ON Σ_1-STRUCTURAL DIFFERENCES AMONG ERSHOV HIERARCHIES

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ABSTRACT. We show that the structure $\mathcal{R}$ of recursively enumerable degrees is not a Σ_1-elementary substructure of $\mathcal{D}_n$, where $\mathcal{D}_n$ ($n > 1$) is the structure of n-r.e. degrees in Ershov hierarchy.

1. Introduction

This paper studies the differences between various substructures in Ershov hierarchy and separates the class of r.e. degrees from n-r.e. ones ($n \geq 2$) by a Σ_1-sentence with r.e. parameters. First of all, let us recall some background and introduce the necessary definitions and notations.

The notion of n-r.e. sets was introduced by Putnam [9] and Gold [8] in the middle of 1960's. The classes of n-r.e.sets form part of a more general structure called *Ershov hierarchy* (see Ershov [5], [6] and [7]).

Definition 1.1. *Let n be a positive natural number.*

(i) *A set A is said to be n-r.e. if there is a recursive function $f : \omega \times \omega \to \omega$ such that for each m,*
  - $f(0, m) = 0$*;*
  - $A(m) = \lim_s f(s, m)$*;*
  - $|\{s | f(s + 1, m) \neq f(s, m)\}| \leq n.$

(ii) *A Turing degree is said to be n-r.e. if it contains an n-r.e. set.*

Our main focus is the partially ordered structures $\mathcal{D}_n = (D_n, \leq)$ where D_n denotes the collection of n-r.e. degrees and $\leq$ is the Turing reducibility. For historical reasons, we use $\mathcal{R}$ to denote $\mathcal{D}_1$, since 1-r.e. sets are exactly the usual recursively enumerable sets. The class of n-r.e. degrees forms a substructure of $\mathcal{D}(\leq \mathbf{0}')$, the Δ^0_2-degrees.

One natural question to ask is whether the degree structure becomes more complicated when one increases the number of changes in the approximation. As we will discuss later, the structure $\mathcal{D}_2$ is significantly more complex than $\mathcal{R}$. The following so-called Downey Conjecture challenges us to explore the structural differences among $\mathcal{D}_n$ and $\mathcal{D}_m$ for $n, m > 1$:

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Conjecture 1.2 (Downey [4]). *For each $n, m > 1$, the structures $\mathcal{D}_n$ and $\mathcal{D}_m$ are elementarily equivalent.*

Recall that two structures $\mathfrak{A}$ and $\mathfrak{B}$ over the same language $\mathcal{L}$ are called *elementarily equivalent*, written $\mathfrak{A} \equiv \mathfrak{B}$, if for every sentence σ over $\mathcal{L}$, $\mathfrak{A} \models \sigma$ if and only if $\mathfrak{B} \models \sigma$. If we restrict the sentences σ to Σ_k-ones, we get the more refined notion of $\mathfrak{A}$ being *Σ_k-elementarily equivalent* to $\mathfrak{B}$, written $\mathfrak{A} \equiv_k \mathfrak{B}$.

Given two structures $\mathfrak{A}$ and $\mathfrak{B}$ over the same language $\mathcal{L}$, we say that $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$, written $\mathfrak{A} \subseteq \mathfrak{B}$ if $A \subseteq B$ (here we use the corresponding roman letter to denote the universe of the structure) and the identity map $id : A \to B$ is a homomorphism, that is, the interpretation of any symbol (in the language $\mathcal{L}$) in $\mathfrak{A}$ is the restriction of its interpretation in $\mathfrak{B}$.

If $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, there is a finer notion to gauge the structural differences by allowing parameters from the universe of $\mathfrak{A}$. More precisely, let $\mathcal{L}_A$ be the extended language $\mathcal{L} \cup \{\mathbf{a} : a \in A\}$ obtained by adding a constant symbol $\mathbf{a}$ for each element a in A.

Definition 1.3. *Let n be a natural number. We say that $\mathfrak{A}$ is a Σ_n-substructure of $\mathfrak{B}$, written $\mathfrak{A} \preceq_n \mathfrak{B}$, if for all Σ_n-formulas $\varphi(x_1, x_2, ..., x_n)$ and all $a_1, a_2, ..., a_n \in A$,*

$$\mathfrak{A} \models \varphi(\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n) \text{ if and only if } \mathfrak{B} \models \varphi(\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n).$$

We now return to the historical development which leads to Downey's Conjecture.

First observe that any elementary differences among $\mathcal{D}_n$ would not occur at the Σ_1-level: For any Σ_1-sentence σ, $\mathcal{D}_n$ or $\mathcal{D}(\leq \mathbf{0}')$ satisfies σ if and only if σ is consistent with the theory of partial orderings (see, for example, some exercises in Soare [14]). Therefore,

Theorem 1.4 (Folklore). *For any $m, n \in \omega$, $\mathcal{D}_m \equiv_1 \mathcal{D}_n$ and $\mathcal{D}_m \equiv_1 \mathcal{D}(\leq \mathbf{0}')$.*

The elementary difference between $\mathcal{D}_n$ and $\mathcal{D}(\leq \mathbf{0}')$ shows up at Σ_2-level: On the one hand, Lachlan observed that $\mathcal{D}_n$ is downward dense; on the other hand, Sacks [10] showed that there are Δ^0_2-minimal degrees.

Theorem 1.5. *For all $n \geq 1$, $\mathcal{D}_n \not\equiv_2 \mathcal{D}(\leq \mathbf{0}')$.*

The elementary difference between $\mathcal{R}$ and $\mathcal{D}_n$ ($n > 1$) was first revealed at Σ_3-level by Arslanov [1] who showed that every element in $\mathcal{D}_n$ is cuppable, whereas in $\mathcal{R}$ there exist noncuppable elements by Cooper and Yates. Later many differences at Σ_2-level were discovered, for example, the following pair of theorems offers perhaps the clearest order-theoretic difference:

Theorem 1.6 (Sacks [11]). *$\mathcal{R}$ is dense.*

Theorem 1.7 (Cooper, Harrington, Lachlan, Lempp and Soare [3]). *For each natural number $n > 1$, maximal elements exist in $\mathcal{D}_n$.*

Therefore,

Corollary 1.8. *For each natural number $n > 1$, $\mathcal{R} \not\equiv_2 \mathcal{D}_n$.*

Historically, it was Downey [4] who demonstrated a Σ_2-difference by showing that diamond exists in $\mathcal{D}_n$, which motivated him to propose his Conjecture 1.2. Recently Arslanov, Kalimullin and Lempp announced a negative solution to Conjecture 1.2. They proved the following result.

Theorem 1.9 (Arslanov, Kalimullin and Lempp [2])**.** $\mathcal{D}_2 \not\equiv_2 \mathcal{D}_3$.

They further conjectured that their technique could be generalized to show that $\mathcal{D}_n \not\equiv_2 \mathcal{D}_m$, thus answer Downey Conjecture negatively. Assuming their conjecture is correct, by Theorem 1.4, Theorem 1.5 and Corollary 1.8, the only remaining question is whether one structure can be a Σ_1-elementary substructure of the other.

The first remarkable result related to the Σ_1-elementary substructure was obtained by Slaman [13] in 1983.

Theorem 1.10 (Slaman)**.**

(i) *There are r.e. sets A, B and C and a Δ_2^0-set X such that*
- $\emptyset <_T X \leq_T A$;
- $C \not\leq_T B \oplus X$;
- *for all r.e. set W $(\emptyset <_T W \leq_T A \Rightarrow C \leq_T B \oplus W)$.*

(ii) *For each natural number $n \geq 1$, $\mathcal{D}_n \not\preceq_1 \mathcal{D}(\leq \mathbf{0}')$.*

Theorem 1.10 naturally leads to the question whether or not $\mathcal{D}_m \preceq_1 \mathcal{D}_n$ for $m < n$, in particular, the following open problem by Slaman (see [3]):

Question 1.11 (Slaman)**.** *Do the r.e. degrees form a Σ_1-substructure of the d.r.e. degrees?*

We gave a negative answer in this paper.

First notice that by Lachlan's observation that every nonrecursive n-r.e. degree bounds a nonrecursive r.e. degree, one cannot hope that any n-r.e. degree D plays the role of X as in Theorem 1.10. However we can introduce another parameter to control the r.e. degrees below the degree of D. More precisely, we have the following result:

Theorem 1.12. *There are r.e. sets A, B, C and E and a d.r.e. set D such that*

(1) *$D \leq_T A$ and $D \not\leq_T E$;*
(2) *$C \not\leq_T B \oplus D$;*
(3) *for all r.e. set W $(W \leq_T A \Rightarrow$ either $C \leq_T B \oplus W$ or $W \leq_T E)$.*

Assuming Theorem 1.12, we can obtain the following result:

Theorem 1.13. *For all $n > 1$, $\mathcal{R} \not\preceq_1 \mathcal{D}_n$.*

Proof. Assume $n > 1$. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$ be the degrees of their corresponding sets as in Theorem 1.12. Note all of them except $\mathbf{d}$ belong to $\mathcal{R}$ and $\mathbf{d}$ belongs to $\mathcal{D}_n$.

Let $\varphi(x_1, x_2, x_3, x_4)$ be the following Σ_1-formula:

$$\exists d \exists g (d \leq x_1 \wedge d \not\leq x_4 \wedge x_2 \leq g \wedge d \leq g \wedge x_3 \not\leq g).$$

By Theorem 1.12 and taking $\mathbf{g} = \mathbf{b} \cup \mathbf{d} \in \mathcal{D}_n$, we have

$$\mathcal{D}_n \models \varphi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}).$$

However, by Theorem 1.12 again,

$$\mathcal{R} \models \forall \mathbf{w} \forall \mathbf{g}((\mathbf{w} \leq \mathbf{a} \wedge \mathbf{g} \geq \mathbf{w} \wedge \mathbf{g} \geq \mathbf{b} \wedge \mathbf{w} \not\leq \mathbf{e}) \implies \mathbf{c} \leq \mathbf{g}).$$

In other words,

$$\mathcal{R} \models \neg\varphi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}).$$

□

The rest of the paper is devoted to the proof of Theorem 1.12. Notations and terminologies are standard and generally follow Soare [14]. The basic knowledge of tree constructions in recursion theory is assumed. It will be helpful if the reader is familiar with the construction of a Slaman triple, see Shore and Slaman [12].

We use capital Greek letters such as Φ to denote Turing functionals, and the corresponding lower case letter $\varphi(A; x)$ to denote the use function for $\Phi(A; x)$. If the Turing functional Φ applies to the join of two sets X and Y, we will write $\Phi(XY)$ instead of $\Phi(X \oplus Y)$. During the course of a construction, whenever we define a parameter as *fresh*, we mean that it is defined as the least natural number which is greater than any number mentioned so far. We assume that the priority tree grows *upwards*.

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2. Description of Strategies

2.1. **List of Requirements.** Fix recursive enumerations of Turing functionals $\{\Phi_e\}_{e\in\omega}$, $\{\Psi_e\}_{e\in\omega}$ and $\{\Theta_e\}_{e\in\omega}$ and of r.e. sets $\{W_i\}_{i\in\omega}$. We have the following requirements:

- M_e: $D \neq \Psi_e(E)$;
- N_e: $C \neq \Theta_e(BD)$;
- $R_{e,i}$: $\Phi_e(A) = W_i$ implies $\exists\Gamma(\Gamma(BW_i) = C)$ or $\exists\Delta(\Delta(E) = W_i)$.

Plus the global requirement:

- P: $\exists\Omega(\Omega(A) = D)$.

2.2. **Description of individual strategies.** The strategy to satisfy P-requirement is to build a Turing functional Ω such that $\Omega(A) = D$. Since D is a d.r.e. set, whenever a number x enters or leaves D, we must guarantee that some number less than or equal to the use $\omega(x)$ enters A. It has a positive effect on A. As we shall see that a number is enumerated into D only by M-strategies, we will let the M-strategies define $\Omega(A)$.

The strategy μ to satisfy M-requirement, say M_e, is the usual Friedberg-Muchnik diagonalization.

(1) Pick a diagonalization witness x, wait for $\Psi_e(E;x)\downarrow=0$. Since M must help P to define $\Omega(A)$, μ picks a fresh number x_0, and saves x_0 and $x_1=x_0+1$ for coding $D(x)$ into A. Define $\Omega(A;x)=0$ with the use block $\{x_0,x_1\}$.
(2) When $\Psi_e(E;x)\downarrow=0$, put x into D, preserve E up to the use $\psi(x)+1$, correct $\Omega(A;x)$ by putting x_1 into A and define $\Omega(A;x)=1$.

It has two outcomes: 0 indicating we have reached step (2); 1 indicating we are waiting forever.

It has finitary positive effect on D and finitary negative effect on E.

As D is a d.r.e. set, the witness x might be extracted out of D at later stages. We need to make sure that this happens only when a higher priority N-strategy acts. Thus M would be initialized. The initialization is done by putting x_0 into A to correct $\Omega(A;x)$ again, discarding the witness x forever and the starting over the M-strategy.

The strategy α for R-requirement, say $R_{e,i}$, originated from the idea of building a Slaman triple as in Shore and Slaman [12]. Thus an R-requirement will be satisfied by an R-strategy α together with infinitely many N-strategies. The R-strategy only deals with the definition of the functional Γ. The correction of Γ and the construction of Δ will be spread out to the subsequent N-strategies.

The strategy α for $R_{e,i}$ measures the length of agreement between $\Phi_e(A)$ and W_i. During the expansionary stages, we extend the definition of $\Gamma(BW_i)$. Let y be the least number not yet in the domain of $\Gamma(BW_i)$. Define $\Gamma(BW_i;y)=C(y)$ with a fresh use $\gamma(y)$.

An R-strategy α has two outcomes: ∞ indicating there are infinitely many α-expansionary stages; 0 finitely many. When α has outcome ∞, it enumerates more axioms into Γ; when it has outcome 0, it adds an finitary restraint on A, which will be done automatically by the design of tree.

The strategy β for an N-requirement, say N_e, has two components: one is a gap/co-gap strategy for the sake of higher priority R-strategies; the other is the Friedberg-Muchnik diagonalization strategy for $\Theta_e(BD)\neq C$.

We begin with describing the action of a single N-strategy with the presence of a single higher priority R-strategy. The complication with more R-strategies is then discussed. After that, we consider the interaction between two N-strategies, where the advantage of having Δ and the roles which the set E plays become apparent.

The N-strategy β operates as follows. We drop the indices e in Θ_e and i in W_i.

(1) Choose a *threshold* parameter z and a *diagonal witness* y.
(2) Wait until a stage s at which $\Theta(BD;y)\downarrow=0[s]$. Preserve B and C on all numbers $\leq s$ and go to Step (3).
(3) *Open an A-gap* by dropping the restraint on A. This allows A and D to change and indirectly W can change also. Wait for the next stage t when β is accessible again (note that it is necessary that t is an R-expansionary stage). Go to Step (4)
(4) There are four possibilities:

(a) $D \upharpoonright (\theta(y)+1)[s] = D \upharpoonright (\theta(y)+1)[t]$ and $W \upharpoonright (\gamma(z)+1)[s] \neq W \upharpoonright (\gamma(z)+1)[t]$.
Go to Step (5).

(b) $D \upharpoonright (\theta(y)+1)[s] = D \upharpoonright (\theta(y)+1)[t]$ and $W \upharpoonright (\gamma(z)+1)[s] = W \upharpoonright (\gamma(z)+1)[t]$.
Then we enumerate $\gamma(z)$ into B. This will make $\Gamma(BW)$ partial. *Close the A-gap* by adding a restraint on A to preserve the length of agreement between $\Phi(A)$ and W for the R-strategy. For each number $v \leq \gamma(z)$ for which $\Delta(E; v)$ is undefined, define $\Delta(E; v) = W(v)$. Here the role which E plays is not significant, we have to wait until we have more than one N-strategies. We use g to indicate this outcome.

(c) $D \upharpoonright (\theta(y)+1)[s] \neq D \upharpoonright (\theta(y)+1)[t]$ and $W \upharpoonright (\gamma(z)+1)[s] = W \upharpoonright (\gamma(z)+1)[t]$.
Do the same as in (b).

(d) $D \upharpoonright (\theta(y)+1)[s] \neq D \upharpoonright (\theta(y)+1)[t]$ and $W \upharpoonright (\gamma(z)+1)[s] \neq W \upharpoonright (\gamma(z)+1)[t]$.
(Under the assumption that no other N-strategies are present) The change of D must have been done by some lower priority M-strategies, which enumerate their diagonalization witnesses into D between the stages s and t. Thus we can extract all these numbers out of D to recover D_s.
Go to Step (5).

(5) Enumerate y into C, preserve the sets D and B up to $\theta(y)$. We have won the diagonalization part of N without interfering R. We use d to indicate this outcome.

The phenomenon in Step (4)(d) will be more and more important in later sections. Thus we introduce the terminology recoverable. We say that a computation $\Theta(BD; y)\downarrow [s]$ is *recoverable* at stage $t > s$ if $B \upharpoonright (\theta(y)+1)[s] = B \upharpoonright (\theta(y)+1)[t]$ and $D \upharpoonright (\theta(y)+1)[s] \subseteq D \upharpoonright (\theta(y)+1)[t]$.

3. Modified Strategies

3.1. **One N- and Many R-strategies.** We now describe a single N-strategy β in the environment of many higher priority R-strategies. Let $\alpha_0 \subset \cdots \subset \alpha_n$ be all active (to be defined precisely later) R-strategies in decreasing order of priority that β has to deal with. Each α_i enumerates Γ_i and maintains equality between $\Gamma(BW_i)$ and C. We further assume that β is accessible only during α_i-expansionary stages for all $i \leq n$.

The N-strategy β acts as follows.

(1) From $i = 0$ to n, choose a fresh threshold parameter z_i for α_i and a fresh diagonal witness y, if they have not been chosen; and go to Step (2).

(2) Wait until a stage s at which $\Theta_e(BD; y)\downarrow = 0[s]$. Preserve B and C on all numbers $\leq s$ and go to Step (3.0).

(3.i) *Open an i-gap* by dropping the restraint on A when β last reached (4.i)(b). Wait until the next stage t_i when β is accessible. Go to Step (4.i).

(4.i) There are four possibilities depending on the combination of whether W_i- or D-changes between stage s and t_i.

(a) $D \upharpoonright (\theta(y)+1)[s] = D \upharpoonright (\theta(y)+1)[t_i]$ and $W_i \upharpoonright (\gamma_i(z_i)+1)[s] \neq W_i \upharpoonright (\gamma_i(z_i)+1)[t_i]$.
Then we cancel the values of z_i. If $i < n$ then go to Step (3.i+1). If $i = n$ then go to Step (5).

(b) $D \upharpoonright (\theta(y)+1)[s] = D \upharpoonright (\theta(y)+1)[t_i]$ and $W_i \upharpoonright (\gamma_i(z_i)+1)[s] = W_i \upharpoonright (\gamma_i(z_i)+1)[t_i]$.
Then for any $j < i$, cancel the values of z_i. *Close the i-gap* by restraining A to preserve the length of agreement between W_i and $\Phi_{e_i}(A)$. For any $j \geq i$, enumerate $\gamma_j(z_j)$ into B; drop all restraint on B and C. For each number $v \leq \gamma_i(z_i)$ for which $\Delta_i(E;v)$ is undefined, define $\Delta_i(E;v) = W_i(v)$. We use g_i to indicate this outcome.

(c) $D \upharpoonright (\theta(y)+1)[s] \neq D \upharpoonright (\theta(y)+1)[t_i]$ and $W_i \upharpoonright (\gamma_i(z_i)+1)[s] = W_i \upharpoonright (\gamma_i(z_i)+1)[t_i]$.
Do the same as in (b).

(d) $D \upharpoonright (\theta(y)+1)[s] \neq D \upharpoonright (\theta(y)+1)[t_i]$ and $W_i \upharpoonright (\gamma_i(z_i)+1)[s] \neq W_i \upharpoonright (\gamma_i(z_i)+1)[t_i]$.
(Under the assumption that no other N-strategies are present) extract all numbers out of D which were enumerated into D between the stages s and t_i, to recover D_s. Cancel the values of z_i. If $i < n$ then go to Step (3.i+1). If $i = n$ then go to Step (5).

(5) Let $t_n = t$. Enumerate y into C. For all $i \leq n$, if $\gamma_{i,t}(z_i)$ is defined, then enumerate it into B and preserve the sets D and B up to $\theta(y)[t]$. We use d to indicate the outcome.

3.2. **More than one N-strategies.** When more than one N-strategies, say N_0 and N_1 with N_0 having higher priority than N_1, are present, the extraction done by N_1 could have irreversible impact on N_0 such that the computation $\Theta_0(BD;y_0)$ at N_0 is not recoverable. Thus we have to make use of the set E so that we can correct the functional $\Delta_0(E)$. Roughly speaking, when N_1's extraction makes a computation $\Theta_0(BD;y_0)$ at N_0 non-recoverable, we will enumerate numbers into E so that $\Delta(E) = W_i$ can be maintained. Roughly speaking, though not entirely true, the set E codes those r.e. set W_i whose changes coincide with elements leaving D.

First we describe the scenario which illustrates the points in the paragraph above.

The setting consists of two R-strategies α_0 and α_1; two N-strategies β_0 and β_1; and one M-strategy μ. We assume that

$$\alpha_0\hat{}\infty \subset \alpha_1\hat{}\infty \subset \beta_0\hat{}g_1 \subset \beta_1\hat{}g_0 \subset \mu.$$

- At stage s_0, β_0's g_1-gap is open; β_1's g_0-gap is open, and μ is accessible. Assume that μ puts its witness x into D.
- At stage $s_1 > s_0$, β_0's g_1-gap is closed as in (4)(a). β_1 is not accessible, hence its g_0-gap remains open.
- At stage $s_2 > s_1$, β_0's g_1-gap is opened again, however, its computation $\Theta(BD;y_0) = 0[s_2]$ has used the information that x is in D; β_1's g_0-gap is

closed as in (4)(d), β_1 extracts x out of D, making $\Theta(BD; y_0)[s_2]$ not recoverable, and has outcome 0 forever.
- At stage $s_3 > s_2$, β_0 wants to close its g_1-gap as in Step (4)(d), however it cannot put x back into D to recover the computation $\Theta(BD; y_0) = 0[s_2]$. Thus β_0 has to act as in Step (4)(b) by putting $\delta(v)$ into E and redefine $\Delta(E; v)$ with fresh use $\delta(v)$. This would cancel the impact of the extraction of N_1.

Now a new type of conflict occurs. By putting $\delta(v)$ into E, β_0 may injure some M-strategy μ'. The worry is that an extraction of a big x always coincide with a small v entering W, thus an M-strategy μ' extending $\beta_0\hat{\ }g_1$ always gets injured by $\delta(v)$. This new conflict can be solved by restraining A bit by bit.

4. The Construction

4.1. **Description of the priority tree T.** Fix a recursive priority list of the requirements

$$R_0, N_0, M_0, R_0, N_1, M_1, \ldots$$

where it is understood that the index d in R_d is the canonical code of the pair $\langle e, i\rangle$. We label T inductively as follows. We label each node on T with a requirement. We identify a node on T with the strategy of satisfying its labelling requirement. The root node on T is labelled R_0. Suppose that τ is a node on T. If τ is labelled R_e then τ has two outgoing edges labelled ∞ and 0, from left to right. We say that a node α labelled R is *active at* τ if for every node α' with $\alpha' \subset \alpha \subset \tau$ there is no β with $\alpha \subset \beta\hat{\ }g_{\alpha'} \subset \tau$. If τ is labelled N_e then τ has $n+2$ outgoing edges $d, g_{\alpha_{n-1}}, \ldots, g_{\alpha_1}, g_{\alpha_0}, w$ from left to right where α_i is labelled R_i and α_i is active at τ. If τ is labelled M_e then τ has two outgoing edges labelled 0 and 1, from left to right.

We say that a requirement U is *represented* by σ at a node τ if $\sigma \subset \tau$ and one of the following conditions holds: U is M_e and σ is labelled M_e; or U is R_e and σ is labelled R_e and is active at τ; or U is N_e and either $\sigma\hat{\ }w$ or $\sigma\hat{\ }d \subseteq \tau$.

Continuing the inductive definition of T, if all $\alpha \subset \tau$ have been labelled, then τ is labelled with the highest priority O such that O is not represented by any $\sigma \subset \tau$.

A left-right order $\leq$ can be naturally put on tree T, namely, for any two nodes σ, τ on T, $\sigma \leq \tau$ if and only if σ is to the left of τ or σ is an initial segment of τ.

4.2. **Parameters and Initialization.** Let τ be a node on T. We now list a collection of parameters associated with τ, which will be used in the construction. During the construction, the value of a parameter p may be updated. Thus, strictly speaking, a parameter is a function of stage s, though we will not mention s explicitly below for simplicity.

(1) If τ is labelled M_e, it has a parameter x targeting D as the witness to diagonalize against $\Psi_e(E)$. For each x, M_e also has a *use block* $\{x_0, x_1 = x_0 + 1\}$ for x which is used for correcting the functional $\Omega(A; x)$.
(2) If τ is labelled R_d and $d = \langle e, i\rangle$, then it has parameters l which is the length of agreement function between $\Phi_e(A)$ and W_i, and a finite restraint r to preserve l. It also builds a functional Γ_τ.
(3) If τ is labelled N_e, then it has the following parameters:

- a stage s^- at which τ was accessible for the last time; if τ has never been accessible before, $s^- = 0$;
- A finite permanent restraint on A, B, C, D and E;
- sequences of alternating restraints on the triple A, D and E, and on the pair B and C;
- a finite set of R-strategies $\alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_n$, which are active at τ;
- for each α in the list of active R-strategies, τ has a threshold z_α trying to drive $\Gamma_\alpha(BW)$ partial at z_α, and builds a functional $\Delta_\alpha(E)$;
- a uniformly recursive set of numbers that will enter B, in fact, the set consists of all uses $\gamma_{\alpha'}(z_{\alpha'})$ for some N-strategy β' such that $\beta'\hat{\ }g_{\alpha'} \subset \beta$;
- a number y, targeting C as the witness to diagonalize against $\Theta_e(BD)$;

To prevent a computation $\Delta_\alpha(E)$ from being injured by that corrections of other $\Delta'_{\alpha'}(E)$, we add the following *convention* on the definition of Δ_α: If a computation $\Delta_\alpha(E; v)$ becomes undefined because of other strategies putting some delta use into E, then we redefine $\Delta_\alpha(E; v)$ to be the same value with the same use. This convention is certainly reasonable, since the more E changes, the easier the defining of $\Delta_\alpha(E)$ becomes.

When a node τ is *initialized* at stage s, all witnesses, thresholds and functionals get cancelled and discarded forever; the stage parameter s^- is set to s, and the computation of expansionary stages will be re-started from stage s.

4.3. **Construction.** We now describe the stage by stage construction. At stage s, we first specify a string TP_s of length less than or equal to s, called the *accessible string*, then act along the accessible string.

The accessible string is defined inductively from the root. The root of the priority tree T is always accessible.

At the inductive step, suppose that the node τ is accessible. If the length of τ is equal to s then we let $\tau = \mathrm{TP}_s$ and go to next stage.

Suppose that the length of τ is less than s. We decide the outcome o, let $\tau\hat{\ }o$ be accessible and take the actions based on the label of τ as follows.

(1) The node is an $R_{\langle e,i\rangle}$-strategy α.
Check if s is an α-expansionary stage. If not, then let $o = 0$ and do nothing. If yes, then let $o = \infty$; choose the least number z such that $\Gamma_\alpha(BW_i; z)[s]$ is undefined, define $\Gamma_\alpha(BW_i; z) = C(z)[s]$ with a fresh use $\gamma(z)$.

(2) The node is an N_e-strategy β.
Let us assume that the diagonalization parameter $y = y_\beta$ and the threshold parameters z_α for each α in its active list are defined, otherwise simply define it to be the least fresh number.
Case (2.1) The outcome at stage s^- was w.
If $\Theta_e(BD; y)\uparrow [s]$ or $\Theta_e(BD; y)\downarrow\neq 0[s]$, then let o be w. Otherwise, that is $\Theta_e(BD; y)\downarrow= 0[s]$, preserve B and C on all numbers less than or equal to s, open an α_0-gap (the action of open gap will be described below) and let g_{α_0} be the outcome.
Case (2.2.j) The outcome at stage s^- was g_{α_j}.

Suppose that the α_j-gap was closed at stage s^-. Then let the outcome be w and do noting.

Suppose that the α_j-gap was opened at stage s^-. Let i be the index of the W_i which the node α_j is working for. There are two cases depending on what happened to W_i and D between stage s^- and s.

(a) The computation $\Theta_e(BD;y)\downarrow= 0[s^-]$ is recoverable and

$$W_i \upharpoonright (\gamma_{\alpha_j}(z_{\alpha_j})+1)[s^-] \neq W_i \upharpoonright (\gamma_{\alpha_j}(z_{\alpha_j})+1)[s].$$

Then we *recover* the computation $\Theta_e(BD;y)\downarrow= 0[s^-]$, that is, we extract all numbers x in $D_s \setminus D_{s^-}$ and enumerate the x_0 in the use block of $\Omega(A;x)$ into A; cancel the values of z_{α_j}. If $j<n$ then open an $\alpha_{(j+1)}$-gap and let $g_{\alpha_{j+1}}$ be the outcome. If $j=n$ then enumerate y into C; for all $k \leq n$, if $\gamma_{\alpha_k}(z_{\alpha_k})[s^-]$ is defined, then enumerate it into B and preserve the sets D and B up to $\theta_e(y)[s]$; let d be the outcome; go to next stage and initialize all nodes $\geq \beta\hat{}d$.

The action of *open an α_j-gap* is simply drop the restraint on A when β last reached (2.2.j)(b).

(b) Otherwise, that is either the computation $\Theta_e(BD;y)\downarrow= 0[s^-]$ is not recoverable and $W_i \upharpoonright (\gamma_{\alpha_j}(z_{\alpha_j})+1)[s^-] \neq W_i \upharpoonright (\gamma_{\alpha_j}(z_{\alpha_j})+1)[s]$; or $W_i \upharpoonright (\gamma_{\alpha_j}(z_{\alpha_j})+1)[s] = W_i \upharpoonright (\gamma_{\alpha_j}(z_{\alpha_j})+1)[s^-]$.

Then for any $k<j$, cancel the values of z_{α_k}. *Close the α_j-gap* by restraining A to preserve the length of agreement between W_i and $\Phi_{\alpha_j}(A)$. For any $k \geq j$, enumerate $\gamma_{\alpha_k}(z_{\alpha_k})$ into B; drop all restraint on B and C. Find the least v (if any) in the domain of Δ_{α_j} such that $\Delta_{\alpha_j}(E;v) \neq W_i(v)$, enumerate $\delta_{\alpha_j}(v)$ into E. Then find the least v such that $\Delta_{\alpha_j}(E;v)$ is undefined, define $\Delta_{\alpha_j}(E;v) = W_i(v)$ with a fresh use $\delta_{\alpha_j}(v)$. Let g_{α_j} be the outcome.

(3) The node is an M_e-strategy μ.

Let us assume that the witness parameter x has been defined together with the use block $\{x_0,x_1\}$ for $\Omega(A;x)=D(x)=0$, otherwise define x fresh, then define $\Omega(A;x)=D(x)=0$ with fresh use block. If $\Psi_e(E;x)\uparrow[s]$ or $\Psi_e(E;x)\downarrow\neq 0[s]$, then let outcome be 1. Otherwise, that is $\Psi_e(E;x)\downarrow= 0[s]$, put x into D, preserve E up to the use $\psi(x)+1$, correct $\Omega(A;x)$ by putting x_1 into A and redefine $\Omega(A;x)=1$, end the stage.

At the end of the stage, initialize all nodes to the right of the accessible string. This finishes the construction.

5. Verification

We now verify that the construction works. We begin with the lemma stating that the true path exists.

Lemma 5.1. *For any $e\in\omega$, there is a unique node α on T such that α is the leftmost one of length e which is accessible infinitely often.*

Let TP be the *true path* in T, that is, TP is the leftmost path which is accessible infinitely often. By Lemma 5.1, TP exists and it is indeed an infinite path.

Lemma 5.2. *For every requirement O there is a node τ on TP such that for all n greater than the length of τ, O is represented by τ at TP$\upharpoonright n$.*

We skip the proof of Lemmas 5.1 and 5.2, as it is essentially the same as Lemma 3.3. in Shore and Slaman [12].

We argue by induction along TP that every requirement is satisfied. We split the proof into a few lemmas.

Lemma 5.3. *Let τ be a node on TP and O be the label of τ. Then*

(a) *Suppose that O is $R_{\langle e,i\rangle}$ and τ is the R-strategy α. Then $\alpha\hat{\ }\infty \subset TP$ if and only if there are infinitely many α-expansionary stages.*

(b) *Suppose that O is N_e and τ is the N-strategy β. Then β's witness parameter y is eventually fixed; the uniformly recursive set in its parameter list is a subset of B. Furthermore*

(b1) *if $\beta\hat{\ }d \subset TP$ then $\Theta_e(BD; y)\downarrow= 0$ and $y \in C$.*

(b2) *if $\beta\hat{\ }g_{\alpha_i} \subset TP$, then its threshold parameter z_{α_i} is eventually fixed. Moreover, the functional $\Delta_{\alpha_i}(E)$ is total and $= W$ which is the r.e. set appeared in the requirement R_{α_i};*

(b3) *if $\beta\hat{\ }w \subset TP$ then ($\Theta_e(BD; y)\uparrow$ or $\Theta_e(BD; y)\downarrow\neq 0$) and $y \notin C$;*

(c) *Suppose that O is M_e and τ is the M-strategy μ. Then μ's witness parameter x is eventually fixed.*
Furthermore

(c1) *if $\mu\hat{\ }0 \subset TP$, then $\Psi_e(E; x)\downarrow= 0$ and $x \in D$;*

(c2) *if $\mu\hat{\ }1 \subset TP$ then ($\Psi_e(E; x)\uparrow$ or $\Psi_e(E; x)\downarrow\neq 0$) and $x \notin D$.*

Proof. We prove (a), (b) and (c) by simultaneous induction.

Statement (a) follows from the construction.

We now prove (b). Let s_0 be the stage after which β will not be initialized. After stage s_0, the parameter y will be fixed. Moreover, for each N-node β' with $\beta'\hat{\ }g_{\alpha'} \subset \beta$, by induction hypothesis the parameters $z_{\alpha'}$ at β' is fixed, thus, the recursive set consists of $\gamma_{\alpha'}(z_{\alpha'})$ is a subset of B.

Suppose that $\beta\hat{\ }d \subset$ TP. Let $s_1 > s_0$ be the stage at which $\beta\hat{\ }d$ is accessible for the first time. By Case (2.2.n)(a) in the construction, we enumerated y into C and recovered the computation $\Theta(BD; y)\downarrow= 0[s^-]$; thus $y \in C$ and $\Theta(BD; y)\downarrow= 0[s_1]$. It remains to show that for all $t > s_1$, the computation $\Theta(BD; y)\downarrow= 0[s_1]$ is preserved at stage t. Since we initialize all nodes $> \beta\hat{\ }d$ at stage s_1, the computation can only be injured by the action of nodes $\leq \beta$. By the choice of s_0, we only need to consider the nodes which are $\subset \beta$. The B-side of the computation is safe: By the choice of the recursive set in the parameter list of β, the elements entering B will not injure the computation. The D-side of the computation is also safe: No M-node $\eta \subset \beta$ can put element into D, and no N-node $\beta' \subset \beta$ would extract elements out of D, as it would have an outcome to the left of β hence initialize β. This establishes (b1).

Suppose that $\beta\hat{\ }g_{\alpha_i} \subset$ TP. Let $s_1 > s_0$ be the stage after which $\beta\hat{\ }g_{\alpha_i}$ never gets initialized. Since we only cancel z_{α_i} when we reach (2.2.j)(b) for some $j > i$, the choice of s_1 guarantees that z_{α_i} is fixed after stage s_1.

By the correction done in case (2.2.i)(b), for any v in domain of $\Delta_{\alpha_i}(E)$, we have $\Delta_{\alpha_i}(E; v) = W(v)$, where W is the r.e. set appeared in the requirement R_{α_i}. We now prove that $\Delta_{\alpha_i}(E)$ is total by contradiction. Suppose that v_0 is the least number such that $\Delta_{\alpha_i}(E; v_0) \neq W(v_0)$. Let s_2 be the stage at which $\Delta_{\alpha_i}(E; v_0)$ is defined and after which no correction happens for $\Delta_{\alpha_i}(E; v)$ for $v < v_0$. By our convention, no other N-requirements will move the use $\Delta_{\alpha_i}(E; v_0)$. Thus, $\Delta_{\alpha_i}(E; v_0)$ can only change at most once, depending if v_0 enters W after s_2. This establishes (b2).

Suppose that $\beta\hat{\ }w \subset$ TP. Let $s_1 > s_0$ be the stage after which no nodes to the left of $\beta\hat{\ }w$ are accessible. By the argument in (b1), if we ever have the outcome d after stage s_0, then we will have outcome d forever. Thus, $y \notin C$. Moreover after stage s_1 we will not see $\Theta(BD; y)\downarrow = 0$, otherwise, we would open an α_0-gap and have outcome g_{α_0}. This establishes (b3) and finishes the proof of statement (b).

We now prove (c). Let s_0 be the stage after which μ will not be initialized. After stage s_0, the parameter x will be fixed.

If at some stage $s_1 > s_0$ at which μ is accessible and $\Psi_e(E; x)\downarrow = 0[s_1]$, then by case (3) in the construction $x \in D$. It remains to show that x is never get extracted out of D and the computation $\Psi_e(E; x)\downarrow = 0[s_1]$ is preserved.

By construction, each N-node β only extracts the numbers which were put into D by some M-nodes extending β. Thus only those $\beta \subset \mu$ may extract x out of D. However such extraction would initialize μ, contradicting to the choice of s_0. Thus x is never extracted out of D.

Since we end the stage s_1 after the action at μ, and all nodes $\geq \mu$ gets initialized at the end of stage s_1. The only possible injury of $\Psi_e(E; x)$ comes from the N-nodes β such that $\beta\hat{\ }g_{\alpha_i} \subset \mu$ and β puts $\delta(v)$ into E in order to correct the error $\Delta_{\alpha_i}(E; v) \neq W(v)$ for some v, where W is the r.e. set appeared in the requirement R_{α_i}. Therefore v must enter W after stage s_1. We argue that this will cause an disagreement forever between $\Phi_{\alpha_i}(A; v)$ and $W(v)$ at α_i. To avoid disagreement, A must change below the use $\varphi(v)$. That can only happen when some x_0, x_1 in the use Ω-block $\{x_0, x_1\}$ for some x enters A after s_1. However x_0, x_1 enters A only when x is out and in D respectively. By the initialization done at s_1, such x must be a witness parameter at some M-nodes $\mu' \subset \mu$. μ' cannot put x into D after stage s_1, otherwise it would initialize μ. On the other hand, no node $\beta' \subset \mu'$ can extract x out of D after s_1 by the same reason. This establishes (c1).

As argued in (b3), after stage s_0 we will never reach outcome 0. Thus $x \notin D$. Furthermore, if $\Psi_e(E; x)\downarrow = 0$ at some stage $t > s_0$ at which μ is accessible, then we would reach outcome 0. This establishes (c2). □

Finally we show that all requirements are satisfied.

Lemma 5.4. *All requirements are satisfied. More specifically,*

(1) *The P-requirement is satisfied, namely, the functional $\Omega(A)$ is total and for all natural number x, $\Omega(A; x) = D(x)$.*
(2) *For each natural number e, the requirement N_e is satisfied.*
(3) *For each natural number e, the requirement R_e is satisfied.*

Proof. Statement (1) follows from the construction.

Statement (2) is essential the argument (b1) and (b3) in Lemma 5.3.

We now verify statement (3). Let α be the node on TP which represents $R_{\langle e,i\rangle}$. Let us assume that $\Phi_e(A) = W_i$.

Fix a stage s_0, after which α never gets initialized. Clearly, since $\Phi_e(A) = W_i$, we know $\alpha\hat{\ }\infty$ is on TP. We consider two cases based on whether R_e has a global Σ_3-outcome.

Case 1. There is a node β on TP labelled N which has α in its parameter list, such that $\beta\hat{\ }g_\alpha \subset$ TP.

Then by statement (b2) in Lemma 5.3, $\Delta_\alpha(E)$ is total and $\Delta_\alpha(E) = W_i$. Thus $R_{\langle e,i\rangle}$ is satisfied by successfully build Δ_α at β.

Case 2. For all nodes β on TP labelled N such that α is in its parameter list, $\beta_p\hat{\ }g_\alpha \not\subset$ TP.

In this case we argue that the Turing functional $\Gamma_\alpha(BW_i)$ is total and equal to C.

Since we always make corrections on $\Gamma_\alpha(BW_i)$, it suffices to show that it is total. We show by induction that for all p, $\Gamma_\alpha(BW_i; p)$ is defined. Suppose that the statement is true for all $p' < p$, let s_0 be the stage after which $B \oplus W_i$ will not change on any number less than $\gamma(p')$ for all $p' < p$. Now as there are only finitely many threshold $z < p'$ and they are all located off the true path, thus eventually they are either getting cancelled or never acting. Hence $\Gamma(BW_i; p)$ is defined eventually.

This ends all verification. $\square$

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