CHARACTERIZING STRONG RANDOMNESS VIA MARTIN-LÖF RANDOMNESS

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ABSTRACT. We introduce two methods to characterize strong randomness notions via Martin-Löf randomness. By applying these methods, we investigate $\emptyset'$-Schnorr randomness.

1. Introduction

The goal of this paper is to characterize strong randomness notions via Martin-Löf randomness.

In the literature, various randomness notions were introduced for different motivations. The most commonly accepted one is Martin-Löf randomness. Martin-Löf randomness has quite a number of nice properties. For example, van-Lambalgen's theorem holds for Martin-Löf randomness and it can be characterized by Kolmogorov complexity, etc. (these results can be found in [5] and [18]). So we view Martin-Löf randomness as the standard one. For the other randomness notions stronger than Martin-Löf's, we call them strong randomness notions.

One of the goals of algorithmic randomness theory is to compare randomness notions. To compare two randomness notions, we often need to show which randomness notion is stronger. But this is not just what we want to know. We need to know not only the question which one is stronger but also the question how strong it is? So we need to measure the strength of randomness notions.

There are many ways to measure the strength of randomness notions. For example, by comparing the Kolmogorov complexity of randomness notions, one may compare their strength. But there are two flaws about the Kolmogorov complexity: One is that it is difficult to describe the exact Kolmogorov complexity of a randomness notion. The only successful example is the characterization of $\emptyset'$-randomness by the prefix free Kolmogorov complexity (see [13]). Moreover, for some randomness notions, we don't even know whether they are closed upward in the K-degrees; Another one is many randomness notions cannot be classified level by level. For example, Chaitin's Ω is Martin-Löf random but not $\emptyset'$-random. However, every $\emptyset'$-random real has an incomparable K-degree with Ω (see [15]).

In this paper, we propose a general way to measure the strength of randomness notions. Because those randomness notions weaker than Martin-Löf's have unusual

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properties and are not commonly considered, or in my opinion at least, as "real randomness", we focus on the stronger ones. The proposed way is to characterize strong randomness notions via Martin-Löf randomness. In other words, given a randomness notion $\mathscr{A}$ stronger than Martin-Löf randomness, can it be described precisely in terms of oracles relativized to Martin-Löf randomness? If this can be done, then we may transfer the studying of $\mathscr{A}$ to the studying of the sets of oracles corresponding to $\mathscr{A}$. Let's use $\Delta(\mathscr{A})$ to denote a set of oracles corresponding to $\mathscr{A}$. So the question can be translated into the question how powerful are the reals in $\Delta(\mathscr{A})$? Or which Turing degrees are in $\Delta(\mathscr{A})$? Then we may apply the results in computability theory, which is well studied, to study algorithmic randomness theory. This kind of characterization has some advantages. For example, by a carefully selection of $\Delta(\mathscr{A})$, we may obtain a Kolmogorov complexity characterization of $\mathscr{A}$ (see Subsection 3.3). Moreover, such characterizations also help to clarify the relationship between lowness and highness properties (see Proposition 3.5) and study the structure of LR-degrees (such results spread throughout the paper).

We organize the paper as follows: In Section 2, we review the definitions and notations; In section 3, we introduce two concrete methods to characterize strong randomness notions by Martin-Löf randomness; In section 4 , we study Π-type characterization for $\emptyset'$-Schnorr randomness; In section 5, by putting all the previous results together, we give a Σ-type characterization for $\emptyset'$-Schnorr randomness; We finish the paper by giving some remarks about characterizing other strong randomness notions in Section 6.

2. Preliminary

Mostly we follow the terminology and notions from [5]. For the facts in algorithmic randomness theory, we refer readers to [5] and [18]. For the facts in computability theory, we refer readers to [20] and [12].

A real x is an element in Cantor space. Given a set of real U, we use $\mu(A)$ to denote the Lebesgue measure of U. $x \oplus y = \{n \mid \exists m \in x(n = 2m) \vee \exists m \in y(n = 2m + 1)\}$. $\oplus_{i\in\omega} z_i = \{\langle i, n\rangle \mid n \in z_i\}$.

Given two reals x and y, $x =^* y$ means that for co-finitely many n's, $x(n) = y(n)$.

For any partial computable function Φ, we use $\Phi(n)[s]$ to denote the n-th value of Φ at stage s (if it is defined; otherwise, we use $\Phi(n)[s] \uparrow$ to denote that it is undefined). Given a c.e. set U, we use $U[s]$ to denote the state of U enumerated up to stage s. For a real x, we use x' to denote the Turing jump of x. x is low if $x' \equiv_T \emptyset'$.

Given two reals x and y, we say that x is c.e. traceable by y if for every function $f \leq_T x$, there is a uniformly y-c.e sequence $\{T_e\}_{e\in\omega}$ and a computable function h so that for every e, $|T_e| \leq h(e)$ and $f(e) \in T_e$.

A Schnorr-test is a uniformly c.e. sequence of open sets $\{U_n\}_{n\in\omega}$ so that $\mu(U_n) = 2^{-n}$. A real x is Schnorr random if and only if for any Schnorr test $\{U_n\}_{n\in\omega}$, $x \notin \bigcap_n U_n$.

A Martin-Löf test is an uniformly c.e. sequence of open sets $\{U_n\}_{n\in\omega}$ so that $\mu(U_n) < 2^{-n}$ for every n. A real x is Martin-Löf random (or 1-random) if for every Martin-Löf test $\{U_n\}_{n\in\omega}$, $x \notin \bigcap_{n\in\omega} U_n$. There exists a universal Martin-Löf test. A very special Martin-Löf random real is Chaitin's Ω.

A generalized Martin-Löf test is an uniformly c.e. sequence of open sets $\{U_n\}_{n\in\omega}$ so that $\lim_{n\to\infty} \mu(U_n) = 0$ for every n. A real x is weakly-2-random if for every generalized Martin-Löf test $\{U_n\}_{n\in\omega}$, $x \notin \bigcap_{n\in\omega} U_n$. There is no universal generalized Martin-Löf test.

We use ML, W2R, Sch to denote the collection of Martin-Löf random, weakly-2-random and Schnorr random reals respectively.

All these notions can be relativized. We use x-randomness to denote Martin-Löf randomness relativized to x.

$x \leq_{LR} y$ if for every y-random real is x-random.

Given two randomness notions R and S, let

$$\mathrm{Low}(\mathrm{R},\mathrm{S}) = \{x \mid \mathrm{R} \subseteq \mathrm{S(x)}\}$$

and

$$\mathrm{High}(\mathrm{R},\mathrm{S}) = \{x \mid \mathrm{R(x)} \subseteq \mathrm{S}\}$$

, where R(x) and S(x) denote R, S relativized to x respectively.

We use C and K to denote Kolmogorov complexity and prefix free Kolmogorov complexity respectively.

$\langle \cdot, \cdot \rangle$ is a recursive 1-1 onto function from $\omega \times \omega$ to ω so that for every pair $\langle i, j\rangle$, $\langle i, j\rangle \leq \max\{i^3, j^3\}$. We also define $\langle \cdot,\cdot,\cdot\rangle = \langle \cdot, \langle \cdot,\cdot\rangle\rangle$. We identify an open set U as a prefix-free subset of $2^{<\omega}$. We also identify a finite string $\sigma \in 2^{<\omega}$ as a natural number.

3. Two methods to characterize strong randomness notions

We introduce two methods to characterize strong randomness notions.

3.1. Π-type characterization.

The first is a Π-type characterization.

Definition 3.1. *Given a randomness notion $\mathscr{A}$ stronger than Martin-Löf randomness, we use $\mathfrak{F}(\mathscr{A})$ to denote the collection of all the classes R's which have the property that for every real z, $z \in \mathscr{A}$ if and only if for every real $x \in R$, $z \in \mathrm{ML(x)}$.*

Intuitively, every class $R \in \mathfrak{F}(\mathscr{A})$ characterizes randomness notion $\mathscr{A}$. For example, let $\mathscr{A} = \mathrm{ML}$, then the Turing degree $\mathbf{0} = \{x \mid x \text{ is computable.}\}$ belongs to $\mathfrak{F}(\mathrm{ML})$. Note that $\mathfrak{F}(\mathscr{A})$ may be empty even if $\mathscr{A}$ is stronger than ML (see the discussion in Section 6).

Suppose that $\mathfrak{F}(\mathscr{A})$ is not empty, then may pick up a special class from $\mathfrak{F}(\mathscr{A})$. Let

$$\widetilde{R} = \bigcup_{R \in \mathfrak{F}(\mathscr{A})} R.$$

Then it is clear that $\widetilde{R} \in \mathfrak{F}(\mathscr{A})$. So $\widetilde{R}$ is the largest member in $\mathfrak{F}(\mathscr{A})$. Thus we may use the unique set $\widetilde{R}$ to characterize $\mathscr{A}$. This defines a partial map Π from strong randomness notions to sets of reals so that

$$\Pi(\mathscr{A}) = \widetilde{R}.$$

There are two problems about the map Π. The first is that $\Pi(\mathscr{A})$ may not exist. Obviously for any randomness notion $\mathscr{A}$ weaker than ML, $\Pi(\mathscr{A})$ is undefined. The second is about the complexity of $\Pi(\mathscr{A})$. By the definition of $\Pi(\mathscr{A})$, $\Pi(\mathscr{A})$ does not appear to be second order arithmetical definable. So even $\Pi(\mathscr{A})$ is defined, $\Pi(\mathscr{A})$ may be rather complicated. But we have a better calculation of the complexity of $\Pi(\mathscr{A})$.

Proposition 3.2. *If $\Pi(\mathscr{A})$ exists, then*

(1) *If $R \in \mathfrak{F}(\mathscr{A})$ and $x \leq_{LR} y$ for some $y \in R$, then $R \cup \{x\} \in \mathfrak{F}(\mathscr{A})$;*
(2) $\Pi(\mathscr{A}) = \mathrm{Low}(\mathscr{A}, \mathrm{ML})$.

Proof. Suppose that $\Pi(\mathscr{A})$ exists.

For (1). Obviously.

For (2). Clearly $\Pi(\mathscr{A}) \subseteq \mathrm{Low}(\mathscr{A}, \mathrm{ML})$.

For any $R \in \mathfrak{F}(\mathscr{A})$ and $x \in \mathrm{Low}(\mathscr{A}, \mathrm{ML})$, we have that $R \cup \{x\} \in \mathfrak{F}(\mathscr{A})$. So $\Pi(\mathscr{A}) = \mathrm{Low}(\mathscr{A}, \mathrm{ML})$. □

So if $\mathscr{A}$ is Σ^1_1, then $\Pi(\mathscr{A})$ is Π^1_1. In some special cases, $\Pi(\mathscr{A})$ can be fairly simple. For example, the set $KT = \{x \mid x \text{ is } K\text{-trvial. }\}$ is arithmetical. But $KT = \mathrm{Low}(\mathrm{ML}, \mathrm{ML})$ (see [17]). So $\Pi(\mathrm{ML})$ is arithmetical.

3.2. **Σ-type characterization.** The second is a Σ-type characterization.

Definition 3.3. *Given a strong randomness notion $\mathscr{A}$ than Martin-Löf randomness, we use $\mathfrak{G}(\mathscr{A})$ to denote the collection of all the classes R's which have the property that for every real z, $z \in \mathscr{A}$ if and only if for there exists some real $x \in R$, $z \in \mathrm{ML}(\mathrm{x})$.*

For example, the Turing degree $\mathbf{0} = \{x \mid x \text{ is computable.}\}$ also belongs to $\mathfrak{G}(\mathrm{ML})$. Note that $\mathfrak{G}(\mathscr{A})$ maybe empty.

Suppose that $\mathfrak{G}(\mathscr{A})$ is not empty, then may also pick up a special class from $\mathfrak{G}(\mathscr{A})$. Let

$$\widehat{R} = \bigcup_{R \in \mathfrak{G}(\mathscr{A})} R.$$

Then it is clear that $\widehat{R} \in \mathfrak{G}(\mathscr{A})$. So $\widehat{R}$ is the largest member in $\mathfrak{G}(\mathscr{A})$. So from the randomness notion $\mathscr{A}$, we may find the unique set $\widehat{R}$ to characterize $\mathscr{A}$. This defines a partial map Σ from strong randomness notions to sets of reals so that

$$\Sigma(\mathscr{A}) = \widehat{R}.$$

Proposition 3.4. *If $\Sigma(\mathscr{A})$ exists, then*

(1) *If $R \in \mathfrak{G}(\mathscr{A})$ and $y \leq_{LR} x$ for some $y \in R$, then $R \cup \{x\} \in \mathfrak{G}(\mathscr{A})$;*
(2) $\Sigma(\mathscr{A}) = \mathrm{High}(\mathrm{ML}, \mathscr{A})$.

Proof. Suppose that $\Sigma(\mathscr{A})$ exists.

For (1). Obviously.

For (2). Clearly $\Sigma(\mathscr{A}) \subseteq \mathrm{High}(\mathrm{ML}, \mathscr{A})$.

For any $R \in \Sigma(\mathscr{A})$ and $x \in \mathrm{High}(\mathrm{ML}, \mathscr{A})$, we have that $R \cup \{x\} \in \mathfrak{F}(\mathscr{A})$. So $\Sigma(\mathscr{A}) = \mathrm{High}(\mathrm{ML}, \mathscr{A})$. □

So if $\mathscr{A}$ is Π^1_1, then $\Sigma(\mathscr{A})$ is Π^1_1. In some special cases, $\Sigma(\mathscr{A})$ can be fairly simple. For example, $2^\omega = \mathrm{High}(\mathrm{ML}, \mathrm{ML})$. So $\Sigma(\mathrm{ML})$ is arithmetical.

The following proposition builds a connection between $\Sigma(\mathscr{A})$ and $\Pi(\mathscr{A})$.

Proposition 3.5. *Suppose that both $\Sigma(\mathscr{A})$ and $\Pi(\mathscr{A})$ are defined. Then a real $x \in \Sigma(\mathscr{A})$ if and only if for every $y \in \Pi(\mathscr{A})$, $y \leq_{LR} x$.*

Proof. If $x \in \Sigma(\mathscr{A})$, then every x-random real z belongs to $\mathscr{A}$. So z must be y-random for every $y \in \Pi(\mathscr{A})$. Thus $x \geq_T y$ for every $y \in \Pi(\mathscr{A})$.

If every $y \in \Pi(\mathscr{A})$, $y \leq_{LR} x$. Then every x-random real must belong to $\mathscr{A}$. In other words, $x \in \mathrm{High}(\mathrm{ML}, \mathscr{A})$. By Proposition 3.4, $x \in \Sigma(\mathscr{A})$. □

3.3. **Characterizing randomness via Kolmogorov complexity.** In [14], it was asked whether some randomness notions can be characterized by Kolmogorov complexity and whether they are closed upwards in the K-degrees or C-degrees. In [9], Hölzl et al prove a number of results related. But their characterization is not very satisfactory. Some of their characterizations don't even guarantee the upward closedness in the K-degrees. For example, it is not even clear, according to their characterization, whether $\mathrm{ML}(\emptyset')$, a very simple randomness notion, is upward closed in the K-degrees. Here we give a program to answer these questions by applying the previous results.

We need the following result.

Theorem 3.6 (Miller and Yu [15])**.** *$x \oplus y$ is Martin-Löf random if and only if there is a constant c such that for every n, $K(x \upharpoonright n) + C(y \upharpoonright n) \geq 2n - c$.*

By applying Theorem 3.6 and the previous discussions, we have the following result.

Proposition 3.7. *Given a randomness notion $\mathscr{A}$ stronger than* ML. *Suppose $R \subseteq$* ML*, then*

(1) *If $R \in \mathfrak{F}(\mathscr{A})$, then $x \in \mathscr{A}$ if and only if for every $y \in R$, there is a constant c so that for every n, $K(x \upharpoonright n) \geq 2n - C(y \upharpoonright n) - c$;*
(2) *If $R \in \mathfrak{G}(\mathscr{A})$, then $x \in \mathscr{A}$ if and only if there is some $y \in R$ and a constant c so that for every n, $K(x \upharpoonright n) \geq 2n - C(y \upharpoonright n) - c$.*

In either case, $\mathscr{A}$ is closed upward in the K-degrees.

Proof. Suppose $R \subseteq$ ML.

For (1). Suppose that $R \in \mathfrak{F}(\mathscr{A})$. By van-Lambalgen's Theorem, $x \in \mathscr{A}$ if and only if for every $y \in R$, $x \oplus y$ is Martin-Löf random and so, by Theorem 3.6, if and only if there is a constant c so that for every n, $K(x \upharpoonright n) \geq 2n - C(y \upharpoonright n) - c$. So if $z \geq_K x$, then $z \oplus y$ is Martin-Löf random for every $y \in R$. Thus $z \in \mathscr{A}$.

For (2). Then $x \in \mathscr{A}$ if and only if there is a $y \in R$, $x \oplus y$ is Martin-Löf random and so, by Theorem 3.6, if and only if there is a constant c so that for every n, $K(x \upharpoonright n) \geq 2n - C(y \upharpoonright n) - c$. So if $z \geq_K x$, then $z \oplus y$ is Martin-Löf random for some $y \in R$. Thus $z \in \mathscr{A}$. □

It is clear that Proposition 3.7 remains true if one interchanges K with C.

For example, $\{\Omega\} \in \mathfrak{F}(\mathrm{ML}(\emptyset')) \cap \mathfrak{G}(\mathrm{ML}(\emptyset'))$, so $\mathrm{ML}(\emptyset')$ is closed upward in the both K-degrees and C-degrees.

In the subsequent sections, we apply the ideas in this section to study some strong randomness notions. In particular, we obtain a complete characterization of $\emptyset'$-Schnorr randomness.

4. The Π-type characterization of $\emptyset'$-Schnorr randomness

In this section, we study $\Pi(\mathrm{Sch}(\emptyset'))$ by applying the methods in Section 3.

4.1. **The collection of low reals belongs to $\mathfrak{F}(\mathrm{Sch}(\emptyset'))$.** We show that $\mathfrak{F}(\mathrm{Sch}(\emptyset'))$ is not empty.

Theorem 4.1. *For every $\emptyset'$-Schnorr test $\{U_e^{\emptyset'}\}_{e\in\omega}$, there is a real z with $z' \leq_T \emptyset'$ such that there is z-Martin-Löf-test $\{V_e^z\}_{e\in\omega}$ so that $\bigcap_{e\in\omega} V_e^z \supseteq \bigcap_{e\in\omega} U_e^{\emptyset'}$.*

Theorem 4.1 also follows the proof in Theorem 4.5. But we give a proof of Theorem 4.1 as a warming up of the proof of Theorem 4.5. Moreover, the proof here is more flexible than there. For example, one may combine it with genericity requirements.[1]

Proof. We prove that for every $\emptyset'$-Schnorr test $\{U_e^{\emptyset'}\}_{e\in\omega}$, there is a real z with $z' \leq_T \emptyset'$ such that there is z-Martin-Löf-test $\{V_e^z\}_{e\in\omega}$ so that $\bigcap_{e\in\omega} V_e^z \supseteq \bigcap_{e\in\omega} U_e^{\emptyset'}$.

The proof is by a finite injury argument.

We will describe the strategies and leave the rest to the reader.

We build a low real z and z-Martin-Löf test $\{V_e^z\}_{e\in\omega}$ by a full approximation priority argument. We need to satisfy two kinds of requirements:

[1]Mr. Higuchi [8] proves that the real z in Theorem 4.1 can be 1-generic.

$N_e : \exists^\infty s \Phi_e^{z_s}(e)[s] \downarrow \implies \Phi_e^z(e) \downarrow$;
$P_e : U_{2e}^{\emptyset'} \subseteq V_e^z$.

It is easy to see that if all the N_e's are satisfied, then $z' \leq_T \emptyset'$ (see [20]).

To satisfy P_e, we need to decompose P_e into infinitely many subrequirements $P_{\langle e,n\rangle}$. For every e, n, let

$$U_e^{\emptyset'} \upharpoonright n = U_e^{\emptyset'} \cap 2^{\leq l_n^e} = \{\sigma \mid |\sigma| \leq l_n^e \wedge \sigma \in U_e^{\emptyset'}\}$$

where l_n^e is the least number l such that $\mu(U_e^{\emptyset'} \cap 2^{\leq l}) > 2^{-e}(1 - 2^{-2^n})$. It is Obvious that $U_e^{\emptyset'} \upharpoonright n \subseteq U_e^{\emptyset'} \upharpoonright (n+1)$ for every n. Notice that since $\{U_e^{\emptyset'}\}_{e\in\omega}$ is a $\emptyset'$-Schnorr test, we may $\emptyset'$-recursively find l_n^e for every n and e.

So there is a computable function $f : \omega \times 2^{<\omega} \times \omega^2 \to 2$ so that for every e, n and σ,

(1) $\lim_s f(e, \sigma, s, n) = 0$ or 1;
(2) $\lim_s f(e, \sigma, s, n) = 1$ if and only if $\sigma \in U_e^{\emptyset'} \upharpoonright n$.

Set

$$P_{\langle e,n\rangle} : U_{2e}^{\emptyset'} \upharpoonright n \subseteq V_e^z.$$

It suffices to satisfy those $P_{\langle e,n\rangle}$'s so that $e \leq n$. Then we may set the priority list as $N_e < P_{\langle 0,e\rangle} < P_{\langle 1,e\rangle} < ...P_{\langle e,e\rangle} < N_{e+1}$, $e \in \omega$.

As in the usual finite injury argument, we build a restriction function $r(e, s) > \phi_e^{z_s}(e)$ for every negative requirement N_e at every stage e where $\phi_e^{z_s}(e)$ is the use function of $\Phi_e^{z_s}(e)[s]$. Set

$$R(e, s) = \sum_{i \leq e} r(i, s).$$

At stage s, N_e requires attention if $\Phi_e^{z_s}(e)[s] \downarrow$ but N_e has not received attention after it has been initialized (if ever) before stage s. Then N_e sets up a restriction $r(e, s)$.

At every stage s, for every e, n, let

$$U_e^{\emptyset'_s}[s] \upharpoonright n = U_e^{\emptyset'_s}[s] \cap 2^{\leq l_n^e[s]} = \{\sigma \in 2^{<\omega} \mid |\sigma| \leq l_n^e[s] \wedge \sigma \in U_e^{\emptyset'_s}[s]\}$$

where $l_n^e[s]$ is the least number l such that $\mu(U_e^{\emptyset'_s}[s] \cap 2^{\leq l}) > 2^{-e}(1-2^{-2^n})$. Obviously $\lim_s l_n^e[s] = l_m^e$. Obviously $U_e^{\emptyset'_s}[s] \upharpoonright n \subseteq U_e^{\emptyset'_s}[s] \upharpoonright (n+1)$ for every n.

The basic strategy for $P_{\langle e,n\rangle}$ is: At any stage s, for each σ, there is a follower $\langle 2^e, \sigma, t_s\rangle$ attached to σ. If σ enters $U_{2^e}^{\emptyset'_s}[s] \upharpoonright n - U_{2^e}^{\emptyset'_s}[s] \upharpoonright (n-1)$ (i.e. $f(2^e, \sigma, s, n) = 1$ but $f(2^e, \sigma, s, n') = 0$ for all $n' < n$), then we set $z_s(\langle 2^e, \sigma, t_s\rangle) = 1$. If σ exits $U_{2^e}^{\emptyset'_s}[s] \upharpoonright n$ (i.e. $f(2^e, \sigma, s, n) = 0$), then we set $z_s(\langle 2^e, \sigma, t_s\rangle) = 0$. So we may define $V_e^{z_s}[s] = \{\sigma \mid z_s(\langle 2^e, \sigma, t_s\rangle) = 1\}$ and $V_e^z = \{\sigma \mid \exists s(z(\langle 2^e, \sigma, t_s\rangle) = 1)\}$.

The rule attributing a follower to $P_{\langle e,n\rangle}$ at stage s is: For any σ with $l_n^e[s] \geq |\sigma| > l_{n-1}^e[s]$, if either there is no a follower attributed to σ at stage $s-1$ or the follower attributed at $s-1$ was initialized, we attribute a new follower $\langle 2^e, \sigma, t_s\rangle$ to σ such that t_s greater than all the parameters mentioned in the higher priority requirements no later than stage s; otherwise, we keep the older attributed follower being unchanged by setting $t_s = t_{s-1}$.

$P_{\langle e,n\rangle}$ requires attention at stage s if either

(1) σ enters $U^{\emptyset'_s}_{2^e}[s]\upharpoonright n - U^{\emptyset'_s}_{2^e}[s]\upharpoonright (n-1)$ but $z_s(\langle 2^e,\sigma,t_s\rangle)=0$. The action is to $z_{s+1}(\langle 2^e,\sigma,t_s\rangle)=1$; or

(2) σ exits $U^{\emptyset'_s}_{2^e}[s]\upharpoonright n$ but $z_s(\langle 2^e,\sigma,t_s\rangle)=1$. The action is to $z_{s+1}(\langle 2^e,\sigma,t_s\rangle)=0$.

To avoid the confliction between $P_{\langle e_0,n_0\rangle}$ and $P_{\langle e_1,n_1\rangle}$, say $P_{\langle e_0,n_0\rangle} < P_{\langle e_1,n_1\rangle}$, we initialize all the parameters for $P_{\langle e_1,n_1\rangle}$ and set $z_{s+1}(\langle 2^{e_1},\sigma,t_s\rangle)=0$ for any parameter $\langle 2^{e_1},\sigma,t_s\rangle$ for $P_{\langle e_1,n_1\rangle}$ once upon $P_{\langle e_0,n_0\rangle}$ receives attention. This cannot happen infinitely often by the property of f and $\{U^{\emptyset'}_e\}_{e\in\omega}$.

Notice that there are at most $2^{-2^e-(2^n-1)}$ measure of clopen sets which can be put into V^z_e by $P_{\langle e,n\rangle}$ for any pair $\langle e,n\rangle$.

Since $\{U^{\emptyset'}_e\}_{e\in\omega}$ is a $\emptyset'$-Schnorr test, by a usual finite injury argument, it is easy to show that N_e will be injured at most finitely many times for every e. Thus N_e is satisfied and so z must be low.

For each $P_{\langle e,n\rangle}$ with $n\geq e$, there are n many negative requirements $\{N_k\}_{k\leq n}$ having higher priority than $P_{\langle e,n\rangle}$. For each $k\leq n$, once N_k set up a restriction $r(k,s)$, then $P_{\langle e,n\rangle}$ cannot change its parameters less than $R(k,s)$ anymore until some $P_{\langle e',n'\rangle}$ higher than N_k receives attention. So $P_{\langle e,n\rangle}$ may make at most 2^n-times mistakes by putting clopen sets into U^z_n. The measure of the sum of these mistakes is no more than $2^n\cdot 2^{-2^e-2^n+1}$. Thus for $e\geq 2$,

$$\mu(V^z_e)\leq \sum_{n\in\omega}(2^n)\cdot 2^{-2^e-2^n+1}\leq \sum_{n\in\omega}2^{-2^e-n+1}=2^{-2^e+2}\leq 2^{-e}.$$

So $\{V^z_e\}_{e\geq 2}$ is a z-Martin-Löf test. By the definition of V^z_e, $U^{\emptyset'}_{2^e}\subseteq V^z_e$ for every e. So $\bigcap_{e\in\omega}U^{\emptyset'}_e\subseteq\bigcap_{e\in\omega}V^z_e$.

This completes the proof. □

Corollary 4.2. [2] *For any reals $x\geq_T\emptyset'$ and z, the followings are equivalent:*

(1) *z is x-Schnorr random;*
(2) *For any real y with $y'\leq_T x$, z is weakly-2-random relativized to y;*
(3) *For any real y with $y'\leq_T x$, z is Martin-Löf-random relativized to y.*

So $L_x=\{y\mid y'\equiv_T x\}$ belongs to $\mathfrak{F}(\mathrm{Sch(x)})$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $y'\leq_T x$ and $z\in\{U^y_e\}_{e\in\omega}$ is a generalized Martin-Löf test relativized to y. Since the statement "$\mu(U^y_e)>p$" is $\Sigma^0_1(y)$ when p ranges over rationals and e ranges over ω, it is not difficult to see that $\{U_e\}^y_{e\in\omega}$ can be covered by a Schnorr test relativized to x. So z must be weakly-2-random relativized to y

(2) $\Longrightarrow$ (3) is obvious.

[2]Mr. Peng, in his Master Thesis [19], studied the so-called **L**-randomness, which is the collection of random reals relativized to all low reals.

We show that (3) $\Longrightarrow$ (1). Since $x \geq_T \emptyset'$, there is a real $z_0 \leq_T x$ so that $z_0' \equiv_T x$. Relativizing the proof of Theorem 4.1 to z_0, every x-Schnorr random real is Martin-Löf-random relativized to y for some y with $z_0 \leq y$ and $y' \leq_T x$. $\square$

It should be pointed out that $\emptyset'$ is the least Turing degree in High(Sch, ML) (see [6]). So Corollary 4.2 characterizes all the relativized Schnorr randomness stronger than Martin-Löf randomness.

We give an application of Theorem 4.1 to LR-degrees.

Corollary 4.3. *For any pair of low reals x and y, there is a low real $z \geq_{LR} x, y$.*

Proof. It is easy to see that given any two low reals x and y and universal x- and y-Martin-Löf test $\{V_n^x\}_{n\in\omega}$ and $\{V_n^y\}_{n\in\omega}$, there is a $\emptyset'$-Schnorr test $\{U_n^{\emptyset'}\}_{n\in\omega}$ so that $\bigcap_{n\in\omega} U_n^{\emptyset'} \supset \bigcap_{n\in\omega} V_n^x \cup \bigcap_{n\in\omega} V_n^y$. Then by Theorem 4.1, there is a real z with $z' \leq_T \emptyset'$ such that there is z-Martin-Löf-test $\{V_n^z\}_{n\in\omega}$ so that $\bigcap_{n\in\omega} V_n^z \supseteq \bigcap_{n\in\omega} U_n^{\emptyset'}$. So every z-random real is both x- and y-random. $\square$

Diamondstone, by a direct argument, proves the following stronger result.

Theorem 4.4 (Diamondstone [3]). *For any pair of low reals x and y, there is a low c.e. real $z \geq_{LR} x, y$.*

4.2. **On low random reals.** We prove the following result.

Theorem 4.5. *For every low real z, there is a low random real $x \geq_{LR} z$.*

The proof of Theorem 4.5 is a combination of Kučera's coding with the proof of low basis theorem. We need a technique lemma.

Lemma 4.6 (Kučera [11] and Gács [7], see Lemma 3.3.1 in [18]). *Suppose $T \subseteq 2^{<\omega}$ is a tree and $\sigma \in 2^{<\omega}$. If $\mu(T \upharpoonright \sigma) \geq 2^{-r-|\sigma|}$ for some $r \in \omega$ where $T \upharpoonright \sigma = \{\tau \in T \mid \tau \prec \sigma \vee \tau \succ \sigma\}$. Then there are two distinct strings $\sigma_0, \sigma_1 \succ \sigma$ with $|\sigma_0| = |\sigma_1| = |\sigma| + r + 1$ so that $\mu(T \upharpoonright \sigma_i) > 2^{-r-1-|\sigma_i|}$ for $i = 0, 1$.*

Proof. of Theorem 4.5.

If z is low and $\{\widetilde{U}_n^z\}_{n\in\omega}$ is a z-Martin-Löf test, then there must exist a $\emptyset'$-Schnorr test $\{U_n^{\emptyset'}\}_{n\in\omega}$ so that $\bigcap_{n\in\omega} \widetilde{U}_n^z \subseteq \bigcap_{n\in\omega} U_n^z$. So it is sufficient to prove that for every $\emptyset'$-Schnorr test $\{U_n^{\emptyset'}\}_{n\in\omega}$, there is a low random real x and an x-Martin-Löf test $\{V_n^x\}_{n\in\omega}$ so that for every n, $U_{2^n}^{\emptyset'} \subseteq V_n^x$.

Fix a computable tree $T \subseteq 2^{<\omega}$ so that $[T] = \{x \in 2^\omega \mid \forall n(x \upharpoonright n \in T)\}$ only contains Martin-Löf random reals. We may assume that $\mu([T]) > 2^{-1}$.

For every e, let $Q_e = \{\sigma \in 2^{<\omega} \mid \Phi_e^\sigma(e)[|\sigma|] \uparrow\}$ be a computable tree. Let $f : \omega \mapsto \mathcal{P}_{<\omega}(2^{<\omega})$ be a computable bijection where $\mathcal{P}_{<\omega}(2^{<\omega})$ is the collection of all finite subsets of $2^{<\omega}$.

Since $\{U_e^{\emptyset'}\}_{e\in\omega}$ is a $\emptyset'$-Schnorr test, there is a $\emptyset'$-computable function $g:\omega\times\omega\mapsto\omega$ so that for any two numbers e,n, $g(e,n)$ is the unique number m so that

$$f(m)=U_e^{\emptyset'}\upharpoonright n-U_e^{\emptyset'}\upharpoonright(n-1)$$

where $U_e^{\emptyset'}\upharpoonright n$ is a finite subsets so that

$$U_e^{\emptyset'}\upharpoonright n=U_e^{\emptyset'}\cap 2^{\leq l_n^e}=\{\sigma\in 2^{<\omega}\mid |\sigma|\leq l_n^e\wedge\sigma\in U_e^{\emptyset'}\}$$

where l_n^e is the least number l such that $\mu(U_e^{\emptyset'}\cap 2^l)>2^{-e}(1-2^{-2^n})$. Note that $\mu(f(m))\leq 2^{-e-2^n+1}$.

We do the coding construction. It is essentially an effective forcing argument.

At level 0, let $T_0=T$, $\sigma_0=\lambda$, $r_0=1$.

Suppose at level s, we have the following parameters: T_s is a computable tree; $\sigma_s\in 2^{<\omega}$ so that $|\sigma_s|\geq s$ and for every $\tau\in T_s$, either $\tau\succ\sigma_s$ or $\tau\prec\sigma_s$; r_s is a natural number so that $\mu(T_s)>2^{-r_s}$.

At level $s+1$, check whether $\mu([Q_s]\cap[T_s])\geq 2^{-r_s-1}$.

Case(1). No. Then $\mu([T_s])-\mu([Q_s]\cap[T_s])\geq 2^{-r_s-1}$. Since $U_s=2^\omega-Q_s$ is a c.e. open set, we may let t be the least level so that $\mu(U_s[t]\cap[T_s])\geq 2^{-r_s-2}$. Let

$$T_s^1=T_s\cap\{\sigma\mid\exists\tau(\tau\in U_s[t]\wedge(\sigma\succ\tau\vee\sigma\prec\tau)\}$$

be a computable tree. Then $\mu(T_s^1)\geq 2^{-r_s-2}$. Pick up the unique pair j_s and n_s so that $\langle j_s,n_s\rangle=s$. Let $e_s=2^{j_s}$. Set $r_s+3+|\sigma_s|=r_0^s<r_1^s<r_2^s<\ldots<r^s_{g(e_s,n_s)}$ to be a finite sequence so that $r^s_{i+1}=r^s_i+r_s+4+i$ for every $i<g(e_s,n_s)$. Note that

$$\mu(T_s^1\upharpoonright\sigma_s)=\mu(T_s^1)\geq 2^{-r_s-2}\geq 2^{-(r_s+2)-|\sigma_s|}.$$

By Lemma 4.6, it is not difficult to see that there is a finite sequence $\sigma_s\prec\tau_0\prec\tau_1\prec\ldots\prec\tau_{g(e_s,n_s)}$ such that

(1) $\forall i\leq g(e_s,n_s)(|\tau_i|=r_i^s)$;
(2) τ_0 is the leftmost $\tau\in T_s^1$ such that $\sigma_s\prec\tau$ of length r_0^s has the property that $\mu(T_s^1\upharpoonright\tau)>2^{-r_s-3-|\tau|}$;
(3) $\forall i<g(e_s,n_s)-1$, τ_{i+1} is the leftmost $\tau\in T_s^1$ such that $\tau_i\prec\tau$ of length r^s_{i+1} has the property that $\mu(T_s^1\upharpoonright\tau)>2^{-r_s-4-i-|\tau|}$;
(4) $\tau_{g(e_s,n_s)}$ is the rightmost $\tau\in T_s^1$ such that $\tau_{g(e_s,n_s)-1}\prec\tau$ of length $r^s_{g(e_s,n_s)}$ has the property that $\mu(T_s^1\upharpoonright\tau)>2^{-r_s-3-g(e_s,n_s)-|\tau|}$.

Let $\sigma_{s+1}=\tau_{g(e_s,n_s)}$, $T_{s+1}=T_s^1\upharpoonright\sigma_{s+1}$ and $r_{s+1}=r_s+3+g(e_s,n_s)+|\sigma_{s+1}|$.

Case(2). Yes. Then let $T_s^1=Q_s\cap T_s$ be a computable tree. Note that $\mu(T_s^1)\geq 2^{-r_s-1}>2^{-r_s-2}$. Then we perform the same construction as in Case(1).

Let $\sigma_{s+1}=\tau_{g(e_s,n_s)}$, $T_{s+1}=T_s^1\upharpoonright\sigma_{s+1}$ and $r_{s+1}=r_s+3+g(e_s,n_s)+|\sigma_{s+1}|$.

This finishes the construction at level $s+1$.

Obviously $\sigma_s\prec\sigma_{s+1}$ for all s. Let $x=\bigcup_{s\in\omega}\sigma_s$.

The construction is $\emptyset'$-computable, so $x \leq_T \emptyset'$. Moreover, to check whether $\Phi_e^x(e)\uparrow$ or not, one just needs to check which case applied at level $e+1$ in the construction. Again, this is $\emptyset'$-decidable. So $x' \leq_T \emptyset'$.

Now we construct an x-Martin-Löf test $\{V_n^x\}_{n\in\omega}$ so that for every n, $U_{2^n}^{\emptyset'} \subseteq V_n^x$. To do this, we decode the coding construction using x.

At level 0, let $T_0^0 = T$, $\sigma_0^0 = \lambda$ and $r_0^0 = 1$. And put nothing into V_n^x for every n. For any level $s \geq 0$, we always keep these parameters unchanged.

For any computable tree T, define $\mu(T[s])$, the measure of T at level s, to be $2^{-s} \cdot |\{\sigma \in 2^s \mid \sigma \in T\}|$.

Without loss of generality, we may assume that $\mu(T_s^0[s]) > 2^{-r_s^0-1}$ for every $s \geq 0$.

Suppose at level s, we have the following parameters: a finite sequence of numbers $\{r_s^i\}_{i\leq s}$; $\{T_s^i\}_{i\leq s}$ is a finite sequence of computable tree so that $\mu(T_s^i[s]) > 2^{-r_s^i}$ and $T_s^{i+1} \subseteq T_s^i$ for every $i \leq s$; $\sigma_s^i \prec x$ so that $|\sigma_s^i| \geq s$ and for every $\tau \in T_s^i$, either $\tau' \succ \sigma_s'$ or $\tau \prec \sigma_s'$. Note that it is not necessary that for every $i \leq s$, the parameters corresponding to i are defined. Some of them may be void. We also have a finite string $\nu_s \in 2^s$ so that for each $i \geq 0$, $\nu_s(i) = 0$ if and only if $\mu(Q_i \cap T_s^i[s]) \geq 2^{-r_s^i-1}$.

At level $s+1$, check whether there is some $i \leq s$ so that the parameters corresponding to i are defined and $\nu_s(i) = 0$ but $\mu(Q_i \cap T_s^i[s+1]) < 2^{-r_s^i-1}$.

Case(1) Yes. Then we say that j is injured at level $s+1$ for every $j > i$. Pick up the least such i and initialize all the parameters for every j with $j > i$. Then we set up $\nu_{s+1}(i) = 1$.

Then $\mu(T_s^i[s]) - \mu(Q_i \cap T_s^i[s]) \geq 2^{-r_s^i-1}$. Let U_i be a c.e. open set which is the complement of Q_i. Then let $t \leq s$ be the least level so that $\mu(U_i[t] \cap T_s[s]) \geq 2^{-r_s^i-2}$. Let

$$T_s^{1,i} = T_s^i \cap \{\sigma \mid \exists\tau(\tau \in U_s[t] \wedge (\sigma \succ \tau \vee \sigma \prec \tau)\}$$

be a computable tree so that $\mu(T_s^{1,i}[s]) \geq 2^{-r_s^i-2}$. Pick up the unique pair j_i and n_i so that $\langle j_i, n_i\rangle = i$. Let $e_i = 2^{j_i}$. We try to x-computably find a finite sequence $\sigma_s^i \prec \tau_0 \prec \tau_1 \prec ... \prec \tau_k \prec x$ such that

(1) $|\tau_0| = |\sigma_s^i| + 3 + r_s^i$;
(2) $\forall j < k-1(|\tau_{j+1}| = |\tau_j| + |\sigma_s^i| + 4 + j)$;
(3) τ_0 is the leftmost $\tau \in T_s^{1,i}$ such that $\sigma_s \prec \tau$ of length $|\tau_0|$ has the property that $\mu(T_s^{1,i} \upharpoonright \tau[s]) > 2^{-r_s^i-3-|\tau|}$;
(4) $\forall i < k-1$, τ_{i+1} is the leftmost $\tau \in T_s^{1,i}$ such that $\tau_i \prec \tau$ of length $|\tau + i + 1|$ has the property that $\mu(T_s^{1,i} \upharpoonright \tau[s]) > 2^{-r_s^i-4-i-|\tau|}$;
(5) τ_k is the rightmost $\tau \in T_s^{1,i}$ such that $\tau_{k-1} \prec \tau$ of length $|\tau_k|$ has the property that $\mu(T_s^{1,i} \upharpoonright \tau[s]) > 2^{-r_s^i-3-k-|\tau|}$.

If these parameters can be found, then we just let $\sigma_{s+1}^{i+1} = \tau_k$, $T_{s+1}^{i+1} = T_s^{1,i} \upharpoonright \sigma_{s+1}$, $r_{s+1}^{i+1} = |\sigma_s| + k$ and $\nu_{s+1}(i) = 1$. Put all $\sigma \in f(k)$ into $V_{j_i}^x$ if $\mu(f(k)) <$

$2^{-e_i-2^{n_i}+1}$; Otherwise, we cancel this all the parameters defined for i and go to next level. Keep all the parameters corresponding to $i' \leq i$ unchanged.

Case(2) Otherwise. Then for every $i \leq s$, if the parameters corresponding to i are defined, then if $\nu_s(i) = 0$ then $\mu(Q_i \cap T^i_s[s]) \geq 2^{-r^i_s-1}$. But by the construction, if $\nu_s(i) = 1$ then $\mu(Q_i \cap T^i_s[s]) < 2^{-r^i_s-1}$ (since "$\mu(Q_i \cap T^i_s) < 2^{-r^i_s-1}$" is a Σ^0_1-fact). Thus, in this case, if the parameters corresponding to i are defined, then $\nu_s(i) = 0$ if and only if $\mu(Q_i \cap T^i_s[s]) \geq 2^{-r^i_s-1}$.

Case(2.1) Every $i \leq s$ is defined. Then keep the parameters for $i = s+1$ be undefined (the slow down construction is to avoid duplicate construction).

Case(2.2) Otherwise. Pick up the least $i \leq s$ so that the parameters corresponding to i are undefined. There are two subcases:

Case(2.2.1) $\nu_s(i-1) = 1$. Then $\mu(Q_{i-1} \cap T^{i-1}_s[s+1]) < 2^{-r^{i-1}_s-1}$. Then just do the same the construction as in Case (1) by replacing i with $i-1$. We can define the parameters corresponding to i and put $\sigma \in f(k)$ into $V^x_{j_i}$ if $\mu(f(k)) < 2^{-e_i-2^{n_i}+1}$.

Case(2.2.2) $\nu_s(i-1) = 0$. Then $\mu(Q_{i-1} \cap T^{i-1}_s[s+1]) \geq 2^{-r^{i-1}_s-1}$. Then let $T^{1,i-1}_s = Q_s \cap T^{i-1}_s$ be a computable tree. Note that $\mu(T^{1,i-1}_s[s+1]) \geq 2^{-r_s-1} > 2^{-r_s-2}$. Then we perform the same construction as in Case(2.2.1), define the corresponding the parameters to i and put $\sigma \in f(k)$ into $V^x_{j_i}$ if $\mu(f(k)) < 2^{-e_i-2^{n_i}+1}$.

This finishes the decoding construction at level $s+1$. □

Obviously $\{V^x_n\}_{n\omega}$ is an x-c.e. sequence of open sets.

Lemma 4.7. (1) *For any $i \in \omega$ and level s, if $\nu_s(i) = 1 > 0 = \nu_{s+1}(i)$, then there must be some $i' < i$ so that $\nu_s(i') \neq \nu_{s+1}(i')$;*

(2) *For any $i \in \omega$, $|\{s \mid \nu_s(i) \neq \nu_{s+1}(i)\}| \leq 2^i$.*

Proof. For (1). For any level s, if s is the first level so that $\nu_s(i) = 1$, then $\mu(Q_i \cap T^i_s[s+1]) < 2^{-r^i_s-1}$ and so $\mu(Q_i \cap T^i_s) < 2^{-r^i_s-1}$. Thus for any level $t > s$, if the parameters corresponding to i are not initialized between any level s and t, then $\nu_t(i) = \nu_s(i)$. This means that $\nu_t(i)$ changes from 1 to 0 at any level $t+1 > s$ only if the parameters corresponding to i are initialized at level $t+1$. Thus there must be some $i' < i$ so that $\nu_t(i') \neq \nu_{t+1}(i')$.

For (2). Immediately from (1). □

Lemma 4.8. (1) *For some j_0, $\{V^x_j\}_{j>j_0}$ is an x-Martin Löf test;*

(2) *For every j, $U^{\emptyset'}_{2^j} \subseteq V^x_j$.*

Proof. For (1). For every j, at any level $s+1$, we put something into V_j only if $\nu_{s+1}(i_n) \neq \nu_s(i_n)$ and $i_n = \langle j, n\rangle$ for some n. Moreover, at each time, we put at most

$2^{-2^j-2^n+1}$ measure of reals into V_j^x. By Lemma 4.7, if j is big enough, then

$$\mu(V_j^x) \leq \sum_{n\in\omega} 2^{i_n} \cdot 2^{-2^j-2^n+1} = \sum_{n\leq j} 2^{i_n} \cdot 2^{-2^j-2^n+1} + \sum_{n\geq j} 2^{i_n} \cdot 2^{-2^j-2^n+1}$$
$$\leq \sum_{n\leq j} 2^{j^3} \cdot 2^{-2^j-2^n+1} + \sum_{n\geq j} 2^{n^3} \cdot 2^{-2^j-2^n+1} \leq 2^{-j}.$$

So $\{V_j^x\}_{j>j_0}$ is an x-Martin Löf test for some big enough j_0.

For (2). For any j and $\sigma \in U_{2^j}^{\emptyset'}$, let n be the unique number so that $\sigma \in U_{2^j}^{\emptyset'} \upharpoonright n - U_{2^j}^{\emptyset'} \upharpoonright (n-1)$. Let $i = \langle j, n\rangle$ and s_i be the last level at which the parameters corresponding to i are defined. If $\sigma \in V_j^x[s_i - 1]$, then we are done. Otherwise, we claim that $\sigma \in V_j^x[s_i]$. Obviously, $\nu_t(k) = \nu_{s_i}(k)$ and $T_t^k = T_{s_i}^k$ for any $k \leq i$ and $t \geq s_i$. Then, by an easy induction on $k \leq i$, T_k, the tree constructed at level k in the coding construction, is the same as $T_{s_i}^k$ for any $k \leq i$. So $T_i^1 = T_{s_i}^{1,i}$. Pick up the unique pair j_i and n_i so that $\langle j_i, n_i\rangle = i$. Let $e_i = 2^{j_i}$. We may x-computably find a finite sequence $\sigma_s^i \prec \tau_0 \prec \tau_1 \prec ... \prec \tau_k \prec x$ such that

(1) $|\tau_0| = |\sigma_s^i| + 3 + r_s^i$;
(2) $\forall j < k-1(|\tau_{j+1}| = |\tau_j| + |\sigma_s^i| + 4 + j)$;
(3) τ_0 is the leftmost $\tau \in T_s^{1,i}$ such that $\sigma_s \prec \tau$ of length $|\tau_0|$ has the property that $\mu(T_s^{1,i} \upharpoonright \tau[s]) > 2^{-r_s^i-3-|\tau|}$;
(4) $\forall i < k-1$, τ_{i+1} is the leftmost $\tau \in T_s^{1,i}$ such that $\tau_i \prec \tau$ of length $|\tau + i + 1|$ has the property that $\mu(T_s^{1,i} \upharpoonright \tau[s]) > 2^{-r_s^i-4-i-|\tau|}$;
(5) τ_k is the rightmost $\tau \in T_s^{1,i}$ such that $\tau_{k-1} \prec \tau$ of length $|\tau_k|$ has the property that $\mu(T_s^{1,i} \upharpoonright \tau[s]) > 2^{-r_s^i-3-k-|\tau|}$.

By the coding construction, k is exactly $g(j,n)$. So $f(k) = f(g(j,n)) = U_{2^j}^{\emptyset'} \upharpoonright n - U_{2^j}^{\emptyset'} \upharpoonright (n-1)$. By the decoding construction, we put all the elements in $f(k)$ into V_j^x at level s_i. So $\sigma \in V_j^x[s_i]$. □

This completes the proof of Theorem 4.5.

By Proposition 3.7 and Theorem 4.5, we have the following conclusion.

Corollary 4.9. $\mathrm{ML} \cap \{x \mid x' \equiv_T \emptyset'\} \in \mathfrak{F}(\mathrm{Sch}(\emptyset'))$. *So* $\mathrm{Sch}(\emptyset')$ *is closed upward in the both K-degrees and C-degrees.*

Proof. Obviously every $\emptyset'$-Schnorr random is x-Martin-Löf random for every $x \in \mathrm{ML} \cap \{y \mid y' \equiv_T \emptyset'\}$. Morover, By Theorem 4.5 and Corollary 4.2, if z is x-Martin-Löf random for every $x \in \mathrm{ML} \cap \{y \mid y' \equiv_T \emptyset'\}$, then z must be $\emptyset'$-Schnorr random. So $\mathrm{ML} \cap \{y \mid y' \equiv_T \emptyset'\} \in \mathfrak{F}(\mathrm{Sch}(\emptyset'))$.

By Proposition 3.7, $\mathrm{Sch}(\emptyset')$ is closed upward in the both K-degrees and C-degrees. □

By the relativization of the proof of Theorem 4.5, we have the following results.

Corollary 4.10. *Suppose both x and z are low, then there is a z-random real y so that $y \oplus z$ is low and $y \oplus z \geq_{LR} x$.*

Corollary 4.11. *There is a sequence of reals $\{z_n\}_{n\in\omega}$ so that for every n,*

(1) *z_{n+1} is $\oplus_{i\leq n} z_i$-random;*
(2) *$\oplus_{i\leq n} z_i$ is low;*
(3) *$z = \oplus_{i\in\omega} z_i$ is LR-above all the low reals.*

4.3. **On** $\Pi(\mathrm{Sch}(\emptyset'))$**.** We characterize $\Pi(\mathrm{Sch}(\emptyset'))$.

Before proceeding with the proof, we need the following technique theorems.

Theorem 4.12 (Nies [18])**.** *If $y \leq_T x'$ and $y \leq_{LR} x$, then $y' \leq_T x'$.*

Theorem 4.13 (Kjos-Hanssen, Miller and Solomon [10])**.** *For any two real x and y, $x \leq_{LR} y$ and $x \leq_T y'$ if and only if for every $\Pi^0_1(x)$ set P, there is a $\Sigma^0_2(y)$ set $Q \subseteq P$ such that $\mu(Q) = \mu(P)$.*

Let $\mathbf{BL} = \{x \mid \exists z(z' \equiv_T \emptyset' \wedge x \leq_{LR} z)\}$. By Theorem 4.12, every Δ^0_2 real in $\mathbf{BL}$ is low.

We remark that $\mathbf{BL}$ contains lots of reals due to the following theorem.

Theorem 4.14 (Barmpalias, Lewis and Stephan [1])**.** *There is a c.e. real x with $x' \leq_T \emptyset'$ so that the set $\{z \mid z \leq_{LR} x\}$ contains a perfect Π^0_1 subset.*

Proposition 4.15. $\mathbf{BL} \in \mathfrak{F}(\mathrm{Sch}(\emptyset'))$.

Proof. It is clear that if $z \in \mathrm{Sch}(\emptyset')$ and $x \leq_{LR} y$ where y is low, then z is Martin-Löf random relativized to x.

By Theorem 4.1, if z is Martin-Löf random relativized to x for every low real x, then $z \in \mathrm{Sch}(\emptyset')$. □

So $\mathfrak{F}(\mathrm{Sch}(\emptyset'))$ exists. We show that $\mathbf{BL} = \Pi(\mathrm{Sch}(\emptyset'))$.

Theorem 4.16. *If $x \notin \mathbf{BL}$, then there is a $\emptyset'$-Schnorr random real which is not x-random.*

We use a forcing argument to prove Theorem 4.16.

Let $\mathbb{P} = (\mathbf{P}, \leq)$ where $\mathbf{P}$ is the collection of $\Pi^0_1(y)$ set of reals having positive measure for some low real y. For $P_1, P_2 \in \mathbf{P}$, $P_1 \subseteq P_2$ if and only if $P_1 \leq P_2$.

Lemma 4.17. *For any low real y, the class*

$$\mathcal{D}_y = \{P \in \mathbf{P} \mid P \text{ only contains Martin-Löf random reals relativized to } y \wedge \mu(P) > 0\}$$

is dense. In other words, for any $P_0 \in \mathbb{P}$, there is a $Q \leq P_0$ in $\mathcal{D}_y$.

Proof. Given a condition $P_0 \in \mathbf{P}$ and a low real y_0 so that P_0 is $\Pi^0_1(y_0)$. By Theorem 4.3, there is a low real z so that every z-random real is both y_0- and y-random.

Let P be a $\Pi^0_1(y)$ set of reals so that P only contains y-random reals and $\mu(P \cap P_0) > \frac{\mu(P_0)}{2}$. Note that $y, y_0 \leq_T \emptyset' \equiv_T z'$ and $y, y_0 \leq_{LR} z$. So by Theorem 4.13, there are

$\Sigma^0_2(z)$ sets $\widetilde{Q}, \widetilde{Q}_0 \subseteq P$ such that $\mu(\widetilde{Q}) = \mu(\widetilde{Q}_0) = \mu(P)$. Then there are $\Pi^0_1(z)$ sets Q and Q_0 so that

(1) $Q_0 \subseteq \widetilde{Q}_0 \subseteq P_0$ and $Q \subseteq \widetilde{Q} \subseteq P$; and
(2) $\mu(P_0 - Q_0) + \mu(P - Q) \leq \frac{\mu(P_0)}{4}$.

Let $Q_1 = Q \cap Q_0 \subseteq P \cap P_0$ be a $\Pi^0_1(z)$ set of reals. Moreover,

$$\mu(Q_0 \cap Q) \geq \mu(P_0 \cap P) - (\mu(P_0 - Q_0) + \mu(P - Q)) \geq \frac{\mu(P_0)}{2} - \frac{\mu(P_0)}{4} = \frac{\mu(P_0)}{4}.$$

Since $Q_1 \subseteq P$ has positive measure, we have that $Q_1 \in \mathcal{D}_y$ and $Q_1 \leq P_0$. $\square$

We need a lemma due to Kučera.

Lemma 4.18 (Kučera[11])**.** *For any Π^0_1 set of reals P and Martin-Löf random real x, there is a real $y \in P$ so that $x =^* y$.*

Fix a universal x-Martin-Löf test $\{U^x_n\}_{n\in\omega}$.

Lemma 4.19. *For any n, the class*

$$\mathcal{D}_n = \{P \in \mathbf{P} \mid P \subseteq U^x_n\}$$

is dense.

Proof. Given a condition $P_0 \in \mathbf{P}$ and a low real y_0 so that P is $\Pi^0_1(y_0)$. Note that we may find a $\Pi^0_1(y_0)$ set P'_0 which only contains y_0-random reals and has big enough measure so that $\mu(P_0 \cap P'_0) > 0$. So we may assume that P_0 only contains y_0-random reals. Note that for every y_0-random real z, there is a real $z_0 \in P_0$ so that $z =^* z_0$.

Since $x \not\leq_{LR} y_0$, there must be a y_0-random real which is not x-random. We claim that for every i, $U^x_i \cap P_0 \neq \emptyset$. Otherwise, there is some i so that $U^x_i \cap P_0 = \emptyset$. Since $\{U^x_i\}_{i\in\omega}$ is a universal x-Martin-Löf test, every real in P_0 is x-random. Since, by Lemma 4.18, for every real z, there is a real $z_0 \in P_0$ so that $z =^* z_0$, then z must be x-random. Thus $x \leq_{LR} y_0$ which contradicts to $x \not\leq_{LR} y_0$.

So there must be some σ with $[\sigma] \subseteq U^x_n$ but $[\sigma] \cap P_0 \neq \emptyset$. Let $P = [\sigma] \cap P_0$. Since P is $\Pi^0_1(y_0)$ and only contains y_0-random reals, $\mu(P) > 0$. Then $P \in \mathcal{D}_n$. $\square$

So if g, as a generic real corresponding to $\mathbb{P}$, meets all the previous dense sets, then g must be (by Lemma 4.17) y-random for every low real y but not (by Lemma 4.19) x-random.

This completes the proof of Theorem 4.16.

By Proposition 3.2, we have the following result.

Corollary 4.20. $\mathbf{BL} = \Pi(\mathrm{Sch}(\emptyset')) = \mathrm{Low}(\mathrm{Sch}(\emptyset'), \mathrm{ML})$.

5. The Σ-type characterization of $\emptyset'$-Schnorr randomness

In this section, we study $\Sigma(\mathrm{Sch}(\emptyset'))$ by applying the methods in Section 3.
We need a technique result due to Miyabe.

Theorem 5.1 (Miyabe [16]). *Given a sequence reals $\{z_n\}_{n\omega}$ so that for every n, z_{n+1} is $\oplus_{i\leq n}z_i$-random. Then there is a sequence $\{z_n^*\}_{n\in\omega}$ so that for every n, $z_n^* =^* z_n$ and $z^* = \oplus_{n\in\omega}z_n^*$ is Martin-Löf random.*

Barmpalias, Miller and Nies give a characterization of $\mathrm{High}(\mathrm{Sch}(\emptyset'), \mathrm{ML})$.

Theorem 5.2 (Barmpalias, Miller and Nies [2]). *For any real x, $x \in \mathrm{High}(\mathrm{ML}, \mathrm{Sch}(\emptyset'))$ if and only if $\emptyset'$ is c.e. traceable by x.*

Then we have the following result characterizing the reals LR-above all the low reals.

Corollary 5.3. *A real z is an upper bound of the collection of low LR-degrees if and only if $\emptyset'$ is c.e. traceable by z.*

Proof. By Corollary 4.2 and Proposition 3.5, z is an upper bound of the collection of low LR-degrees if and only if $z \in \mathrm{High}(\mathrm{ML}, \mathrm{Sch}(\emptyset'))$. Then, by Theorem 5.2, $z \in \mathrm{High}(\mathrm{ML}, \mathrm{Sch}(\emptyset'))$ if and only if $\emptyset'$ is c.e. traceable by z. □

Finally by putting all the previous results together, we prove the the following theorem.

Theorem 5.4. (1) $\mathrm{ML} \cap \mathrm{High}(\mathrm{ML}, \mathrm{Sch}(\emptyset')) \in \mathfrak{G}(\mathrm{Sch}(\emptyset'))$;
(2) $\Sigma(\mathrm{Sch}(\emptyset')) = \mathrm{High}(\mathrm{ML}, \mathrm{Sch}(\emptyset'))$.

Proof. For (1). It suffices to show that for every real $x \in \mathrm{Sch}(\emptyset')$, there is real Martin-Löf random real $z^* \in \mathrm{High}(\mathrm{ML}, \mathrm{Sch}(\emptyset'))$ so that x is z^*-random. Fix a real $x \in \mathrm{Sch}(\emptyset')$ and a real $z = \oplus_{n\in\omega}z_n$ as in Corollary 4.11. Since z is LR above all the low reals, by Corollary 5.3, $z \in \mathrm{High}(\mathrm{ML}, \mathrm{Sch}(\emptyset'))$.

Note that x is $\oplus_{i\leq n}z_i$-random for every n. So by van-Lambalgen's Theorem, z_{n+1} is $x \oplus (\oplus_{i\leq n}z_i)$-random for every n. By Theorem 5.1, there is a Martin-Löf random real $x^* \oplus z^* = x \oplus (\oplus_{n\in\omega}z_n^*)$ as in Theorem 5.1 (viewing x as z_{-1}). Obviously z^* is LR-above all the low reals. By Corollary 4.2, $z^* \in \mathrm{High}(\mathrm{ML}, \mathrm{Sch}(\emptyset'))$. By van-Lambalgen Theorem, x^* is z^*-random. Since $x =^* x^*$, x is also z^*-random.

For (2). By (1), $\Sigma(\mathrm{Sch}(\emptyset'))$ exists. Thus by Proposition 3.4, $\Sigma(\mathrm{Sch}(\emptyset')) = \mathrm{High}(\mathrm{ML}, \mathrm{Sch}(\emptyset'))$. □

6. Some remarks on other randomness notions

It is clear that both Π and Σ are undefined over $\mathscr{A}$ if $\mathscr{A}$ is weaker than ML. One may ask whether both maps Π and Σ are defined over all the randomness notions stronger than ML. The answer is no.

Theorem 6.1 (Downey, Nies, Weber and Yu [4]). $\mathrm{Low}(\mathrm{W2R}, \mathrm{ML}) = \mathrm{Low}(\mathrm{ML}, \mathrm{ML})$.

Suppose that $\Pi(\mathrm{W2R})$ exist,, then $\Pi(\mathrm{W2R}) = \mathrm{Low}(\mathrm{ML}, \mathrm{ML})$. Pick up a Martin-Löf random real x which is not weakly-2-random, then x is Martin-Löf random relative to any real in $\mathrm{Low}(\mathrm{ML}, \mathrm{ML})$, a contradiction.

We don't know whether $\Sigma(\mathscr{A})$ can be undefined for some randomness notion $\mathscr{A}$ stronger than Martin-Löf randomness. For the weak-2-randomness, Barmpalias et al have the following theorem.

Theorem 6.2 (Barmpalias, Miller and Nies [2]). *For any real x, $x \in \mathrm{High}(\mathrm{ML}, \mathrm{W2R})$ if and only if for any function $f \leq_T \emptyset'$, there is a number n so that $\Phi_n^x(n) \downarrow$ and $f(n) = \Phi_n^x(n)$.*

But we don't know whether Theorem 6.2 can be used to show the existence of $\Sigma(\mathrm{W2R})$.[3]

A plenty of highness properties related to other randomness notions stronger than Martin-Löf randomness were explored in [2]. But we don't know whether the characterizations exist.

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