Some questions in higher randomness

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Given a class $\Gamma \subseteq \mathcal{P}(2^\omega)$, a real $x$ is $\Gamma$ random if $x \notin P$ for all null set $V \in \Gamma$.

$x$ is $\Delta^1_1$-random if $x \notin P$ for any $\Delta^1_1$-null set $P$.

$x$ is $\Pi^1_1$-random if $x \notin P$ for any $\Pi^1_1$-null set $P$.

**Theorem (Chong, Nies and Yu)**

$x$ is $\Pi^1_1$-random if and only if $x$ is $\Delta^1_1$-random and $x \not\geq_h \emptyset$. 

Given a class $\Gamma \subseteq \mathcal{P}(2^\omega)$, a real $x$ is $\Gamma$ random if $x \notin P$ for all null set $V \in \Gamma$.

$x$ is $\Delta_1^1$-random if $x \notin P$ for any $\Delta_1^1$-null set $P$.

$x$ is $\Pi_1^1$-random if $x \notin P$ for any $\Pi_1^1$-null set $P$.

**Theorem (Chong, Nies and Yu)**

$x$ is $\Pi_1^1$-random if and only if $x$ is $\Delta_1^1$-random and $x \not\geq_h O$.
Theorem (Chong, Nies, Yu)

There are $2^{\aleph_0}$ many reals low for $\Delta^1_1$-randomness.
Lowness for randomness

$x$ is low for $\Gamma$-random if every $\Gamma$-random real is $\Gamma(x)$-random.

**Theorem (Chong, Nies, Yu)**

\[There are \, 2^{\aleph_0} \, many \, reals \, low \, for \, \Delta^1_1 \,-\,randomness.\]
A real $x$ is $\Gamma$-random cuppable if there is a $\Gamma$-random real $y$ so that $x \oplus y \geq_h \emptyset$.

Since every hyperdegree greater or equal to $\emptyset$ contains a $\Delta^1_1$-random real, we have that every real is $\Delta^1_1$-random cuppable.

**Question**

Is every nonhyperarithmetic real $\Pi^1_1$-random cuppable?
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**Question**

*Is every nonhyperarithmetic real $\Pi^1_1$-random cuppable?*
Theorem (Harrington, Nies and Slaman)

A real is low for $\Pi^1_1$-randomness if and only if it is low for $\Delta^1_1$-randomness and non-$\Pi^1_1$-random cuppable?

Proof.

$\quad \Rightarrow: \{y \mid y \oplus x \geq_h x\}$ is a $\Pi^1_1(x)$-null set. The set $C = \bigcup\{B \mid B \in \Delta^1_1 \land \mu(B) = 0\}$ is a $\Pi^1_1$-null set. So if $2^\omega - C$ contains a non-$\Delta^1_1(x)$ random real, then it must contain a non-$\Delta^1_1(x)$ random $z$ so that $x \oplus z \not\geq O$. Then $z$ must be $\Pi^1_1$-random, a contradiction.

Another direction. The largest $\Pi^1_1(x)$-null set $Q$ is a union of countably many $\Delta^1_1(x)$-null set $Q_n$ with $P = \{y \mid y \oplus x \geq_h O\}$. Some questions in higher randomness.
Lowness for $\Pi^1_1$-randomness

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Another direction. The largest $\Pi^1_1(x)$-null set $Q$ is a union of countably many $\Delta^1_1(x)$-null set $Q_n$ with $P = \{y \mid y \oplus x \geq_h \mathcal{O}\}$. 
Question

Is there a nonhyparithmetic real low for $\Pi^1_1$-randomness?
A real $x$ is strong $\Pi^1_1$-ML-random if $x$ passes all the generalized $\Pi^1_1$-ML-test.

**Theorem (Yu)**

No left-$\Pi^1_1$-random real can be strongly $\Pi^1_1$-ML-random.
Strong $\Pi_1^1$-ML-randomness

A real $x$ is strong $\Pi_1^1$-ML-random if $x$ passes all the generalized $\Pi_1^1$-ML-test.

Theorem (Yu)

No left-$\Pi_1^1$-random real can be strongly $\Pi_1^1$-ML-random.
Every $\Pi^1_1$-random real is strongly $\Pi^1_1$-ML-random real.

Question

Is there a strongly $\Pi^1_1$-ML-random real which is not $\Pi^1_1$-random? Or is there a strongly $\Pi^1_1$-ML-random real $x$ with $x \geq_h \emptyset$?
Strong $\Pi_1^1$-ML-randomness vs $\Pi_1^1$-randomness

Every $\Pi_1^1$-random real is strongly $\Pi_1^1$-ML-random real.

Question

Is there a strongly $\Pi_1^1$-ML-random real which is not $\Pi_1^1$-random? 
Or is there a strongly $\Pi_1^1$-ML-random real $x$ with $x \geq_h \emptyset$?
A real $x$ is $\Gamma$-Kurtz random if $x$ does not belong to any $\Gamma$ closed null set.

By effective descriptive set theory, a real $x$ is $\Delta^1_1$-Kurtz random if and only if $x$ does not belong to any closed null set with a $\Delta^1_1$-code. This greatly simplifies the study of $\Delta^1_1$-Kurtz randomness.

**Theorem (Kjos-Hanssen, Nies, Stephan and Yu)**

If $\omega^x_1 = \omega^{CK}_1$, then $x$ is $\Delta^1_1$-Kurtz random if and only if $x$ is $\Pi^1_1$-Kurtz random.
A real $x$ is $\Gamma$-Kurtz random if $x$ does not belong to any $\Gamma$ closed null set. By effective descriptive set theory, a real $x$ is $\Delta^1_1$-Kurtz random if and only if $x$ does not belong to any closed null set with a $\Delta^1_1$-code. This greatly simplifies the study of $\Delta^1_1$-Kurtz randomness.

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**Theorem (Kjos-Hanssen, Nies, Stephan and Yu)**

If $\omega^x_1 = \omega^\text{CK}_1$, then $x$ is $\Delta^1_1$-Kurtz random if and only if $x$ is $\Pi^1_1$-Kurtz random.
Lowness for $\Pi_1^1$-Kurtz randomness

Theorem (KH, Nies, Stephan and Yu)

Every low for $\Pi_1^1$-Kurtz randomness real is low for $\Delta_1^1$-Kurtz randomness.

Proof.

The set $C = \bigcup\{B \mid B$ is closed, null and $\Delta_1^1\}$ is $\Pi_1^1$ and null.

Question

Is there a nonhyperarithmetic real which is low for $\Pi_1^1$-Kurtz randomness?
Theorem (KH, Nies, Stephan and Yu)

Every low for $\Pi_1^1$-Kurtz randomness real is low for $\Delta_1^1$-Kurtz randomness.

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Question

Is there a nonhyperarithmetic real which is low for $\Pi_1^1$-Kurtz randomness?
Thanks