

DEGREE SPECTRA OF EQUIVALENCE RELATIONS

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ABSTRACT. In [2] and [3], Greenberg, Montalbán and Slaman investigated both hyperarithmetic and constructibility degree spectra of countable structures. Inspired by their results, we push them to a more general setting by investigating degree spectra of equivalence relations.

1. INTRODUCTION

Definition 1.1. *For any equivalence relation E , reduction $\leq_r$ over 2^ω and real $x \in 2^\omega$, let*

$$\text{Spec}_{E,r}(x) = \{y \mid \exists z \leq_r y (E(z, x))\}$$

be the (E, r) -spectrum of x .

In [2] and [3], Greenberg, Montalbán and Slaman investigated $\text{Spec}_{\cong,h}(x)$ and $\text{Spec}_{\cong,L}(x)$, where $\cong$ denotes the isomorphism relation, and $\leq_h$ and $\leq_L$ denote hyperarithmetic and constructibility reducibility respectively. They prove the following result.

Theorem 1.2 (Greenberg, Montalbán and Slaman, [2] and [3]). (1) *For the countable structures of partial ordering language, there is a linear ordering structure $\mathcal{M}$ so that $\text{Spec}_{\cong,h}(\mathcal{M}) = \{y \mid y \text{ is not hyperarithmetic}\}$.*
 (2) *Assume that ω_1 is inaccessible. For any recursive language and any countable structure $\mathcal{M}$ of the language, if $\text{Spec}_{\cong,L}(\mathcal{M})$ contains all the nonconstructible reals, then it contains all the reals.*

The motivation to classify degree spectra of an equivalence relation is, as in the introduction in [3], to find which recursion theoretical aspects of a set x of natural numbers are reflected in the equivalence class of x . Moreover, pushing the results to the general setting may give some clearer explanation why the argument used in the classical setting works. For example, (1) in Theorem 1.2 can be viewed as a result for Σ_1^1 -equivalence relations. One may wonder whether there is a Π_1^1 -equivalence relation so that the conclusion of (1) remains true. We refute this by showing Proposition 2.1. And the genius method used in the proof of (2) of Theorem 1.2 by Greenberg, Montalbán and Slaman is much more powerful than it looked. Actually the method in the proof can be viewed as a generalization of the proof of the classical result that every nontrivial upper cone of Turing degrees is null. We show that, in Theorem 3.1, the conclusion remains true for any Σ_2^1 -equivalence relation under a fairly weak set theoretical assumption. Moreover, we prove that the relativization of the conclusion

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does not require any large cardinal assumption by showing Corollary 3.2 (The existence of an inaccessible cardinal seems necessary to relativize their original proof to arbitrary countable language and structures of (2) of Theorem 1.2). Both the proofs use some ideas from [2].

Mostly we follow the notations from [7]. Readers should be familiar with higher recursion and set theory.

We enumerate some classical results which are needed later.

We say that a real x codes a well ordering if the relation $R(n, m) \leftrightarrow x(2^n \cdot 3^m) = 1$ is a well ordering of ω .

For $n \in \mathcal{O}$, H_n^x is a $\Pi_2^0(x)$ -singleton. Actually it is the $|n|$ -th Turing jump relative to x . If $\omega_1^x = \omega_1^{\text{CK}}$, then each real hyperarithmetic in x is recursive in H_n^x for some $n \in \mathcal{O}$. If a real x codes a well ordering of order type α , then we use $|x|$ to denote α .

For any $\sigma \in 2^{<\omega}$, $[\sigma] = \{x \in 2^\omega \mid x \succ \sigma\}$.

Theorem 1.3 (Sacks [6]). *Let μ be the Lebesgue measure, then $\mu(\{x \mid \omega_1^x = \omega_1^{\text{CK}}\}) = 1$.*

Theorem 1.4 (Sacks [6]). *For any Π_1^1 set $B \subseteq 2^\omega \times 2^\omega \times \omega$, the set $\{(y, p) \mid p \text{ is a rational} \wedge \mu(\{x \mid (x, y) \in B\}) > p\}$ is Π_1^1 .*

Theorem 1.5 (Sacks [6] and Tanaka [8]). *Every Π_1^1 positive measure set $A \subseteq 2^\omega$ contains a hyperarithmetic real.*

For the set theory notions, we follow from the book [4]. We use $\dot{x}, \dot{y}, \dots$ to denote names over a forcing language.

Others can be found in [4], [5], [7], and the forthcoming book [1].

2. ON Π_1^1 -EQUIVALENCE RELATIONS

Proposition 2.1. *For any Π_1^1 -equivalence relation E and real x , if $\text{Spec}_{E,h}(x) \supseteq \{z \mid z \notin \Delta_1^1\}$, then $\text{Spec}_{E,h}(x) = 2^\omega$.*

Proof. Suppose that E is a Π_1^1 -equivalence relation. Fix a real x so that $\text{Spec}_{E,h}(x) \supseteq \{z \mid z \notin \Delta_1^1\}$. So $\mu(\text{Spec}_{E,h}(x)) = 1$. For any $n \in \mathcal{O}$ and Turing oracle functional Φ_e^* with $e \in \omega$, the set

$$A_{n,e} = \{z \mid E(\Phi_e^{H_n^z}, x)\}$$

is a $\Pi_1^1(x)$ subset of $\text{Spec}_{E,h}(x)$ and so measurable. By Theorem 1.3,

$$\mu\left(\bigcup_{n \in \mathcal{O}, e \in \omega} A_{n,e}\right) = 1.$$

So there must be some $n \in \mathcal{O}$ and e so that the set $A_{n,e}$ has positive measure. By the Lebesgue density theorem, there must be some $\sigma \in 2^{<\omega}$ so that

$$\mu(A_{n,e} \cap [\sigma]) > \frac{3}{4} \cdot 2^{-|\sigma|}.$$

Let

$$B_{n,e} = \{y \succ \sigma \mid \mu(\{z \succ \sigma \mid E(\Phi_e^{H_n^z}, \Phi_e^{H_n^y})\}) > \frac{3}{4} \cdot 2^{-|\sigma|}\}.$$

Then by Theorem 1.4, $B_{n,e}$ is a Π_1^1 set. Moreover $B_{n,e} = A_{n,e} \cap [\sigma]$. So $B_{n,e}$ has positive measure. Then by Theorem 1.5, $B_{n,e}$ contains a hyperarithmetic real. Thus $A = 2^\omega$. $\square$

Note that Proposition 2.1 fails for Σ_1^1 -equivalence relations due to (1) of Theorem 1.2. Here we give a much simpler example. Let $E(x, y)$ if and only if $x = y$ or $x \notin \Delta_1^1$ and $y \notin \Delta_1^1$. Then E is a Σ_1^1 -equivalence relation and for any nonhyperarithmetic real x , $\text{Spec}_{E,h}(x) = \{z \mid z \notin \Delta_1^1\}$.

3. ON Σ_2^1 -RELATIONS

For any real x , let $\mathbb{P}_x = (\mathbf{P}_x, \leq)$ be the random forcing over $L[x]$, the constructible universe relative to x .

Theorem 3.1. *Assume that $\mu(\{x \mid x \text{ is } L\text{-random}\}) = 1$. Then for any Σ_2^1 -relation E and real x , if $\text{Spec}_{E,L}(x) \supseteq \{z \in 2^\omega \mid z \notin L\}$, then $\text{Spec}_{E,L}(x) = 2^\omega$.*

Proof. Note that by the assumption, for almost every real x , $\mu(\{y \mid y \text{ is } L[x]\text{-random}\}) = 1$.

Let E be a Σ_2^1 -relation and x be a real so that $\text{Spec}_{E,L}(x) \supseteq \{z \in 2^\omega \mid z \notin L\}$. Since E is Σ_2^1 , there must be some Π_1^1 -relation $R_0 \subseteq (2^\omega)^3$ so that

$$\forall y \forall z (E(y, z) \leftrightarrow \exists s R_0(y, z, s)).$$

By the Shoenfield absoluteness theorem,

$$\forall y \forall z (E(y, z) \leftrightarrow \exists s \in L_{\omega_1^{L[y \oplus z]}}[y \oplus z] R_0(y, z, s)).$$

In particular,

$$\forall y (E(y, x) \leftrightarrow \exists s \in L_{\omega_1^{L[y \oplus x]}}[y \oplus x] R_0(y, x, s)).$$

Whence

$$z \in \text{Spec}_{E,L}(x) \leftrightarrow \exists t \exists y \exists s (t \text{ codes a well ordering} \wedge y \in L_{|t|}[z] \wedge s \in L_{|t|}[y \oplus x] \wedge R_0(y, x, s)).$$

Note that, by the assumption, the set $\text{Spec}_{E,L}(x)$ is $\Sigma_2^1(x)$ and conull.

By the assumption, there are conull many L -random reals. Since random forcing does not collapse cardinals, the set $\{y \mid \omega_1^L = \omega_1^{L[y]}\}$ is conull. Thus, without loss of generality, we may also assume that $\omega_1^{L[x]} = \omega_1^L$ and the set of $L[x]$ -random reals is of measure 1. So the set $\{y \mid \omega_1^{L[x \oplus y]} = \omega_1^L\}$ is also conull.

For any real t coding a well ordering, let

$$z \in R_{1,t} \leftrightarrow \exists y \in L_{|t|}[z] \exists s \in L_{|t|}[y \oplus x] (R_0(y, x, s)).$$

Then $R_{1,t} \subseteq \text{Spec}_{E,L}(x)$ is a $\Pi_1^1(t \oplus x)$ -set and so measurable. Moreover, if z is $L[x]$ -random, then $z \in \text{Spec}_{E,L}(x)$ if and only if $z \in R_{1,t}$ for some real $t \in L$ coding a well ordering. Since $\mu(\text{Spec}_{E,L}(x)) = 1$ and the set of $L[x]$ -random reals is of measure 1, there must be some $L[x]$ -random real z and a real $t \in L$ coding a well ordering so that $z \in R_{1,t}$. Then there must be some condition $p \in \mathbf{P}_x$ so that $z \in p$ and

$$p \Vdash_{\mathbb{P}_x} \exists \dot{y} \in L_{|t|}[\dot{z}] \exists \dot{s} \in L_{|t|}[\dot{y} \oplus x] (R_0(\dot{y}, x, \dot{s})).$$

Since $\mu(p) > 0$ and almost every real in p is $L[x]$ -random, we have that $\mu(R_{1,t}) > 0$. Fix such a real $t \in L$ to be t_0 . By the countable additivity of Lebesgue measure, there must be some formula φ in the set theory language so that the set

$$R_{1,t_0,\varphi} = \{z \mid \exists y \exists s \in L_{|t_0|}[y \oplus x] (\forall n (n \in y \leftrightarrow L_{|t_0|}[z] \models \varphi(n)) \wedge R_0(y, x, s))\}$$

has positive measure. Then there must be some $\sigma \in 2^{<\omega}$ so that

$$\mu(R_{1,t_0,\varphi} \cap [\sigma]) > \frac{3}{4} \cdot 2^{-|\sigma|}.$$

Now we try to get rid of the parameter x .

Let

$$S = \{r \mid \mu(\{z \succ \sigma \mid \exists y \exists s \in L_{|t_0|}[y \oplus r] (\forall n (n \in y \leftrightarrow L_{|t_0|}[z] \models \varphi(n)) \wedge R_0(y, r, s))\}) > \frac{3}{4} \cdot 2^{-|\sigma|}\}.$$

Then S is a $\Pi_1^1(t_0)$ -set and every real in S is E -equivalent to x . Since $x \in S$, we have that S is not empty. Thus there must be some t_0 -constructible, and so constructible, real in S .

This completes the proof. $\square$

Corollary 3.2. (1) Assume $\omega_1^L < \omega_1$. Then for any Σ_2^1 -relation E and real x , if $\text{Spec}_{E,L}(x) \supseteq \{z \in 2^\omega \mid z \notin L\}$, then $\text{Spec}_{E,L}(x) = 2^\omega$.
(2) Assume that $MA + 2^{\aleph_0} > \aleph_1$, where MA is Martin's axiom. Then for any reals x_0, x and $\Sigma_2^1(x_0)$ -relation E , if $\text{Spec}_{E,L}(x) \supseteq \{z \in 2^\omega \mid z \notin L[x_0]\}$, then $\text{Spec}_{E,L}(x) = 2^\omega$.

Proof. (1). If $\omega_1^L < \omega_1$, then the set of L -random random reals is conull. Thus the assumption of Theorem 3.1 is satisfied.

(2). We prove the lightface version. The boldface version follows immediately by a relativization. By $MA + 2^{\aleph_0} > \aleph_1$, a union $\aleph_1$ -many null sets is null. So for any real x , the set of $L[x]$ -random reals is of measure 1. Thus the assumption of Theorem 3.1 is satisfied. $\square$

Note the conclusion of Theorem 3.1 cannot be proved under ZFC . If $V = L[g]$ for a Sacks generic real g , then there are only two constructible degrees in V . Let $E(x, y)$ if and only if $x = y$. Then for any nonconstructible real x , $\text{Spec}_{E,L}(x) = \{z \mid z \notin L\}$.

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