty)$ *and which consist of degrees r.e.a. 1-generics.*

Proof. We may fix $A \in \mathbf{a} \notin \mathbf{GL}_2$ with $\mathbf{a}$ incomparable with $\mathbf{0}'$. Then we may build an A-pointed perfect tree T such that all branches of T are $\leq_{\mathrm{T}}$-incomparable with $\emptyset'$ and mutually $\leq_{\mathrm{T}}$-incomparable. Thus the upper cones determined by the degrees of branches of T are as desired. □

4. Remarks

One may find that degrees proved r.e.a. in this paper are always r.e.a. 1-generics. Proposition 3.4 plays an important role here by allowing relativization arguments over 1-generics. Actually there are many interesting connections between r.e.a. and 1-generic degrees. For example, Jockusch [3] proved that each 1-generic degree is r.e.a., Kumabe [5] improved this by showing that n-generics are always r.e.a. other n-generics, and, in some unpublished notes, Liang Yu showed that every r.e.a. degree is a join of two 1-generics.

One more closed relation between r.e.a. and 1-generic degrees is demonstrated below.

Proposition 4.1. *If* $\mathbf{a}$ *is r.e.a. then it is r.e. in some 1-generic* $\mathbf{g}$.

Proof. If $\mathbf{a}$ is r.e.a. $\mathbf{b}$, then by combining a trick of Shore (see [11, VI.3.9]) and the coding strategy in the proof of Proposition 3.3, we may find a 1-generic $\mathbf{g}_0 < \mathbf{a}$ such that $\mathbf{g}_0' = \mathbf{a} \vee \mathbf{0}'$.

Now relativized Corollary 2.5 to get $\mathbf{a}$ r.e. in $\mathbf{g}_0 \vee \mathbf{g}_1$ with $\mathbf{g}_1$ 1-generic in $\mathbf{g}_0$. By Proposition 3.4, $\mathbf{g} = \mathbf{g}_0 \vee \mathbf{g}_1$ is 1-generic. □

These facts together lead to the following question.

Question 4.2. *Is every r.e.a. degree r.e.a. some 1-generic?*

On the other hand, it follows from Sacks Density Theorem for r.e. degrees that there are no minimal r.e.a. degrees. Now let

$$\mathcal{B} = \{\mathbf{b} | \forall \mathbf{a} \geq \mathbf{b}(\mathbf{a} \text{ is r.e.a.})\}.$$

Question 4.3. *Does* $\mathcal{B}$ *have minimal elements?*

Finally Slaman in discussions with Wang and Yu proposed that 2-generic degrees are also bases of upper cones of r.e.a. degrees. So one may wonder what $\mathcal{B}$ is.

Question 4.4. *Characterize the degrees in* $\mathcal{B}$.

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