A REDUCIBILITY RELATED TO BEING HYPERIMMUNE-FREE

FRANK STEPHAN AND LIANG YU

Abstract. The main topic of the present work is the relation that a set X is strongly hyperimmune-free relative to Y. Here X is strongly hyperimmune-free relative to Y if and only if for every partial X-recursive function p there is a partial Y-recursive function q such that every a in the domain of p is also in the domain of q and satisfies $p(a) < p(a)$, that is, p is majorised by q. For X being hyperimmune-free relative to Y, one also says that X is pmaj-reducible to Y ($X \leq_{pmaj} Y$). It is shown that between degrees not above the Halting problem the pmaj-reducibility coincides with the Turing reducibility and that therefore only recursive sets have a strongly hyperimmune-free Turing degree while those sets Turing above the Halting problem bound uncountably many sets with respect to the pmaj-relation. So pmaj-reduction is identical with Turing reduction on a class of sets of measure 1. Moreover, every pmaj-degree coincides with the corresponding Turing degree. The pmaj-degree of the Halting problem is definable with the pmaj-relation.

1. Introduction

Post [7] introduced the notions of immune and hyperimmune sets in order to search for conditions on the complements of r.e. sets which guarantee incompleteness for certain reducibilities. In the subsequent study [5, 6, 8, 10, 15] the notion of hyperimmunity played a central role and also the discovery that there are Turing degrees which do not contain a hyperimmune set, these are called the hyperimmune-free Turing degrees. This was generalised by saying that X is hyperimmune-free relative to Y if every total X-recursive function is majorised by a total Y-recursive function. One could generalise this notion as follows, where one says that a partial function q dominates a partial function p iff for almost all x in the domain of p it holds that x is also in the domain of q and $p(x) < q(x)$; furthermore, q majorises p if the domain of p is a subset of the domain of q and $p(x) < q(x)$ for all x in the domain of p. Now, the following six notions can arise in principle:

- **1d:** One total Y-recursive function dominates every total X-recursive function;
- **1m:** Every total X-recursive function is majorised by a total Y-recursive function;
- **2d:** One total Y-recursive function dominates every partial X-recursive function;

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2m: Every partial X-recursive function is majorised by a total Y-recursive function;

3d: One partial Y-recursive function dominates every partial X-recursive function;

3m: Every partial X-recursive function is majorised by a partial Y-recursive function.

Notion **1m** is closely related to the above discussed notion of hyperimmune-freeness. Notion **1d** has a strong resemblance to the low-high hierarchy: If $X \leq_T Y$ then Y dominates X in sense **1d** iff $X'' \leq_T Y'$ by a result of Martin; in particular fixing X as recursive would give that the Y range over the high degrees while fixing $Y = K$ and taking X to be Δ^0_2 would imply that X is dominated in sense 1d by Y iff X is low$_2$. The notions **2d**, **2m** and **3d** all coincide and are the strongest form of domination which one can get and imply that $X' \leq_T X \oplus Y$. Notion **3m** is the notion of pmaj-reducibility which will be discussed in the following.

So, X is strongly hyperimmune-free relative to Y iff every partial X-recursive function is majorised by a partial Y-recursive function. In contrast to hyperimmune-free degrees, it turns out that no nonrecursive Turing degree is strongly hyperimmune-free. So the more interesting part is the overall relation between sets X and Y than the special case $X \leq_{\mathrm{pmaj}} \emptyset$.

Concerning the downward closure the following result is obtained: If $X \not\geq_T K$ then $\{Y : Y \leq_{\mathrm{pmaj}} X\} = \{Y : Y \leq_T X\}$ else $\{Y : Y \leq_{\mathrm{pmaj}} X\}$ is uncountable and has measure 1. In particular, $\{Y : Y \leq_{\mathrm{pmaj}} K\}$ contains all sets which are Martin-Löf random relative to K, all sets which are low for Ω, which are jump traceable and which are Turing reducible to K.

Furthermore, if a single function q majorises all partial-recursive functions then the Turing degree of q is at least K; hence K is the first dominant degree for the pmaj-degrees. This stands in contrast to Chong's notion of pdomination: He defined that a set X is pdominant iff there is a single partial X-recursive function q such that for all partial-recursive functions p and almost all x in the domain of p there is a $y \leq x$ in the domain of q with $p(x) < q(y)$. This type of domination is easier to obtain and Chong, Hoi, Stephan and Turetsky [1] study in detail which degrees are pdominant in this sense: for example some low r.e. degrees are pdominant while other high r.e. degrees fail to be pdominant. The notion of pdominance is quite orthogonal to many known recursion-theoretic classes of oracles.

Although Shore and Slaman [13] prove that the Turing degree of Halting problem is definable and subsequently Shore [12] found a more direct definition, however both the definitions are quite sophisticated. The pmaj-reducibility can be viewed as a natural and slight modification of the Turing reducibility (in the sense of measure theory, they are identical almost everywhere). With this reduction, one may prove that the Turing degree of Halting problem can be defined in a significantly simpler way.

2. Basic facts

Now the formal definition of when a set X is strongly hyperimmune-free relative to another set Y is given.

Definition 2.1. *X is strongly hyperimmune-free relative to Y, written $X \leq_{\mathrm{pmaj}} Y$, if for every partial X-recursive function p there is a partial Y-recursive function q which majorises p, that is, which satisfies for all $n \in dom(p)$ that $n \in dom(q)$ and $p(n) < q(n)$.*

Note that here only $dom(p) \subseteq dom(q)$ is required and not $dom(p) = dom(q)$; the latter would imply that every X-r.e. set is Y-r.e. and thus $X \leq_T Y$ what would make the notion trivial and equal to Turing reducibility.

The next results give some basic facts about the notion of a set X being strongly hyperimmune-free relative to another set Z.

Theorem 2.2. *If $X \leq_{\mathrm{pmaj}} Z$ then either $X \leq_T Z$ or $K \leq_T Z$.*

Proof. Suppose that $X \leq_{\mathrm{pmaj}} Z$. Let $Y = \{X \upharpoonright n \mid n \in \mathbb{N}\}$. Note that $Y \equiv_T X$. Define a partial X-recursive function p as follows:

$$p(\sigma) = \begin{cases} \mu s(\varphi_{|\sigma|}(|\sigma|)[s] \downarrow), & \sigma \in Y; \\ 0, & \sigma \notin Y. \end{cases}$$

So there is a partial Z-recursive function q such that for any σ, if $p(\sigma)$ is defined, then $q(\sigma)$ is defined and $q(\sigma) > p(\sigma)$. Now one defines a Z-recursive function f so that $f(n)$ is the least stage s so that there are at least $2^n - 1$ many $\sigma \in \{0,1\}^n$ so that $q(\sigma)$ is defined at stage s and the value of $q(\sigma)$ is below s.

If there are only finite many n's so that $f(n)$ is below the stage s where $\varphi_n(n)$ becomes defined, then $Z \geq_T f \geq_T K$.

If there are infinitely many n's ssuch that $\varphi_n(n)$ is defined at some stage $s > f(n)$ then one can find relative to Z an increasing infinite sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$ of binary strings such that for each i and $n = |\sigma_i|$ it holds $\varphi_n(n)$ is defined but not within $f(n)$ steps and σ_i is the unique string of length n such that $q(\sigma_i)$ is either undefined or not below $f(n)$. This Z-recursive sequence of strings consists only of members of Y and so this Z-recursive sequence contains arbitrary long prefixes of X and no other strings. Hence $X \leq_T Z$. □

Theorem 2.3. *If $X \leq_{\mathrm{pmaj}} Z$ then either $X' \leq_T Z'$ or every partial X-recursive function is majorised by a total Z-recursive function.*

Proof. The proof is almost the same as Theorem 2.2. Let $Y = \{X \upharpoonright n \mid n \in \mathbb{N}\}$ and p be a partial X-recursive function as follows:

$$p(\sigma) = \begin{cases} \mu s(\varphi^X_{|\sigma|}(|\sigma|)[s] \downarrow), & \sigma \in Y; \\ 0, & \sigma \notin Y. \end{cases}$$

If there is a total Z-recursive function h majorising p, then it is clear that every partial X-recursive function is majorised by a total Z-recursive function. Otherwise, there is a partial Z-recursive function q with a coinfinite domain majorising p. Then there is a Z'-recursive strictly increasing infinite sequence of numbers $n_0, n_1, n_2, \ldots$ so that for every i, there is some unique $\sigma_i \in 2^{n_i}$ for which $q(\sigma_i)$ is undefined. Then $\sigma_i \in Y$. So there is a Z'-recursive infinite sequence of strings $\sigma_0, \sigma_1, \sigma_2, \ldots$ with $\sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \ldots \prec X$. Hence $X \leq_T Z'$. Since $X \leq_{\text{pmaj}} Z$, we have that $X' \leq_T Z'$. □

Remark 2.4. One can simplify the relation as follows: There is a partial function ψ computed using an oracle such that one can say the following: If ψ^X is majorised by a partial Y-recursive function with co-infinite domain then $X' \leq_T Y'$ and if ψ^X is majorised by a total Y-recursive function f^Y then every partial X-recursive function is majorised by a total Y-recursive function. Indeed, one can in that latter case find one fixed total Y-recursive function which dominates every partial X-recursive function.

A way to define such a ψ^X would be to do the following: $\psi^X(\langle e, a_0, a_1, \ldots, a_x \rangle)$ is $\varphi_e^X(x)$ in the case that $a_y = X(y)$ for $y = 0, 1, \ldots, x$ and is 0 in all other cases.

Furthermore, $X \leq_{\text{pmaj}} Y$ implies $X' \leq_T X \oplus Y'$. In the case that ψ^X is majorised by a partial function with coinfinite domain, it holds that $X' \not\leq_T Y'$ and hence $X' \leq_T X \oplus Y'$. In the case that some Y-recursive function majorises ψ_X, the Halting problem X' of X can be solved by simulating computations relative to X up to that Y-recursive bound, hence $X' \leq_T X \oplus Y$. Thus the formula $X' \leq_T X \oplus Y'$ is in both cases true.

One can use the previous results in order to prove that two sets are pmaj-equivalent iff they are Turing equivalent.

Theorem 2.5. *$X \equiv_{\text{pmaj}} Y$ iff $X \equiv_T Y$.*

Proof. As $X \leq_T Y \Rightarrow X \leq_{\text{pmaj}} Y$, only the other direction of the equivalence has to be proven. For that, assume $X \leq_{\text{pmaj}} Y$ and $Y \leq_{\text{pmaj}} X$.

By Theorem 2.3, as $X \leq_{\text{pmaj}} Y$, either $X' \leq_T Y'$ or every partial X-recursive function is majorised by a total Y-recursive function. So, if $X' \not\leq_T Y'$, then the partial X-recursive function $p(n) = \varphi_n^X(n)$ is majorised by a Y-recursive function h. Since $Y \leq_{\text{pmaj}} X$, there is a total X-recursive function φ_n^X so that for every k, $\varphi_n^X(k) > h(k) + 1$. Now one has that $h(n) > p(n) > h(n) + 1$, a contradiction. So $X' \equiv_T Y'$.

Hence Y has an X-recursive approximation $Y_0, Y_1, \ldots$ and one can define the following partial X-recursive function:

$$c(\sigma) = \min\{s \geq |\sigma| : \sigma \preceq Y_s\}.$$

This partial X-recursive function is majorised by a partial Y-recursive function $\tilde{c}$. Now one can define another function $\hat{c}$ with

$$\hat{c}(n) = \tilde{c}(Y(0)Y(1)\dots Y(n)).$$

The function $\hat{c}$ is majorised by a total X-recursive function f and it holds that for every n there is an s with

- $n \leq s \leq f(n)$;
- $Y_s(0)Y_s(1)\dots Y_s(n) = Y(0)Y(1)\dots Y(n)$.

As the approximation $Y_0, Y_1, \dots$ converges pointwise to Y, $Y(m)$ is the unique value a such that there is an $n > m$ with $Y_s(m) = a$ for $s = n, n+1, \dots, f(n)$. The search for this n converges always and hence $Y \leq_T X$. The converse direction $X \leq_T Y$ follows from the symmetry between X and Y in the above arguments. □

Note that this implies that the pmaj-relation cannot go into the opposite direction as the Turing reducibility.

Corollary 2.6. *There are no sets X, Y with $X <_T Y$ and $Y <_{\mathrm{pmaj}} X$.*

Proof. The Turing reducibility implies the pmaj-relation and therefore $X \leq_{\mathrm{pmaj}} Y \wedge Y \leq_{\mathrm{pmaj}} X$ what on its turn implies, as just seen, $X \equiv_T Y$. This would stand in contrast to the assumption $X <_T Y$. □

Remark 2.7. A set X is called jump traceable iff there is a recursive function f and a uniformly r.e. family $A_0, A_1, \dots$ such that for every $e \in X'$ it holds that $\varphi_e^X(e) \in A_e$ and $|A_e| \leq f(e)$. Nies [5] showed that there is a Π_1^0 class P in which every set is nonrecursive and jump traceable. Hence every $X \in P$ satisfies $X \leq_{\mathrm{pmaj}} K$. Thus, there is a set $X \leq_{\mathrm{pmaj}} K$ which has hyperimmune-free and nonrecursive Turing degree. Furthermore, there are $2^{\aleph_0}$ sets which are strongly hyperimmune-free relative to K.

Actually, one can say even more about them. Dobrinen and Simpson [2] showed that there is a function $f \leq_T K$ so that the class $\{X \mid \forall e(\varphi_e^X(e)\downarrow \implies \varphi_e^X(e) < f(e))\}$ has measure 1. Indeed, one can take f to be the convergence module of Chaitin's Ω. Then any set X which is low for Ω is strongly hyperimmune-free relative to K. These sets include all sets which are Martin-Löf random relative to K.

3. The degree of the Halting problem is definable

The degree of K plays a special role in the pmaj-degrees as it is the least degree bounding an uncountable number of other pmaj-degrees. Therefore one might ask whether one can define K within the pmaj-degrees. The main result of this section is the following one.

Theorem 3.1. *The pmaj-degree of K is definable in the partial order of the pmaj-degrees.*

The proof is based on the following definitions and subsequent results of this section.

- Call X_0, X_1 a bounding pair iff there are Y_0, Y_1 such that $Y_0 \leq_{\mathrm{pmaj}} X_0$, $Y_0 \leq_{\mathrm{pmaj}} X_1$, $Y_1 \leq_{\mathrm{pmaj}} X_0$, $Y_1 \leq_{\mathrm{pmaj}} X_1$ and there is no Z with $Y_0 \leq_{\mathrm{pmaj}} Z$, $Y_1 \leq_{\mathrm{pmaj}} Z$, $Z \leq_{\mathrm{pmaj}} X_0$ and $Z \leq_{\mathrm{pmaj}} X_1$.
- Every bounding pair X_0, X_1 satisfies $K \leq_{\mathrm{pmaj}} X_0 \vee K \leq_{\mathrm{pmaj}} X_1$.
- If L satisfies for every bounding pair X_0, X_1 that $L \leq_{\mathrm{pmaj}} X_0 \vee L \leq_{\mathrm{pmaj}} X_1$ then $L \leq_{\mathrm{pmaj}} K$.

So a bounding pair X_0, X_1 is a pair of common upper bounds of two sets Y_0, Y_1 in the pmaj-degrees for which there is no set Z being on one hand an upper bound of Y_0, Y_1 and on the other hand a lower bound of X_0, X_1. Note that the definition implies that Y_0 is pmaj-incomparable to Y_1 and X_0 is pmaj-incomparable to X_1. The reason is that if $Y_0 \leq_{\mathrm{pmaj}} Y_1$ then one could choose $Z = Y_1$ and similarly in the other cases. Now the pmaj-degrees of the class

$$\{S \mid S \leq_{\mathrm{pmaj}} X_0 \text{ or } S \leq_{\mathrm{pmaj}} X_1 \text{ for every bounding pair } X_0, X_1\}$$

has a greatest degree and that degree is exactly the one of K. The verification of this definition follows from the next two propositions.

Proposition 3.2. *For every bounding pair X_0, X_1 it holds that either $K \leq_{\mathrm{pmaj}} X_0$ or $K \leq_{\mathrm{pmaj}} X_1$.*

Proof. Let Y_0, Y_1 witness that X_0, X_1 is a bounding pair. Then $Z = Y_0 \oplus Y_1$ cannot be a common lower bound of X_0, X_1, hence one of them, say X_0, is not Turing above $Y_0 \oplus Y_1$. It follows that at least one set Y_a ($a \in \{0, 1\}$) is not Turing reducible to X_0 although Y_a is strongly hyperimmune-free relative to X_0. This can only happen when $X_0 \geq_T K$ by Theorem 2.2. Hence $K \leq_{\mathrm{pmaj}} X_0$ and every bounding pair satisfies that one of its halves is pmaj-above the Halting problem. □

Proposition 3.3. *If $L \not\leq_{\mathrm{pmaj}} K$ then there is a bounding pair X_0, X_1 with $L \not\leq_{\mathrm{pmaj}} X_0 \wedge L \not\leq_{\mathrm{pmaj}} X_1$.*

Proof. Let L be given such that $L \not\leq_{\mathrm{pmaj}} K$. Let H be a set such that $H \geq_T K$, H has hyperimmune-free and minimal Turing degree relative to K and $H \not\geq_T L''$. This set H can be obtained by relativizing the construction of uncountably many minimal hyperimmune-free degrees to the Halting problem. Note that $H \not\leq_T K'$, as $K' \leq_T L''$. By Friedberg's jump inversion theorem there is a set $G \leq_T H$ such that $G' \equiv_T G \oplus K \equiv_T H$ [3].

Now one can take in the world relative to G a high r.e. and incomplete set I and split it using Sack's splitting theorem [9] into two low halves I_0 and I_1. Let $Y_0 = G \oplus I_0$ and $Y_1 = G \oplus I_1$. Y_0 and Y_1 are low relative to G and thus there is a total H-recursive function f majorising the partial universal functions $e, x \mapsto \varphi_e^{Y_a}(x)$ for $a = 0, 1$. This function f is majorised by a total K-recursive function g. Hence $Y_0 \leq_{\mathrm{pmaj}} K$ and $Y_1 \leq_{\mathrm{pmaj}} K$. Let $X_0 = G \oplus I$ and $X_1 = K$. As Y_0, Y_1 are Turing reducible to $G \oplus I$,

$Y_0 \leq_{\mathrm{pmaj}} X_0 \wedge Y_1 \leq_{\mathrm{pmaj}} X_0$. Furthermore, as seen, $Y_0 \leq_{\mathrm{pmaj}} X_1 \wedge Y_1 \leq_{\mathrm{pmaj}} X_1$.

Note that the convergence module of G' relative to G is majorised by an X_1-recursive function and, as X_0 has incomplete r.e. Turing degree relative G, $X_1 \not\leq_{\mathrm{pmaj}} X_0$. Furthermore, the halting problem of X_0 cannot be solved by an X_1-recursive function, as $X_0' \geq_T G'' > K = X_1$. Assume now by way of contradiction that $X_0 \leq_{\mathrm{pmaj}} X_1$. As $X_0' \not\leq_T X_1'$ by choice, it follows from Theorem 2.3 that every partial X_0-recursive function would be majorised by a total X_1-recursive function. However, applying this result to the partial function which gives the time to enumerate elements into the Halting set relative to X_0, one could decide the Halting problem of X_0 relative to $X_0 \oplus X_1$. As X_0, X_1 are both below G', this implies $X_0' \leq_T G'$ and X_0 would be low relative to G in contradiction to the choice. Hence $X_0 \not\leq_{\mathrm{pmaj}} X_1$.

Now assume that Z is a common lower bound of X_0 and X_1. It cannot be that $Z >_T K$ as then $Z \not\leq_{\mathrm{pmaj}} X_1$. It also cannot be that $Z \equiv_T K$, as $X_0 \not\leq_{\mathrm{pmaj}} X_1$. Hence $Z \not\geq_T K$. As $X_0 \not\geq_T K$, it follows that $Z \leq_T X_0$ and therefore $Z \not\leq_T K$. Hence $Z' \leq_T H$ and Z is low relative to G. Hence $Y_0, Y_1 \leq_{\mathrm{pmaj}} Z$ only if $Y_0, Y_1 \leq_T Z$. The latter is wrong as the join of Y_0 and Y_1 is Turing equivalent to $I \oplus G$ and not low (relative to G). So Z cannot be above Y_0 and Y_1; in particular, X_0, X_1 are a bounding pair.

By choice, $L \not\leq_{\mathrm{pmaj}} X_1$. If $L >_T K$, then $L \not\leq_{\mathrm{pmaj}} X_0$ as no set above K is pmaj-reducible to X_0. If $L \not\geq_T K$ then $L \leq_{\mathrm{pmaj}} X_0$ implies $L \leq_T X_0$ and $L \leq_T H$. As H has minimal Turing degree above K, and $L \not\leq_T K$, it follows that $H \leq_T L'$ in contradiction to the choice of H. Thus $L \not\leq_{\mathrm{pmaj}} X_0$ in both cases. Hence L is not the lower bound of any half of this bounding pair. □

4. Pmaj-bases and their applications

The notion of a pmaj-basis permits to relativise the definition of the results of the previous sections to the pmaj-degree above this basis. The main idea of a pmaj-basis is that its upper cone is the same for the pmaj-relation and the Turing reduction.

Definition 4.1. *A set S is called a* pmaj-basis *iff every set Z satisfies $S \leq_{\mathrm{pmaj}} Z \Leftrightarrow S \leq_T Z$.*

Remark 4.2. The notion of a pmaj-basis permits to bring over various results from the previous section to the cone above the basis. So, in the following let S be a pmaj-basis. Then the following are some sample results which generalise.

Let X be pmaj-above S. If $X \not\geq_T S'$ then $\{Y : S \leq_{\mathrm{pmaj}} Y \leq_{\mathrm{pmaj}} X\} = \{Y : S \leq_T Y \leq_T X\}$ else $\{Y : S \leq_{\mathrm{pmaj}} Y \leq_{\mathrm{pmaj}} X\}$ is uncountable.

If $S \leq_{\mathrm{pmaj}} X \leq_{\mathrm{pmaj}} Y$ then either $X' \leq_T Y'$ or every partial X-recursive function is majorised by a total Y-recursive function.

One can also relativise the notion of a bounding pair to the pmaj-degres above S and define that using the parameter S.

Definition 4.3. *Call a pair X_0, X_1 is a bounding pair above S iff there are Y_0, Y_1 satisfying $S \leq_{\mathrm{pmaj}} Y_0$, $S \leq_{\mathrm{pmaj}} Y_1$, $Y_0 \leq_{\mathrm{pmaj}} X_0$, $Y_0 \leq_{\mathrm{pmaj}} X_1$, $Y_1 \leq_{\mathrm{pmaj}} X_0$, $Y_1 \leq_{\mathrm{pmaj}} X_1$ and there is no set Z satisfying $Y_0 \leq_{\mathrm{pmaj}} Z$, $Y_1 \leq_{\mathrm{pmaj}} Z$, $Z \leq_{\mathrm{pmaj}} X_0$, $Z \leq_{\mathrm{pmaj}} X_1$.*

This permits now to define the following result.

Theorem 4.4. *Assume that S is a pmaj-basis. Then S' is a member of greatest pmaj-degree in the class $\{T : S \leq_{\mathrm{pmaj}} T$ and for every bounding pair X_0, X_1 above S either $T \leq_{\mathrm{pmaj}} X_0$ or $T \leq_{\mathrm{pmaj}} X_1\}$. In particular, if S is definable and a pmaj-basis then S' is definable in the partial order of the pmaj-degrees.*

The proof is a direct relativisation of the previous proof, working in the world Turing above S instead of the unrelativised world. The corresponding results used relativise. Now it is interesting to know which pmaj-degrees can be a pmaj-basis. Indeed, there are only countably many of them and every Δ^0_2 set is a pmaj-basis.

Proposition 4.5. *For every recursive ordinal α, $\emptyset^{(\alpha)}$ is a* pmaj-*basis.*

Proof. For every recursive ordinal α, there is a function $f \in \omega^\omega$ such that $f \equiv_T \emptyset^{(\alpha)}$ for which f is the unique infinite path in some recursive tree $T \subseteq \mathbb{N}^*$ [10]. Then for every function g majorising f, $g \geq_T f \equiv_T \emptyset^{(\alpha)}$. So $\emptyset^{(\alpha)}$ is a pmaj-basis. □

Corollary 4.6. *The finite jumps K, K', K'', ... are definable in the pmaj-degrees.*

Remark 4.7. There is a close connection between pmaj-bases and recursively encodable sets. Here a set X is called *recursively encodable* if for every infinite set Z there is a set $Y \subseteq Z$ so that $X \leq_T Y$. Obviously, every pmaj-basis is recursively encodable. Solovay [16] showed that a set is recursively encodable if and only if it is hyperarithmetic. So every pmaj-basis is hyperarithmetic and X is hyperarithmetic iff $X \leq_{\mathrm{pmaj}} Y$ for a pmaj-basis Y.

5. Weak Truth-Table Reduction

One can also consider partial functions which are wtt-reducible to an oracle. Here $p \leq_{wtt} X$ iff there is a recursive function f and a partial X-recursive function ψ^X such that, for all x, $\psi^X(x) = \psi^{X \cap \{0,1,\ldots,f(x)\}}(x)$, where either both sides of $=$ are defined and equal or both sides are undefined. In other words, $f(x)$ denotes the largest relevant query of the algorithm computing ψ^X from X on input x.

Definition 5.1. *The set X is strongly hyperimmune-free relative to the set Y with respect to weak truth-table reducibility ($X \leq_{\mathrm{wttpmaj}} Y$) iff every partial p wtt-reducible to X is majorised by a partial q wtt-reducible to Y.*

Note that this relation is transitive and satisfies $X \leq_{wtt} Y \Rightarrow X \leq_{\mathrm{wttpmaj}} Y$. One important difference to the pmaj-relation is that the sets wtt-above K form a single uncountable degree which sits above all other degrees.

Theorem 5.2. *$K \leq_{wtt} X$ iff $\forall Y\, [Y \leq_{\mathrm{wttpmaj}} X]$.*

Proof. First assume that $K \leq_{wtt} X$ and Y is any further set. Let $p \leq_{wtt} Y$ be a partial function. There is a recursive bound $f(x)$ on the largest query used to compute $p(x)$, whenever that is defined. Hence there is a total function $q \leq_{wtt} K$ which returns for input x the largest possible value which the algorithm for $p(x)$ with input x can produce on any oracle queries $Z(0), Z(1), \ldots, Z(f(x))$ where it terminates. Obviously $q \leq_{wtt} X$ and q majorises p. So $Y \leq_{\mathrm{wttpmaj}} X$.

For the converse assume that $Y \leq_{\mathrm{wttpmaj}} X$ for all Y. Then in particular $K \leq_{\mathrm{wttpmaj}} X$. Consider the function p where for $n \in K$ the value $p(n)$ is the least number $s+1$ such that n is enumerated into K within s steps and for $n \notin K$ it holds that $p(n) = 0$. Then p is total and $p \leq_{wtt} K$. Now p is majorised by a total function $q \leq_{wtt} X$ and $n \in K \Leftrightarrow n$ is enumerated into K within $q(n)$ steps. Thus $K \leq_{wtt} X$. $\square$

So one has that $X \leq_{wtt} Y \vee K \leq_{wtt} Y \Rightarrow X \leq_{\mathrm{wttpmaj}} Y$ and one may ask whether the converse of this result holds. The converse of it holds in a restricted way where one replaces on the right hand side weak truth-table reducibility by Turing reducibility.

Theorem 5.3. *If $K \not\leq_T Y$ then $X \leq_{\mathrm{wttpmaj}} Y$ implies $X \leq_{\mathrm{pmaj}} Y$ and $X \leq_T Y$; in particular,*

$$X \leq_{\mathrm{wttpmaj}} Y \Rightarrow X \leq_T Y \vee K \leq_T Y.$$

Proof. Assume that $X \leq_{\mathrm{wttpmaj}} Y$. Let $c_K(y)$ be the time to enumerated y into K and note that $c_K(y)$ is undefined for $y \notin K$; Now consider the following partial X-recursive function with domain $\mathbb{N} \times \mathbb{N} \times \{0,1\}$: If $X(x) = a$ then let $p(x,y,a) = c_K(y)$ else let $p(x,y,a) = 0$.

It is clear that $p \leq_{wtt} X$ and therefore there is a function $q \leq_{wtt} Y$ majorising p. One can compute relative to Y for each input pair (x,y) a value $a(x,y) \in \{0,1\}$ such that $q(x,y,a(x,y))$ is defined. There are now two cases.

Case (a): For every x one can, using Y, find a value $y(x)$ such that $y(x) \in K$ and $c_K(y(x)) > (x, y(x), a(x, y(x)))$. Then $a(x, y(x)) \neq X(x)$ and $X(x) = 1 - a(x, y(x))$. It follows that $X \leq_T Y$.

Case (b): There is an x such that for all $y \in K$, $c_K(y) \leq q(x,y,a(x,y))$. Now one fixes this x and $K \leq_T Y$ as one can check for each y whether y is enumerated into K within $q(x,y,a(x,y))$ steps; if so then $y \in K$ else $y \notin K$.

It is easy to see that always either (a) or (b) applies. This case distinction then shows the statement of the theorem. $\square$

Furthermore, for X and Y of hyperimmune-free Turing degree one has that $X \leq_{\mathrm{pmaj}} Y \Leftrightarrow X \leq_{\mathrm{wttpmaj}} Y \Leftrightarrow X \leq_{wtt} Y \Leftrightarrow X \leq_T Y$. The anonymous referee observed that the same applies if X and Y have strongly contiguous r.e. Turing degree. The next result shows that one cannot strengthen the right hand side of Theorem 5.3 to wtt reducibility.

Theorem 5.4. *There are sets X and Y with*

$$X \leq_{\mathrm{wttpmaj}} Y \not\Rightarrow X \leq_{wtt} Y \vee K \leq_{wtt} Y.$$

Proof. Let f be a K-recursive function which grows faster than every recursive function and which satisfies $f(u+1) > f(u)+2$ for all u. Now let $a_0 = 0$, $a_{n+1} = f(f(f(a_n)))$. Let Z be any set not Turing reducible to K and define X, Y as follows:

$$X(x) = \begin{cases} 1 & \text{if } x = a_n \vee x = f(a_n) \vee x = f(f(a_n)) \text{ for some } n; \\ Z(n) & \text{if } x = a_n + 1 \text{ for that } n; \\ 0 & \text{otherwise.} \end{cases}$$

$$Y(\langle x, y\rangle) = \begin{cases} 1 & \text{if } x = a_n \vee x = f(a_n) \vee x = f(f(a_n)) \text{ for some } n; \\ K(y) & \text{if } x = a_n + 1 \text{ and } y \leq f(f(a_n)) \text{ for some } n; \\ Z(n) & \text{if } x = f(a_n) + 1 \text{ for that } n; \\ 0 & \text{otherwise.} \end{cases}$$

Now $X \not\leq_{wtt} Y$ as for retrieving $X(a_n+1)$ one has to access $Y(\langle f(a_n), 1\rangle)$ or beyond for infinitely many n as otherwise $Z \leq_T K$ would follow from $f \leq_T K$ in contradiction to the choice of f.

Furthermore, one can show that $K \not\leq_{wtt} Y$ as follows. There is a function $g \leq_{wtt} K$ such that the Kolmogorov complexity of $g(m)$ is at least $5m$ for each m. If $g \leq_{wtt} Y$ then the Kolmogorov complexity of $g(f(f(a_n)))$ would, for all sufficiently large n, be bounded by $3f(f(a_n))+2n+5$ as one would take into account that the values $Y(\langle x, y\rangle)$ with $x, y < a_{n+1}$ can be computed from the values $Y(\langle x, y\rangle)$ with $x, y < f(f(a_n))$ and those one can describe by describing n with $n+1$ bits, $Z(0), \ldots, Z(n)$ with $n+1$ bits, $f(f(a_n))$ with $f(f(a_n))+1$ bits and $\{a_0, f(a_0), f(f(a_0)), a_1, f(a_1), f(f(a_1)), a_2, \ldots, a_n, f(a_n), f(f(a_n))\}$ with $f(f(a_n))+1$ bits and K up to $f(f(a_n))$ by $f(f(a_n))+1$ bits. These information would, for almost all n, permit to reconstruct Y up to the bound queried when calculating $g(f(f(a_n)))$ and therefore the Kolmogorov complexity of $g(f(f(a_n)))$ would be strictly less than $5f(f(a_n))$ for almost all n in contradiction to the choice of g. Hence g is not wtt-reducible to Y and $K \not\leq_{wtt} Y$.

So it remains to show that $X \leq_{\text{wttpmaj}} Y$. So let p be a partial function which is wtt-reducible to X with use h; h is a recursive function majorised by f. Let $\tilde{h}(x)$ be a recursive use function so that one can wtt-compute an upper bound for $p(x)$ from the oracle K with use $\tilde{h}$; without loss of generality $\tilde{h}(x) \geq x + h(x)$ for all x. Now one defines the function $q \leq_{wtt} Y$ by the first of the following cases which applies; note that querying $Y(\langle u, 0\rangle)$ for $u = 0, 1, \ldots, \tilde{h}(x)$ reveals which case applies.

- If $f(f(a_n)) < \tilde{h}(x) < a_{n+1}$ for an n then one sets $q(x) = p(x)$ and $p(x)$ can be computed relative to Y by inspecting $Y(\langle u, 0\rangle)$ for $u = 0, 1, \ldots, \tilde{h}(x)+1$ and reconstructing $X(u)$ for the same u.
- If $a_n \leq \tilde{h}(x) \leq f(f(a_n))$ for an n then one can compute an upper bound of $p(x)$ by searching for a_n via inspecting $Y(\langle u, 0\rangle)$ for $u = 0, 1, \ldots, \tilde{h}(x)+1$ and then retrieving the corresponding values of K by inspecting $Y(\langle a_n+1, u\rangle)$ for $u = 0, 1, \ldots, \tilde{h}(x)$.

- In the finitely many remaining cases, $q(x) = p(x)$ through a corresponding lookup in a finite table.

One can easily see that the use for the computation of the so defined function q is in the first two cases bounded by $\langle \tilde{h}(x) + 1, \tilde{h}(x) + 1\rangle$ and in the third case the oracle is not queried at all. Hence one can see that the so defined function q is weak truth-table reducible to Y and majorises p. $\square$

6. Conclusion

The relation studied is when X is strongly hyperimmune-free relative to Y, that is, when every partial X-recursive function p is majorised by a partial Y-recursive function q. Here q majorises p if every x in the domain of p is also in the domain of q and satisfies $p(x) < q(x)$. An initial result states that $X \leq_{\mathrm{pmaj}} Y$ if either $X' \leq_T Y'$ or every partial X-recursive function is majorised by a total Y-recursive function. It is natural to ask whether this relation can be strengthened.

Question 6.1. *Does $X \leq_{\mathrm{pmaj}} Y$ hold iff $X \leq_T Y$ or every partial X-recursive function is majorised by a total Y-recursive one?*

It was shown that $X \equiv_{\mathrm{pmaj}} Y$ iff $X \equiv_T Y$. So the degrees coincide, but not the ordering between them; only the implication $X \leq_T Y \Rightarrow X \leq_{\mathrm{pmaj}} Y$ holds but not its converse. It is interesting to know how related these degrees (with their partial orderings) are.

Question 6.2. *Is the Turing reducibility definable in the pmaj-degrees? Is the Turing jump definable in the pmaj-degrees?*

One could also ask the reverse question whether one can define in the Turing degrees that a degree is strongly hyperimmune-free relative to another one.

Question 6.3. *Is the pmaj reducibility definable in the Turing degrees?*

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F. Stephan: Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Block S17, Singapore 119076, Republic of Singapore, *Email address* fstephan@comp.nus.edu.sg

L. Yu: Institute of Mathematical Science, Nanjing University, P.R. of China 210093, *Email address* yuliang.nju@gmail.com