Lowness for weakly 1-generic and Kurtz-random*

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Abstract. We prove that a set is low for weakly 1-generic iff it has neither dnr nor hyperimmune Turing degree. As this notion is more general than being recursively traceable, we refute a recent conjecture on the characterization of these sets. Furthermore, we show that every set which is low for weakly 1-generic is also low for Kurtz-random.

1 Introduction

Given a notation G and its relativized notation G^x. A real z is called *low for* G if each real satisfying G satisfies G^z. In this paper, we study the lowness for weakly 1-generic and Kurtz-random.

Lowness has been studied by lots of people and is one of the main topics in the theory of algorithmic randomness. We first summarize some known facts.

Theorem 1. *Let x be a set of natural numbers.*

1. (Nies [8]) *x is low for 1-randomness iff x is H-trivial iff x is low for Ω and Δ_2.*
2. (Nies [8]) *x is low for recursively random iff x is recursive.*
3. (Terwijn and Zambella [12]; Kjos-Hanssen, Nies and Stephan [9]) *x is low for Schnorr-random iff x is recursively traceable.*
4. (Greenberg, Miller and Yu [13]) *x is low for* 1*-generic iff x is recursive.*

From the theorem above, we see that there are some deep connections between computability theory and other mathematical branches (or theoretical computer sciences). In this paper, we answer the following conjecture which was raised by several people.

Conjecture 2 (Downey, Griffiths and Reid [3], Miller and Nies [7], Yu [13]). *Is a set x low for weakly* 1*-generic iff x is recursively traceable? Is x low for Kurtz-random iff x is recursively traceable?*

We refute the conjecture. Further, we obtain a characterization of being low for weakly 1-generic.

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Notation 3. We follow the standard notation. We list some notations below. For other terminology, we refer the reader to [1], [6], [10] and [11].

In this paper, a real means an element in Cantor space $\{0,1\}^\omega$. By identifying subsets of natural numbers with their characteristic function, we obtain that reals and subsets of natural numbers are the same. The basic open classes in Cantor space are of the form $\sigma \cdot \{0,1\}^\omega$ and have the measure $2^{-|\sigma|}$ where $|\sigma|$ is the length of σ. We use $\mu(S)$ to denote the measure of a class $S \subseteq \{0,1\}^\omega$.

Furthermore, we use x, y, z for reals, S, T, V for classes of reals, f, g, h for functions and all other lower case letters for natural numbers. We use C to denote the plain Kolmogorov complexity and H to denote the prefix free Kolmogorov complexity. Strings are denoted by greek letters σ, τ. The string $x(0)x(1)\dots x(n)$ is denoted by $x \upharpoonright n+1$.

Definition 4. Given reals x, y,

1. x is 1-y-generic if for every $\Sigma^0_1(y)$ class $S \subseteq \{0,1\}^\omega$ either $x \in S$ or there is an n such that $(x \upharpoonright n) \cdot \{0,1\}^\omega$ is disjoint to S.
2. x is weakly y-generic if $x \in S$ for every dense $\Sigma^0_1(y)$ class $S \subseteq \{0,1\}^\omega$.

Definition 5. Given reals x, y,

1. y is said to be x-random if $y \notin \bigcap_{n\in\omega} V_n$ for each uniform collection $\{V_n | n \in \omega\}$ of $\Sigma^0_1(x)$ classes with $\mu(V_n) \leq 2^{-n}$ for all n.
2. y is said to be Kurtz-random relative to x if $y \in S$ for each $\Sigma^0_1(x)$ class S with $\mu(S) = 1$.

Note that Kurtz-random relative to the halting problem K is not the same as what Downey calls "Kurtz-2-random" as the there are Σ^0_2 classes of measure 1 which are not a $\Sigma^0_1(K)$ class.

2 Recursively Traceable and Diagonally Non-Recursive Reals

In this section, we study the basic properties of the recursively traceable and diagonally non-recursive reals.

Definition 6. 1. Given an infinite set $x = \{n_0 < n_1 < n_2 < \dots\}$, its principal function p_x is defined by $p_x(m) = n_m$. The principal functions of finite sets are partial and have a finite domain.
2. A function f majorizes an infinite set x if $\forall n(f(n) > p_x(n))$.
3. Given a real y, an infinite set x is y-hyperimmune if no y-computable function f majorizes x. Particularly, we say that x is hyperimmune if it is $\emptyset$-hyperimmune.
4. Given a real y, x is said to have y-hyperimmune degree if there is an infinite $z \leq_T x$ which is y-hyperimmune. Otherwise it is said that x has y-hyperimmune-free degree. In particular, x has hyperimmune-free degree if it has $\emptyset$-hyperimmune-free degree.
5. We say that a real x is recursively traceable iff there is a recursive function h, called a bound, such that for all $f \leq_T x$ there is a recursive function g such that the $g(n)$-th canonical finite set $D_{g(n)}$ satisfies the following two properties:
   - $|D_{g(n)}| \leq h(n)$;
   - $f(n) \in D_{g(n)}$.

   Furthermore, x is r.e. traceable if $W_{g(n)}$ is used instead of $D_{g(n)}$ in the definition above.

6. A real x is diagonally nonrecursive (dnr) iff there is a total function $f \leq_T x$ such that for all n either $\varphi_n(n)$ is undefined or different from $f(n)$.
7. A real x is high iff there is a function $f \leq_T x$ which majorizes all infinite recursive sets y. Otherwise x is called non-high.

Clearly, every recursively traceable x is also r.e. traceable. Indeed x is recursively traceable iff it is r.e. traceable and has hyperimmune-free degree. Note that every x of hyperimmune-free degree is non-high. One can combine results of Kjos-Hanssen and Merkle to the following theorem.

Theorem 7 (Kjos-Hanssen; Merkle, Kjos-Hanssen and Stephan [4]). *Let x be not high. Then the following are equivalent:*

1. *x is not dnr;*
2. *x is not autocomplex, that is, there is no $f \leq_T x$ such that $C(x \upharpoonright m) \geq n$ whenever $m \geq f(n)$;*
3. *for every $g \leq_T x$ there is a recursive function h such that $g(n) = h(n)$ infinitely often;*
4. *x is infinitely often traceable in the sense that there is a recursive function h such that for all $f \leq_T x$ there is a recursive function g with $\forall n\,(|D_{g(n)}| \leq h(n))$ and $\exists^\infty n\,(f(n) \in D_{g(n)})$;*
5. *for every unbounded and nondecreasing recursive function h and every function $g \leq_T x$ there are infinitely many n with $C(g(n)) < h(n)$.*

Furthermore, if the Turing degree of x is neither hyperimmune nor dnr, then one can strengthen the third point as follows: for every $g \leq_T x$ there are recursive functions $\tilde{h}, h$ such that

$$\forall n \exists m \in \{n, n+1, \ldots, \tilde{h}(n)\}\ (h(m) = g(m)).$$

Autocomplex sets are not r.e. traceable and vice versa. But these notion do also not partition the class of all reals; the next result shows that there is a whole Π^0_1 class containing reals which are neither r.e. traceable nor autocomplex. This result covers the well-known examples of reals which are neither r.e. traceable nor autocomplex: (a) there is an x of r.e. degree which is neither r.e. traceable nor autocomplex; (b) there is an x of hyperimmune-free degree which is neither r.e. traceable nor autocomplex. Result (a) is quite direct as every r.e. set which is neither Turing complete nor low$_2$ has this property. Result (b) can be obtained by considering sets which are generic for "very strong array forcing" as considered by Downey, Jockusch and Stob [2, 9]; as Kjos-Hanssen pointed out to the authors, those sets are neither autocomplex nor r.e. traceable nor do they have hyperimmune Turing degree. An application of the following result would be that there are reals which are low for Ω but neither recursively traceable nor dnr.

Proposition 8. *There is a partial-recursive $\{0,1\}$-valued function with coinfinite domain such that ever x extending ψ is neither autocomplex nor r.e. traceable.*

Proof. The function ψ is constructed such that

1. $\psi(2^n)$ is undefined for infinitely many n;
2. if $\psi(2^n)$ is undefined and x is a total extension of ψ and $m \geq 2^{n+1}$ then $C(x \upharpoonright m) \geq n - 1$;
3. if $\psi(2^n)$ is undefined, x is a total extension of ψ and $\varphi^x_e(3^n)$ terminates such that the maximum of its computation-time, largest query and computation-result is s for some $e \leq n$ and $s \geq 2n$ then $\psi(m)$ is defined for $m = 2^n, 2^n + 1, \ldots, 2^s - 1$.

Now these three conditions are verified to show that a given total extension x of ψ is neither r.e. traceable nor autocomplex.

Assume a recursive bound h be given. Let $f(m) = C(x \upharpoonright 2^{h(m+1)+m+4})$. Choose m, n such that $h(m)+m+3 \leq n < h(m+1)+m+4$ and $\psi(n)$ is undefined. There are infinitely many m for which there is such an n by the first condition above. Now $C(f(m)) > n$ and $C(f(m) \mid m) > h(m)$, for infinitely many m, thus x is not r.e. traceable with bound h. So x is not r.e. traceable at all.

Furthermore, if φ_e^x it total and $n > e$ then $\varphi_e^x(3^n)$ queries x at places m where either $\psi(m)$ is defined or $m < 2^{n+1}$. Therefore, one can compute $\varphi_e^x(3^n)$ and $x \upharpoonright \varphi_e^x(3^n)$ from n and $x \upharpoonright 2^{n+1}$, thus $C(x \upharpoonright \varphi_x^e(3^n)) < 3^n$ and φ_e^x does not witness that x is autocomplex. So x is neither autocomplex nor dnr.

It remains to show that the considered ψ really exists. Let U be a universal machine for the complexity C and for a string τ in the domain of U, let $\mathrm{bv}(\tau)$ be the value of binary number 1τ. Now one constructs ψ in stages as follows. ψ_0 is everywhere undefined and in stage $s+1$ the following is done.

1. Begin Stage $s+1$.
2. Find the smallest n for which there are e, m, x, t such that $e \leq n$, $2^{n+1} \leq m \leq t \leq s$, x extends ψ_s, $\psi_s(2^n)\uparrow$, $\psi_s(m)\uparrow$ and $\varphi_e^x(3^n)$ terminates such that the maximum of its computation-time, largest query and computation-result is exactly t.
3. If n with e, m, x, t are found in Step 2 then let, for all $k \in \{2^n, 2^n+1, \ldots, 2^{s+1}-1\}$ where $\psi_s(k)$ is undefined, $\psi_{s+1}(k) = x(k)$.
4. For all $\tau \in \{0,1\}^*$ and i, j such that $\mathrm{bv}(\tau) < 2^i < 2^s$, $U_s(\tau)\downarrow$, $j = 2^i + \mathrm{bv}(\tau) < |U_s(\tau)|$ and $\psi_s(j)\uparrow$, let $\psi_{s+1}(j) = 1 - U_s(\tau)(j)$.
5. End Stage $s+1$.

Note that $\psi(2^n)$ can only become defined by activities in Step 3 of some stage. One can show by the usual finite injury arguments that there are infinitely many n for which $\psi(2^n)$ remains undefined. Furthermore, whenever $\psi(2^n)$ is undefined and $|\tau| < n-1$ then $j = 2^n + \mathrm{bv}(\tau)$ satisfies that either $U(\tau)$ is undefined or $U(\tau)(j), \psi(j)$ are both undefined or $U(\tau)(j), \psi(j)$ are both defined and different where, for the string $U(\tau)$, $U(\tau)(j)$ is the bit at position $j+1$ if the length is at least $j+1$ and is undefined if the length is at most j. As the mapping $\tau, n \to 2^n+\mathrm{bv}(\tau)$ is one-one on the domain of all τ, n with $\mathrm{bv}(\tau) < 2^n$ and as Step 3, for every n, makes either ψ either on a whole interval $\{2^n, 2^n+1, \ldots, 2^{n+1}-1\}$ or does not change ψ on the interval at all, it follows that if $\psi(2^n)$ is undefined then Step 4 guarantees that $C(x \upharpoonright m) \geq n-1$ for all $m \geq 2^{n+1}$. Furthermore, compactness ensures that after finitely many stages, Step 3 of the construction has ensured that the third condition on ψ is also satisfied. This completes the verification of the construction of ψ. ■

3 Lowness for Weakly 1-Generic

The next result characterizes when a set is low for weakly 1-generic.

Theorem 9. *The following statements are equivalent for every real x,*

1. *Every dense $\Sigma_1^0(x)$ class $S^x \subseteq \{0,1\}^\omega$ has a dense Σ_1^0 subclass.*
2. *x is low for weakly 1-generic.*
3. *The degree of x is hyperimmune-free and each 1-generic real is weakly 1-x-generic.*

4. The degree of x is hyperimmune-free and not dnr.

Proof. Obviously, the first statement implies the second. Kurtz [5] showed that every hyperimmune degree contains a weakly 1-generic real and thus the second statement implies the third. Proposition 10 below proves that the third statement implies the fourth. The implication from the fourth to the first condition follows from Theorem 11 below. ■

Proposition 10. *If each 1-generic real is weakly 1-x-generic, then x is not dnr.*

Proof. Assume by way of contradiction that x is dnr and every 1-generic set y is also weakly 1-x-generic. Nies [8] showed that there exists a 1-generic and H-trivial real y. Furthermore, as x is dnr, x is autocomplex [4]. So there is an x-recursive function f such that $H(x \upharpoonright m) \geq n$ for all $m \geq f(n)$. Without loss of generality, $f(n)$ queries x only below $f(n)$ when computing this value. Now one defines S as

$$S = \{\sigma(x \upharpoonright f(|\sigma|)) : \sigma \in \{0,1\}^*\}$$

and observes that S is dense. By assumption, y is weakly x-generic. So there are infinitely many n such that $(y \upharpoonright n)(x \upharpoonright f(n)) \preceq y$. Given such an n, one can compute $f(n)$ relative to y by querying $y(m+n)$ whenever the original computation of f queries $x(m)$, the reason is that whenever $x(m)$ is queried in this computation, then $m < f(n)$ and $y(m+n) = x(m)$. As y is H-trivial and autocomplex,

$$H^y(x \upharpoonright f(n)) \leq H^y(n, y \upharpoonright n + f(n)) + c_1 \leq H^y(n, f(n)) + c_2 \leq H^y(n) + c_3 \leq H(n) + c_4$$

for some constants c_1, c_2, c_3, c_4 and the infinitely many n with $(y \upharpoonright n)(x \upharpoonright f(n)) \preceq y$. It follows that $H(n) \geq n - c_4$ for infinitely many n, a contradiction. ■

4 Low for Kurtz-Random

Downey, Griffiths and Reid [3] conjectured that every low for Kurtz-random real is recursively traceable. The following theorem refutes the conjecture.

Theorem 11. *Let x have neither hyperimmune nor dnr Turing degree. Then the following two statements hold.*

1. *Every $\Sigma_1^0(x)$ class S^x of measure 1 has a Σ_1^0 subclass T of measure 1.*
2. *Every dense $\Sigma_1^0(x)$ class has a dense Σ_1^0 subclass.*

In particular, x is low for Kurtz-random and low for weakly 1-generic.

Proof. If S^x has measure 1 then S^x is dense: otherwise there would be a σ such that $\sigma \cdot \{0,1\}^\omega$ is disjoint to S^x and $\mu(S^x) \leq 1 - 2^{-|\sigma|}$. The proof is now given for the first statement where S^x has measure 1 and is dense. The proof for the second statement where S^x is only dense can be obtained from this proof by just omitting all conditions and constraints dealing with the measure of classes.

The argument in the proof is somewhat similar to the one in [13]. But the proof is greatly simplified due to Proposition 7. Fix x such that the Turing degree of x is neither hyperimmune nor dnr and consider any dense Σ_1^0 class S^x. For S^x, there is a function $\hat{f} \leq_T x$ such that, for all n,

- $\hat{f}(n) > n$;
- $\forall \sigma \in \{0,1\}^n \exists \tau \in \{0,1\}^{\hat{f}(n)}\ (\sigma \preceq \tau \wedge \tau \cdot \{0,1\}^\omega \subseteq S^x)$;
- $\mu(\{y \in S^x : (y \upharpoonright \hat{f}(n)) \cdot \{0,1\}^\omega \subseteq S^x\}) \geq 1 - 2^{-n}$.

Since x has hyperimmune-free Turing degree, there is a recursive function f such that, for all n, $f(n+1) > \hat{f}(f(n))$. Then there is a x-recursive function g such that, for all n,

- $g(n) \subseteq \{0,1\}^{f(n+1)}$;
- $\forall \sigma \in \{0,1\}^{f(n)} \exists \tau \in g(n)\ (\sigma \preceq \tau)$;
- $\mu(g(n) \cdot \{0,1\}^\omega) \geq 1 - 2^{-n}$;
- $g(n) \cdot \{0,1\}^\omega \subseteq S^x$.

As the Turing degree of x is neither dnr nor hyperimmune, there are recursive functions $h, \tilde{h}$ such that, for all n,

- $h(n) \subseteq \{0,1\}^{f(n+1)}$;
- $\forall \sigma \in \{0,1\}^{f(n)} \exists \tau \in h(n)\ (\sigma \preceq \tau)$;
- $\mu(h(n) \cdot \{0,1\}^\omega) \geq 1 - 2^{-n}$;
- $\exists m \in \{n, n+1, \ldots, \tilde{h}(n)\}\ (h(m) = g(m))$.

Now one can define the Σ^0_1 class T as

$$T = \{x : \exists n \forall m \in \{n, n+1, \ldots, \tilde{h}(n)\}\ (x \upharpoonright m \in h(m))\}.$$

The class T is dense because for every $\sigma \in \{0,1\}^n$ there is a $\tau_{n-1} \in \{0,1\}^{f(n)}$ extending σ as $f(n) > n$ and a sequence of $\tau_m \in h(m)$ each extending τ_{m-1} for $m = n, n+1, \ldots, \tilde{h}(n)$. Then $\tau_{\tilde{h}(n)} \cdot \{0,1\}^\omega \subseteq T$. The measure of T is 1 as

$$\mu(\{x : \forall m \in \{n, n+1, \ldots, \tilde{h}(n)\}\ (x \upharpoonright m \in h(m))\}) \geq 1 - 2^{-n-1}$$

for all n. Furthermore, for every $x \in T$ there is an n and $m \in \{n, n+1, \ldots, \tilde{h}(n)\}$ such that $h(m) = g(m)$ and $x \in uh(m)$. Thus $x \in g(m) \cdot \{0,1\}^\omega$ and $T \subseteq S^x$. ■

It is open whether lowness for Kurtz-random is equivalent to lowness for weakly 1-generic.

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