ELEMENTARY DIFFERENCES AMONG FINITE LEVELS OF THE ERSHOV HIERARCHY

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Abstract. We study the differences among finite levels of the Ershov hierarchies. We also give a brief survey on the current state of this area. Some questions are raised.

1. Preliminary

Putnam [9] is the first one who introduced the n-r.e. sets.

Definition 1.1. (i) *A set A is n-r.e. if there is a recursive function $f : \omega \times \omega \to \omega$ so that for each m,*
- $f(0, m) = 0$.
- $A(m) = \lim_s f(s, m)$.
- $|\{s | f(s + 1, m) \neq f(s, m)\}| \leq n$.

- *A Turing degree is n-r.e. if it contains an n-r.e. set.*

We use D_n to denote the collection of n-r.e. degrees. For simplicity, we redefine $D_0 = D_1$ which is a little unusual.

For other recursion notations, please refer to Soare [13].

In this paper, we work in the partially ordered language, $\mathcal{L}(\leq)$, through out. $\mathcal{L}(\leq)$ includes variables $a, b, c, x, y, z, ...$ and a binary relation $\leq$ intended to denote a partial order. Atomic formulas are $x = y$, $x \leq y$. Σ_0 formulas are built by the following induction definition.

- Each atomic formula is Σ_0.
- $\neg\psi$ for some Σ_0 formula ψ.
- $\psi_1 \vee \psi_2$ for two Σ_0 formula ψ_1, ψ_2.
- $\psi_1 \wedge \psi_2$ for two Σ_0 formula ψ_1, ψ_2
- $\psi_1 \implies \psi_2$ for two Σ_0 formula ψ_1, ψ_2.

A formula φ is Σ_1 if it is of the form $\exists x_1 \exists x_2 ... \exists x_n \psi(x_1, x_2, ..., x_n)$ for some Σ_0 formula ψ.

For each $n \geq 1$, a formula φ is Π_n if it is the form $\neg\psi$ for some Σ_n formula ψ and a formula φ is Σ_{n+1} if it is the form $\exists x_1 \exists x_2 ... \exists x_m \psi(x_1, x_2, ..., x_m)$ for some Π_n formula ψ .

A sentence is a formula without free variables.

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Given two structures $\mathfrak{A}(A, \leq_A)$ and $\mathfrak{B}(B, \leq_B)$ for $\mathcal{L}(\leq)$, we say that $\mathfrak{A}(A, \leq_A)$ is a substructure of $\mathfrak{B}(B, \leq_B)$, write $\mathfrak{A}(A, \leq_A) \subseteq \mathfrak{B}(B, \leq_B)$, if $A \subseteq B$ and the interpretation $\leq_A$ is a restriction to A of $\leq_B$.

Definition 1.2. *For $n \geq 0$. Given structures $\mathfrak{A}(A, \leq_A)$ and $\mathfrak{B}(B, \leq_B)$ for $\mathcal{L}(\leq)$.*

(i) *We say that $\mathfrak{A}(A, \leq_A)$ is a Σ_n substructure of $\mathfrak{B}(B, \leq_B)$, write $\mathfrak{A}(A, \leq_A) \preceq_{\Sigma_n} \mathfrak{B}(B, \leq_B)$, if $\mathfrak{A}(A, \leq_A) \subseteq \mathfrak{B}(B, \leq_B)$ and for all Σ_n formulas $\varphi(x_1, x_2, ..., x_n)$ and any $a_1, a_2, ..., a_n \in A$,*

$$\mathfrak{A}(A, \leq_A) \models \varphi(a_1, a_2, ..., a_n) \text{ if and only if } \mathfrak{B}(B, \leq_B) \models \varphi(a_1, a_2, ..., a_n).$$

(ii) *We say that $\mathfrak{A}(A, \leq_A)$ is Σ_n-elementary-equivalent to $\mathfrak{B}(B, \leq_B)$, write $\mathfrak{A}(A, \leq_A) \equiv_{\Sigma_n} \mathfrak{B}(B, \leq_B)$, if for all Σ_n sentences φ,*

$$\mathfrak{A}(A, \leq_A) \models \varphi \text{ if and only if } \mathfrak{B}(B, \leq_B) \models \varphi.$$

In this paper, we study the model theoretical properties of Δ^0_2 Turing degrees as the structure $\mathcal{D}(\leq \mathbf{0}') = (D(\leq \mathbf{0}'), \leq)$ of $\mathcal{L}(\leq)$. We are interested in various substructure of $\mathcal{D}(\leq \mathbf{0}')$, particularly, the structures of n-r.e. degrees $\mathcal{D}_n = (D_n, \leq)$. [1] For two degrees $\mathbf{a}$ and $\mathbf{b}$ in $\mathcal{D}_n$ (or $\mathcal{D}(\leq \mathbf{0}')$), we use $\mathbf{a} \cup \mathbf{b}$ and $\mathbf{a} \cap \mathbf{b}$ to denote their least upper bound and the largest lower bound (if exists) in $\mathcal{D}_n$ (or $\mathcal{D}(\leq \mathbf{0}')$) respectively.

For more model theoretic facts, please refer to [7].

2. Elementary difference among Ershov hierarchies

Comparing the structure difference between Ershov hierarchies has a long history beginning with Cooper(1970's) and Lachlan's (1968) unpublished work. They proved the following theorem.

Theorem 2.1 (Lachlan, Cooper)**.** (i) *For each $n \geq 1$, $D_n \subset D_{n+1}$.*

(ii) *For each non-recursive $n+1$-r.e. degree $\mathbf{a}$, there is a non-recursive n-r.e. degree $\mathbf{b} \leq \mathbf{a}$.*

For any Σ_1-sentence φ, $\mathcal{D}_n$ or $\mathcal{D}(\leq \mathbf{0}')$ satisfies φ if and only if φ is consistent with the theory of partial orderings (see, for example, some exercises in Soare [13]). Therefore,

Theorem 2.2 (Folklore)**.** *For all $n \in \omega$, $\mathcal{D}_n \equiv_{\Sigma_1} \mathcal{D}(\leq \mathbf{0}')$.*

Thus elementary differences would not occur at the Σ_1-level.

By improving a technique due to Spector, Sacks proved the following result.

Theorem 2.3 (Sacks [10])**.** *There is a Δ^0_2 minimal degree.*

Comparing with Theorem 2.1, the elementary difference between $\mathcal{D}_n$ and $\mathcal{D}(\leq \mathbf{0}')$ shows up at Σ_2-level.

The elementary difference between $\mathcal{D}_1$ and $\mathcal{D}_n (n > 1)$ was first revealed at Σ_3-level by Arslanov [2] who showed that for every element $\mathbf{a}$ in $\mathcal{D}_n$, there is an element $\mathbf{b} \in \mathcal{D}_n$ of which the supreme is $\mathbf{0}'$, whereas in $\mathcal{D}_1$ this is not true due to Cooper and Yates. Later many differences at Σ_2-level were discovered, for example, the following pair of theorems offers perhaps the clearest order-theoretic difference:

[1]We use "1-r.e." to denote "r.e."

Theorem 2.4 (Sacks[11]). *$\mathcal{D}_1$ is dense.*

Theorem 2.5 (Cooper, Harrington, Lachlan, Lempp, Soare[5]). *For each natural number $n > 1$, there is a maximal element in $\mathcal{D}_n$.*

So the following results can be obtained.

Corollary 2.6. *For each natural number $n > 1$, $\mathcal{D}_1 \not\equiv_{\Sigma_2} \mathcal{D}_n$.*

A further question is how difference between $\mathcal{D}_n$ and $\mathcal{D}_{n+m}$ for $n > 1$. Downey formulated the following ambitious question which is now known as Downey Conjecture.

Conjecture 2.7 (Downey [6]). *For each $n > 1$ and $k \geq 0$, $\mathcal{D}_n \equiv_{\Sigma_k} \mathcal{D}_{n+m}$.*

Though Downey Conjecture looks too optimal to be true, it remained open more than fifteen years. The difficulty of Conjecture 2.7 lies in the technique used in the local theory of $\mathcal{D}_n$. Usually one can generalize a (local) result in $\mathcal{D}_2$ to $\mathcal{D}_n$ without any difficult.

Recently, Arslanov, Kalimullin and Lempp announced a negative solution to Conjecture 2.7. They proved the following result.

Theorem 2.8 (Arslanov, Kalimullin, Lempp [3]). *$\mathcal{D}_2 \not\equiv_{\Sigma_2} \mathcal{D}_3$.*

But the question whether $\mathcal{D}_n \not\equiv_{\Sigma_2} \mathcal{D}_{n+m}$ is true for some very large numbers n, m still remains open.

3. Σ_1-SUBSTRUCTURES OF $\mathcal{D}(\leq \mathbf{0}')$

As we have seen that $\mathcal{D}_n \equiv_{\Sigma_1} \mathcal{D}(\leq \mathbf{0}')$(Theorem 2.2), it is natural to ask whether $\mathcal{D}_n \preceq_{\Sigma_1} \mathcal{D}(\leq \mathbf{0}')$. This was eventually negatively answered by Slaman in 1983.

Theorem 3.1 (Slaman).
(i) *There are r.e. sets A, B and C and a Δ^0_2 set E such that*
  - *$\emptyset <_T E \leq_T A$;*
  - *$C \not\leq_T B \oplus E$;*
  - *For all r.e. set W $(\emptyset <_T W \leq_T A \Rightarrow C \leq_T W \oplus B)$.*

(ii) *For each natural number $n \geq 1$, $\mathcal{D}_n \not\preceq_{\Sigma_1} \mathcal{D}(\leq \mathbf{0}')$.*

Proof. We just show how to deduce (ii) from (i). Take a Σ_1 formula

$$\varphi(x_1, x_2, x_3) \equiv \exists e \exists y \exists z (e \leq x_1 \wedge e \geq y \wedge e \neq y \wedge z \geq x_2 \wedge z \geq e \wedge z \not\geq x_3).$$

Take the r.e. degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and a Δ^0_2 degree $\mathbf{e}$ as in (i). Fix $Z = B \oplus E$.

Then $\mathcal{D}(\mathbf{0}') \models \varphi(\mathbf{a}, \mathbf{b}, \mathbf{c})$ since $\mathbf{e} \leq \mathbf{a} \wedge \mathbf{e} > \mathbf{0} \wedge \mathbf{z} \not\geq \mathbf{c}$.

Then for each $n \geq 1$, $\mathcal{D}_n \not\models \varphi(\mathbf{a}, \mathbf{b}, \mathbf{c})$. If not, then there is an n-r.e. degree $\mathbf{f} > \mathbf{0}$ so that $\mathbf{f} \leq \mathbf{a}$ and $\mathbf{f} \cup \mathbf{b} \not\geq \mathbf{c}$. But, by Theorem 2.1, there is a non-recursive r.e. degree $\mathbf{w} \leq \mathbf{f}$. So $\mathbf{w} \cup \mathbf{b} \not\geq \mathbf{c}$. This is impossible by (i). ☐

Having proved Theorem 3.1, Slaman raised the following conjecture which remained open more than twenty years.

Conjecture 3.2 (Slaman [5]). *For each $n > 1$, $\mathcal{D}_1 \preceq_{\Sigma_1} \mathcal{D}_n$?*

Furthermore, Lempp raised the following conjecture.

Conjecture 3.3 (Lempp). *For all $n > m$, $\mathcal{D}_m \preceq_{\Sigma_1} \mathcal{D}_n$?*

To solve conjecture 3.2, one possible argument is to build a finite array just as Slaman did in Theorem 3.1. However, by the Cooper and Lachlan observation that every nonrecursive n-r.e. degree bounds a nonrecursive r.e. degree, we cannot hope that any n-r.e. degree D plays the role of E as in Slaman Theorem.

We first explain that it is necessary to build a complicated formula to refute Slaman's conjecture.

A formula is called positive if it is built by the following induction definition.

- Each atomic formula is positive.
- $\psi_1 \vee \psi_2$ for two positive formula ψ_1, ψ_2.
- $\psi_1 \wedge \psi_2$ for two positive formula ψ_1, ψ_2.

A formula is called p-Σ_1 if it is the form $\exists x_1 \exists x_2 ... \exists x_n \varphi(x_1, x_2, ..., x_n)$ for some positive formula φ.

We say that $\mathfrak{A}(A, \leq_A)$ is a p-Σ_1 substructure of $\mathfrak{B}(B, \leq_B)$, write $\mathfrak{A}(A, \leq_A) \preceq_{p\text{-}\Sigma_1} \mathfrak{B}(B, \leq_B)$, if $\mathfrak{A}(A, \leq_A) \subseteq \mathfrak{B}(B, \leq_B)$ and for all p-Σ_1 formulas $\varphi(x_1, x_2, ..., x_n)$ and any $a_1, a_2, ..., a_n \in A$,

$$\mathfrak{A}(A, \leq_A) \models \varphi(a_1, a_2, ..., a_n) \text{ if and only if } \mathfrak{B}(B, \leq_B) \models \varphi(a_1, a_2, ..., a_n).$$

We have the following proposition

Proposition 3.4. *$\mathcal{D}_n \preceq_{p\text{-}\Sigma_1} \mathcal{D}_m$ for all $n \leq m$. Furthermore, $(D_1, \leq, \cup, \cap) \preceq_{p\text{-}\Sigma_1} (D_n, \leq, \cup, \cap)$ for all $n > 1$.*

Thus to refute Slaman Conjecture, it is necessary to consider some negative statement.

Eventually we obtained the following formula.

$$\varphi(x_1, x_2, x_3, x_4) \equiv \exists d \exists g (d \leq x_1 \wedge d \not\leq x_4 \wedge g \geq x_2 \wedge g \geq d \wedge x_3 \not\leq g).$$

The solution to Conjecture 3.2 follows from the following technical result:

Theorem 3.5 (Yang and Yu [15]). *There are r.e. sets A, B, C and E and a d.r.e. set D such that*

1. *$D \leq_T A$ and $D \not\leq_T E$;*
2. *$C \not\leq_T B \oplus D$;*
3. *For all r.e. sets W ($W \leq_T A \Rightarrow$ either $C \leq_T W \oplus B$ or $W \leq_T E$).*

Assuming Theorem 3.5, we can obtain the following result to refute Slaman conjecture:

Theorem 3.6 (Yang and Yu [15]). *For all $n > 1$, $\mathcal{D}_1 \not\preceq_{\Sigma_1} \mathcal{D}_n$.*

Proof. Assume $n > 1$.Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$ be the degrees of their corresponding sets as in Theorem 3.5. Note all of them except $\mathbf{d}$ belong to $\mathcal{D}_1$ and $\mathbf{d}$ belongs to $\mathcal{D}_n$. By Theorem 3.5, just take $\mathbf{g} = \mathbf{b} \cup \mathbf{d} \in \mathcal{D}_n$,

$$\mathcal{D}_n \models \varphi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}).$$

However, by Theorem 3.5 again,

$$\mathcal{D}_1 \models \forall \mathbf{w} \forall \mathbf{g}((\mathbf{w} \leq \mathbf{a} \wedge \mathbf{g} \geq \mathbf{w} \wedge \mathbf{g} \geq \mathbf{b} \wedge \mathbf{w} \not\leq \mathbf{e}) \implies \mathbf{c} \leq \mathbf{g}).$$

In other words,

$$\mathcal{D}_1 \models \neg\varphi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}).$$

□

Although Slaman Conjecture is not true, we can ask where the abnormal parameters refuting the conjecture exist. Inspired by Shore and Slaman [12], we conjecture that each high r.e. degree bounds the four parameters as in Theorem 3.5 so that Slaman conjecture fails. But is there a fragment $\mathcal{E} \subset \mathcal{D}_1$ so that $\mathcal{E} \preceq_{\Sigma_1} \mathcal{D}_2$? A critical part of the argument used in the proof of Theorem 3.5 is a modification of the construction of Slaman triple. A triple $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ in $\mathcal{D}_n$ is called *Slaman triple* if $\mathbf{0} < \mathbf{a}, \mathbf{c} \not\leq \mathbf{b}$ and for all non-recursive $\mathbf{x} \in \mathcal{D}_n$ below $\mathbf{a}$, $\mathbf{c} \leq \mathbf{b} \cup \mathbf{x}$. Shore and Slaman [12] showed that a Slaman-triple can be found below each high r.e. degree in $\mathcal{D}_1$. However, Harrington, and Bickford and Mills, showed independently that no low$_2$ r.e. degree bounds a Slaman triple in $\mathcal{D}_1$. Thus it sounds reasonable to conjecture that there is fragment $\mathcal{E} \subset \mathcal{D}_1$ in which all of elements are low$_2$ so that $\mathcal{E} \preceq_{\Sigma_1} \mathcal{D}_2$. A non-recursive degree $\mathbf{a} \in \mathcal{D}_n$ is said to be *almost deep* if for each low $\mathbf{b} \in \mathcal{D}_n$, $\mathbf{a} \cup \mathbf{b}$ is low. Cholak et al [4] proved that almost deep degrees exist in $\mathcal{D}_1$. Hence it is natural to ask whether the almost deep degrees in $\mathcal{D}_1$ form a Σ_1-substructure of $\mathcal{D}_2$.

The last question in this section was raised by Khoussainov.

Question 3.7 (Khoussainov)**.** *For $n > 1$, is there a function $f : D_1 \to D_n$ so that for any Σ_1-formula $\varphi(x_1, ..., x_m)$,*

$$\mathcal{D}_1 \models \varphi(\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_m}) \text{ iff } \mathcal{D}_n \models \varphi(f(\mathbf{x_1}), f(\mathbf{x_2}), ..., f(\mathbf{x_m})),$$

where $\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_m}$ range over D_1?

4. Definable ideals and filters

Recently, Wang and Yu [14] proved that each non-principal ideal in $\mathcal{D}_1$ is a Σ_1-substructure of $\mathcal{D}_1$. But the question whether any non-principal ideal in $\mathcal{D}_2$ is a Σ_1-substructure of $\mathcal{D}_2$ is unknown. A set $\mathcal{A} \subseteq \mathcal{D}_n$ is said to be definable in $\mathcal{D}_n$ if there is a formula ψ so that $\mathbf{a} \in \mathcal{A}$ if and only if $\mathcal{D}_n \models \psi(\mathbf{a})$. For $\mathcal{D}_1$, by the recent work due to Nies [8], Yang and Yu [16], there are many definable non-principal ideals in $\mathcal{D}_1$. A natural question is what about $\mathcal{D}_2$? To construct a non-principal ideal in $\mathcal{D}_n$, we just need to take a non-principal ideal $\mathcal{I}$ in $\mathcal{D}_1$ and then build a non-principal ideal $\mathcal{J} = \{\mathbf{b} | \exists \mathbf{a} \in \mathcal{I}(\mathbf{b} \leq \mathbf{a})\}$. The problem is whether it is definable in $\mathcal{D}_n$. We formulate the following questions which we are very interested in.

Question 4.1. *For $n > 1$, is there a non-trivial definable Σ_1-substructure of $\mathcal{D}_n$?*

From the discussion above, we have seen that the definable ideals play a critical role in the study of global theory. Although there are some non-trivial definable ideals in $\mathcal{D}_1$. It is unknown whether there are infinitely many definable ones in $\mathcal{D}_1$. For $\mathcal{D}_2$, we don't even know whether there is a non-trivial definable ideal in it.

Wang also recently studied definable filters in $\mathcal{D}_1$. It is unknown whether there is a non-trivial definable filter in $\mathcal{D}_2$. We say that a non-zero degree $\mathbf{a} \in \mathcal{D}_n$ is *cappable* if there is a non-zero degree $\mathbf{b} \in \mathcal{D}_n$ so that the infimum of them is the recursive degree $\mathbf{0}$. Otherwise, $\mathbf{a}$ is said to be *non-cappable*. A possible candidate of definable filters is the collection of non-cappable degrees in $\mathcal{D}_2$. Ambos-Spies et al [1] proved that the collection of non-cappable degrees form a filter in $\mathcal{D}_1$. Thus, to construct a definable filter in $\mathcal{D}_2$, it suffices to prove that each non-cappable 2-r.e. degree bounds a non-cappable r.e. degree. So the following technique question is raised.

Question 4.2. *Can each 2-r.e. non-cappable degree compute an r.e. non-cappable degree?*

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