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Longest Paths and Cycles in Bipartite Oriented Graphs

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ABSTRACT

In this paper we obtain two sufficient conditions, Ore type (Theorem 1) and Dirac type (Theorem 2), on the degrees of a bipartite oriented graph for ensuring the existence of long paths and cycles. These conditions are shown to be the best possible in a sense.

An oriented graph is a digraph without loops, multiple arcs, or cycles of length two. We will refer to an oriented complete bipartite graph as a bipartite tournament. Let $R = (V(R), A(R))$ be a bipartite oriented graph, where $V(R)$ and $A(R)$ denote the set of vertices and the set of arcs of R, respectively. For $v \in V(R)$ and $B \subseteq V(R)$, define $N_B^-(v)$ and $N_B^+(v)$ to be the set of vertices of B, which dominate, and are dominated by, the vertex v, respectively. Put $N_B(v) = N_B^-(v) \cup N_B^+(v)$. We shall refer to $N_B^-(v) = |N_B^-(v)|$, $d_B^+(v) = |N_B^+(v)|$ and $d_B(v) = |N_B(v)|$ as the in-degree, the out-degree, and the degree of v in B, respectively. Put $d_R^-(v) \equiv d_{V(R)}^-(v)$, $d_R^+(v) \equiv d_{V(R)}^+(v)$, $d_R(v) \equiv d_{V(R)}(v)$. Let $\delta^-(R) = \mathrm{Min}_{v \in V(R)}\{d_R^-(v)\}$, $\delta^+(R) = \mathrm{Min}_{v \in V(R)}\{d_R^+(v)\}$, $\delta(R)\delta(R) = \mathrm{Min}_{v \in V(R)}\{d_R(v)\}$. The oriented graph R is said to be k-diregular if $d_R^-(v) = k = d_R^+(v)$, for any $v \in V(R)$, and m-diconnected if, for any distinct pair of vertices $u, v \in V(R)$, there exist m internally disjoint paths from u to v. If R is 1-disconnected, we shall say simply that R is disconnected. We follow [2] for other terminology and notation.

The longest directed paths and cycles in oriented graph were discussed in [1, 3, 4]. In this paper we obtained further results on this problem. We shall prove several Lemmas before proving our main results.

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Lemma 1. Let C be the longest cycle in a bipartite oriented graph R and $P = v_1v_2 \cdots v_m$ be the longest path in R-C, then

$$N_R^-(v_1) \subseteq V(P) \cup V(C)\,,$$
$$N_R^+(v_m) \subseteq V(P) \cup V(C)\,.$$

Proof. It's clear, since P is the longest path in R-C. ∎

Lemma 2. Let $P = v_1v_2 \cdots v_m$ be the longest path in a bipartite oriented graph R such that $d_R^-(v_1) \geqslant 1$ (resp. $d_R^+(v_m) \geqslant 1$), then R contains a cycle of length at least $2d_R^-(v_1) + 2$ (resp. $2d_R^+(v_m) + 2$).

Proof. It's clear, since P is the longest path in R. ∎

Lemma 3. Let R be a bipartite oriented graph with $\delta \geqslant 2$, such that whenever $uv \notin A(R)$,

$$d_R^+(u) + d_R^-(v) \geqslant \delta - 1\,,$$

and C be the longest cycle in R, then $|V(C)| \geqslant \delta + 1$.

Proof. Let $P = v_1v_2 \cdots v_m$ be the longest path in R. If $v_mv_1 \in A(R)$, then $N_R(v_1) \subseteq V(P)$, since P is the longest path in R. So $|V(P)| \geqslant 2|N_R(v_1)| \geqslant 2\delta$ and $|V(C)| \geqslant |V(P)| \geqslant 2\delta > \delta + 1$, since $v_1v_2 \cdots v_mv_1$ is a cycle in R. If $v_mv_1 \notin A(R)$, then $d_R^+(v_m) + d_R^-(v_1) \geqslant \delta - 1$. Hence $\mathrm{Max}\{d_R^+(Vm), d_R^-(v_1)\} \geqslant \{1/2(\delta - 1)\}$, then $|V(C)| \geqslant 2\{1/2(\delta - 1)\} + 2 \geqslant \delta + 1$ by Lemma 2.

Lemma 4. Let C be the longest cycle in a diconnected bipartite oriented graph R and $P = v_1v_2 \cdots v_m$ be the longest path in R-C. If the induced subgraph $R[V(P)]$ is Hamiltonian, then $|V(P) \cup V(C)| \geqslant 2\delta + 2$.

Proof. Without loss of generality, we may assume $v_1v_2 \cdots v_mv_1$ is a Hamiltonian cycle in $R[V(P)]$. Let $C = c_1c_2, \cdots c_tc_1$. For any $v_i \in V(P)$, by Lemma 1, we have $N_R(v_i) \subseteq V(P) \cup V(C)$. Hence $|V(P) \cup V(C)| \geqslant 2|N_R(v_i)| \geqslant 2\delta$. If $|V(P) \cup V(C)| = 2\delta$, then for any $v_i \in V(P)$ and any $c_j \in V(C)$, v_i and c_j are adjacent if and only if they belong to the different parts of the bipartition of $V(R)$. Suppose that $c_1v_1 \in A(R)$ (similarly, $v_1c_1 \in A(R)$). If $v_ic_2 \in A(R)$, $1 < i \leqslant m$ and i is even, then R contains a longer cycle than C, this contradicts the hypothesis. So $c_2v_i \in A(R)$, $1 < i \leqslant m$, and i is even. Similarly by repeating the above argument, we may deduce that if $\forall\, c_i \in V(C)$, $\forall\, v_j \in V(P)$, and c_i, v_j belong to the different parts of the bipartition of $V(R)$, then $c_iv_j \in A(R)$. On the other hand there always exists a path $P(v_i, c_j)$ from P to C since R is diconnected. Hence $P(v_i, c_j) \cup (c_jc_{j+1} \cdots c_tc_1 \cdots c_{j-1}v_rv_i)$, where $r = i - 1$ or i depending on the value of index r such that $c_{j-1}v_r \in A(R)$

is a longer cycle than C. This contradiction establishes the Lemma since $|V(P) \cup V(C)|$ is even. ∎

Lemma 5. Let C be a longest cycle in a bipartite oriented graph R and $P = v_1 v_2 \cdots v_m$ be the longest path in R-C with m = odd. If there exist two spanning paths P', P'' in $R[V(P)]$ and $\exists v_i \in V(P)$, which is both an initial vertex of P' and a terminal vertex of P'', then $|V(P) \cup V(C)| \geqslant 2\delta + 1$.

Proof. By Lemma 1, $N_R(v_i) \subseteq V(P) \cup V(C)$. Hence $|V(P) \cup V(C)| \geqslant 2|N_R(v_i)| + 1 \geqslant 2\delta + 1$ since m is odd. ∎

Lemma 6. [4, Lemma 2]. Let R be a bipartite oriented graph. If the length of the longest path in R is $m - 1 \geqslant 5$, then there exists a longest path $P = v_1 v_2 \cdots v_m$ in R such that $d_R^-(v_i) + d_R^+(v_m) \leqslant m - 4$.

Definition. A bipartite oriented graph R is quasi-Hamiltonian if for any longest cycle C in R, the length of any longest path in R-C is equal to one and R-C doesn't contain any isolated vertices.

Example. Figure 1 is a quasi-Hamiltonian, where A and B are independent sets in the graph, and every vertex of A is dominated by every vertex of B.

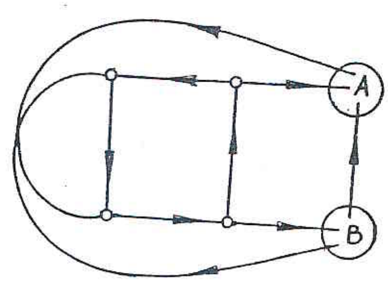

FIGURE 1.

Theorem 1. Let R be a bipartite oriented connected graph with $\delta \geqslant 2$, such that whenever $uv \notin A(R)$,

$$d_R^+(u) + d_R^-(v) \geqslant \delta - 1 .$$

Then R contains a path of length at least 2δ, except if R is Hamiltonian or quasi-Hamiltonian, and in this case R contains a path of length at least $2\delta - 1$.

Proof. If R isn't diconnected, we denote by R_1, R_2 the dicomponents corresponding to, respectively, the initial and terminal vertices of a longest path in the condensation (see [2, exercises 10.1.9]) of R. Then for $\forall\, u_1 \in R_1$, $\forall\, u_2 \in$

R_2, we have $u_2u_1 \notin A(R)$, and $d_R^+(u_2) + d_R^-(u_1) \geq \delta - 1$. If $d_R^+(u_2) = 0$ (similarly, $d_R^-(u_1) = 0$), then $d_R^-(u_1) \geq \delta - 1$. Hence R_1 has a cycle of length at least $2d_R^-(u_1) + 2 \geq 2\delta$ by Lemma 2. Note that there is a path in R from this cycle to u_2, so R has a path of length at least 2δ. If $d_R^-(u_1) \geq 1$, $d_R^+(u_2) \geq 1$, then by Lemma 2, R_1, R_2 contain cycles of lengths at least $2d_R^-(u_1) + 2$ and $2d_R^+(u_2) + 2$, respectively. Note that there is a path in R from one cycle to another, hence R has a path of length at least $(2d_R^-(u_1) + 1) + 1 + (2d_R^+(u_2) + 1) = 2(d_R^-(u_1) + d_R^+(u_2)) + 3 \geq 2(\delta - 1) + 3 = 2\delta + 1$.

Thus we may suppose that R is diconnected. Let $C = c_1c_2 \cdots c_lc_1$ be the longest cycle in R and $P = v_1v_2 \cdots v_m$ be the longest path in R-C. If $m = 0$, then R is Hamiltonian. Obviously, $|V(R)| \geq 2\delta$, hence R has a path of length at least $2\delta - 1$. If $m = 1$, then $\exists v_0 \in R\text{-}C$, $N_R(v_0) \subseteq V(C)$. Thus $|V(C)| \geq 2|N_R(v_0)| \geq 2\delta$, and hence $C \cup \{c_iv_0, v_0c_j\}$ contains a path of length at least 2δ in R.

If $m = 2$, and R is quasi-Hamiltonian, then $N_R^-(v_1) \cup N_R^+(v_2) \subseteq V(C)$ by Lemma 1. Suppose $c_iv_1, v_2c_j \in A(R)$, $j > i, j - i = kl + r$, $0 \leq r < l$, then $r \geq 3$, since C is the longest cycle. Note that $d_R^+(v_2) + d_R^-(v_1) \geq \delta - 1$, hence $|V(C)| \geq 2\delta - 2$, thus $C \cup P \cup \{c_iv_1, v_2c_j\}$ contains a path of length at least $2\delta - 1$ in R. If R isn't quasi-Hamiltonian, we distinquish the following two cases: (a) For some C there is an isolated vertex in R-C. By a similar proof of $m = 1$, R has a path of length at least 2δ. (b) For some C there is a path with length ≥ 2 in R-C; we shall discuss this case in the following case of $m \geq 3$.

If $m \geq 3$. Note that it is enough to prove $|V(P) \cup V(C)| \geq 2\delta + 1$. We may suppose that $R[V(P)]$ is non-Hamiltonian and in particular $v_mv_1 \notin A(R)$. If not, the Theorem is true by Lemma 4. Without loss of generality we may also suppose that:

$$|N_C^-(v_1)| \leq |N_C^+(v_m)|.$$

Let $B = \{(c_i, c_j) | c_i \neq c_j,\ c_i \in N_C^-(v_1),\ c_j \in N_C^+(v_m)$, and $\{c_{i+1}, c_{i+2}, \ldots, c_{j-1}\} \cap (N_C^-(v_1) \cup N_C^+(v_m)) = \varnothing\}$, $|B| = b$, where subscripts of c_{i+j} are taken modulo l. If $(c_i, c_j) \in B$, we have $|\{c_{i+1}, c_{i+2}, \ldots, c_{j-1}\}| \geq m$ since C is a longest cycle.

Let

$$D = \{c_i, c_{i+1} | c_i \in N_C^-(v_1)\},$$
$$E = \{c_{j-2}, c_{j-1} | c_j \in N_C^+(v_m)\}.$$

Clearly, $D \cap E = \varnothing$, and

$$|V(C)| \geq |D| + |E| + b(m - 3) \geq 2(d_C^-(v_1) + d_C^+(v_m)) + b(m - 3).$$

On the other hand, $v_mv_1 \notin A(R)$, so $d_C^-(v_1) \geq d_R^-(v_1) - 1/2(m - 3)$, $d_C^+(v_m) \geq d_R^+(v_m) - 1/2(m - 3)$. Hence $|V(C)| \geq 2(d_R^-(v_1) + d_R^+(v_m)) - 2(m - 3) + b(m - 3) \geq 2(\delta - 1) + (b - 2)(m - 3)$.

For $b \geqslant 1$, R contains a path of length at least 2δ, since $|V(P) \cup V(C)| \geqslant m + 2(\delta - 1) - (m - 3) \geqslant 2\delta + 1$.

Thus we may suppose that $b = 0$. If $N_C^-(v_1) = \{c_i\} = N_C^+(v_m)$, thus we have $d_p^-(v_1) + d_p^+(v_m) \geqslant \delta - 3$. Hence $\mathrm{Max}\{d_p^-(v_1), d_p^+(v_m)\} \geqslant \{1/2(\delta - 3)\}$. Note that at this case m must be odd, so $|V(P)| \geqslant 2\{1/2(\delta - 3)\} + 3 \geqslant \delta$. Thus $|V(P) \cup V(C)| \geqslant \delta + (\delta + 1) = 2\delta + 1$ by Lemma 3. Hence $d_C^-(v_1) = 0$.

If $d_C^+(v_m) = 0$, then $d_R^-(v_1) = d_p^-(v_1)$, $d_R^+(v_m) = d_p^+(v_m)$. Hence $d_p^-(v_1) + d_p^+(v_m) \geqslant \delta - 1$ by the hypothesis, thus $\mathrm{Max}\{d_p^-(v_1), d_p^+(v_m)\} \geqslant \{1/2(\delta - 1)\}$, so R has a cycle C_1 of length at least $2\{1/2(\delta - 1)\} + 2 \geqslant \delta + 1$ by Lemma 2. And $|V(C)| \geqslant \delta + 1$ by Lemma 3. Therefore, $|V(P) \cup V(C)| \geqslant 2\delta + 2$. On the other hand, R always has a path P_0 from C_1 to C since R is diconnected. Hence $C_1 \cup P_0 \cup C$ contains a path of length at least $2\delta + 1$. Hence $d_C^+(v_m) > 0$.

Let

$$h = \mathrm{Max}\{1, i \mid v_i \in N_R^-(v_1)\},$$

$$q = \mathrm{Min}\{m, i \mid v_i \in N_R^+(v_m)\},$$

and

$$K(P) = \begin{cases} \{v_{q+1}, v_{q+2}, \ldots, v_{h-1}\} & \text{if } q < h, \\ \emptyset & \text{if } q \geqslant h. \end{cases}$$

In the following, the proof splits into two cases.

(1) m = odd.

If $d_p^+(v_m) = 0$, then $|V(C)| \geqslant 2d_R^+(v_m)$ and $|V(P)| \geqslant 2d_p^-(v_1) + 3 = 2d_R^-(v_1) + 3$. Hence $|V(P) \cup V(C)| \geqslant 2(d_R^-(v_1) + d_R^+(v_m)) + 3 \geqslant 2(\delta - 1) + 3 = 2\delta + 1$.

If $d_p^-(v_1) = 0$, then $d_R^+(v_m) \geqslant \delta - 1$ by the hypothesis. Since $|V(C)| \geqslant 2d_C^+(v_m)$, $|V(P)| \geqslant 2d_p^+(v_m) + 3$, we have $|V(P) \cup |V(C)| \geqslant 2d_R^+(v_m) + 3 \geqslant 2(\delta - 1) + 3 = 2\delta + 1$.

Thus $d_p^+(v_m) > 0$, $d_p^-(v_m) > 0$. Let $v_m v_i \in A(R)$. Two cases must be considered. (a) $v_{m-1}V_1 \in A(R)$ (similarly, $v_m v_2 \in A(R)$), then R-C has a longest path $v_m v_i \cdots v_{m-1} v_1 \cdots v_{i-1}$. So replacing v_i in Lemma 5 by v_m, we have $|V(P) \cup V(C)| \geqslant 2\delta + 1$. (b) $v_{m-1}v_1, v_m v_2 \notin A(R)$, hence $q \geqslant 4$, $h \leqslant m - 3$. If $v_{i-1}v_1, v_m v_{i+1} \in A(R)$, where $q < i < h$, then $v_i \cdots v_m v_q \cdots v_{i-1} v_1 \cdots v_{q-1}$ and $v_{h+1} \cdots v_m v_{i+1} \cdots v_h v_1 \cdots v_i$ are the longest path in R-C, hence $|V(P) \cup V(C)| \geqslant 2\delta + 1$ by Lemma 5. If at most one of $\{v_{i-1}v_1, v_m v_{i+1}\}$ belongs to $A(R)$, where $q < i < h$, then $|V(P) \geqslant 2(d_p^-(v_1) + d_p^+(v_m)) + 3$, $|V(C)| \geqslant 2d_C^+(v_m)$, $d_p^-(v_1) = d_R^-(v_1)$, hence $|V(P) \cup V(C)| \geqslant 2(d_R^-(v_1) + d_R^+(v_m)) + 3 \geqslant 2\delta + 1$.

Therefore the conclusion of the theorem is true for m = odd.

(2) m = even.

If $K(P) = \emptyset$. Clearly, $|V(P)| \geq 2(d_p^-(v_1) + d_p^+(v_m)) + 2$, $|V(C)| \geq 2d_C^+(v_m)$, hence $|V(P) \cup V(C)| \geq 2(d_R^-(v_1) + d_R^+(v_m)) + 2 \geq 2(\delta - 1) + 2 = 2\delta$.

If $K(P) \neq \emptyset$, transform from $P = v_1v_2 \cdots v_m$ into $P_1 = v_{h+1} \cdots v_m v_q \cdots v_h v_1 \cdots v_{q-1}$ in R-C and denote it by $P \to P_1$. Without loss of generality, we may assume that $d_C^+(v_{q-1}) = 0$ or $d_c^-(v_{h+1}) = 0$ since other cases have been considered before. Obviously, $K(P) \subseteq K(P_1)$. If $K(P) \neq K(P_1)$, consider $P_1 \to P_2$. Continuing this process, we may deduce that $P \to P_1 \to P_2 \to \cdots \to P_r \to \ldots$. Hence there always exists some k, $P_{k-1} \to P_k$ with $K(P_{k-1}) = K(P_k)$, since $V(P)$ is a finite set. Without loss of generality we may assume $k = 1$, namely, $K(P) = K(P_1)$. For $v_i \in K(P)$, we distinguish the following two cases. (a) $v_iv_1, v_{q-1}v_{i+1} \in A(R)$ (similarly, $v_mv_{i+1}, v_iv_{k+1} \in A(R)$), then $|V(P) \cup V(C)| \geq 2\delta + 2$ by Lemma 4. (b) In both $\{v_iv_1, v_{q-1}v_{i+1}\}$ and $\{v_mv_{i+1}, v_iv_{h+1}\}$ at most one arc belongs to $A(R)$. Then we have:

$$|(N_p^-(v_1) \cup N_p^+(v_m)) \cap K(P)| + |(N_p^+(v_{q-1}) \cup N_p^-(v_{h+1})) \cap K(P)| \leq h - q - 1,$$

Let $K^c(P) = V(P) \setminus K(P)$, thus

$$|N_p^+(v_m) \cap K^c(P)| \leq \tfrac{1}{2}(m - h),$$
$$|N_p^-(v_1) \cap K^c(P)| \leq \tfrac{1}{2}(q - 1),$$
$$|N_p^+(v_{q-1}) \cap K^c(P)| \leq \tfrac{1}{2}(q - 1),$$
$$|N_p^-(v_{h+1}) \cap K^c(P)| \leq \tfrac{1}{2}(m - h).$$

Hence we have the following estimate:

$$d_p^-(v_1) + d_p^+(v_m) + d_p^-(v_{h+1}) + d_p^+(v_{q-1}) \leq (q - 1) + (h - q - 1) + (m - h) = m - 2.$$

Thus at least one of $\{P, P_1\}$, say P, has the following inequality:

$$d_p^-(v_1) + d_p^+(v_m) \leq \tfrac{1}{2}(m - 2).$$

Note that $|V(C)| \geq 2d_C^+(v_m)$, and $d_C^- v_1) = 0$, thus

$$(m - 2) + |V(C)| \geq 2(d_R^-(v_1) + d_R^+(v_m)) \geq 2(\delta - 1).$$

Hence $|V(P) \cup V(C)| \geq 2\delta$.

Therefore we always have $|V(P) \cup V(C)| \geq 2\delta$ in (2), if the equality holds, and we assume that $\{c_1, c_3, \ldots, c_{l-1}\}$ is dominated by v_m. Since R is diconnected, there exists a path $P(c_i, v_j)$ from C to P. Thus $v_j \cdots v_m c_{2[(i+1/2)]+1} \cdots c_l c_1 \cdots c_i \cup P(c_i, v_j)$ is a longer cycle than C. This contradicts the choice of C. So we always have $|V(P) \cup V(C)| \geq 2\delta + 2$ in (2).

Therefore, the conclusion of the theorem is true for m = even.

Up to now we have exhausted all possible cases of R. Therefore the proof of Theorem 1 is completed. ∎

Corollary. If a bipartite tournament T with 2δ vertices and $\delta \geqslant 2$ is such that whenever $uv \notin A(T)$,

$$d_T^+(u) + d_T^-(v) \geqslant \delta - 1 .$$

Then T has either a Hamiltonian cycle or a Hamiltonian path.

Proof. T hasn't any path with length 2δ since $|V(T)| = 2\delta$; hence, by Theorem 1, the corollary is true. ∎

If we consider the condition of Dirac type instead of the Ore type, we have:

Theorem 2. Let R be a bipartite oriented graph such that for any $u \in V(R)$, $d_R^-(u) \geqslant h$, and $d_R^+(u) \geqslant k$, then R has either a cycle of length at least $2(h + k)$ or a path of length at least $2(h + k) + 3$.

Proof. If $h = 0$, $k > 0$ or $h > 0$, $k = 0$, R has a cycle of length at least $2(h + k) + 2$ by Lemma 2.

If $h, k \geqslant 1$, R is connected and not diconnected. We consider the dicomponents R_1, R_2 corresponding to the initial and terminal vertices of a longest path in the condensation of R. Clearly, $\delta^-(R_1) \geqslant h$, $\delta^+(R_2) \geqslant k$, thus R_1 (resp. R_2) has a cycle of length at least $2h + 2$ (resp. $2k + 2$) by Lemma 2. Note that there is a path in R between these cycles; hence R has a path of length at least $(2h + 1) + 1 + (2k + 1) = 2(h + k) + 3$.

Thus we may assume that $h, k \geqslant 1$ and R is diconnected. Let $C = c_1c_2 \cdots c_tc_1$ be the longest cycle in R and $P = v_1v_2 \cdots v_m$ be the longest path in R-C. By Lemma 2, $|V(C)| \geqslant 2\,\mathrm{Max}\{h, k\} + 2$. Let (V_1, V_2) be the bipartition of R, for any $v \in V_1$, $N_R^-(v) \cup N_R^+(v) \subseteq V_2$; hence $|V_2| \geqslant h + k$. Similarly, $|V_1| \geqslant h + k$, therefore $|V(R)| \geqslant 2(h + k)$.

If $m = 0$, then C is a Hamilton cycle in R, so $|V(C)| = |V(R)| \geqslant 2(h + k)$.

If $m \geq 1$, let $s = d_C^-(v_1)$, $r = d_C^+(v_m)$. If $s = 0$ or $r = 0$, then R-C has a cycle C_1 of length at least $2\,\mathrm{Min}\{h, k\} + 2$. Since R is diconnected, there exists a path P_1 between C and C_1. Hence $C \cup P_1 \cup C_1$ has a path of length at least $(2\,\mathrm{Max}\{h, k\} + 1) + 1 + (2\,\mathrm{Min}\{h, k\} + 1) = 2(h + k) + 3$.

From the proof above we may assume that $m \geqslant 1$ and $r, s \geqslant 1$. Note that C is the longest cycle in R and P is the longest path in R-C. By Lemma 1, we can show that (see [1, Theorem 2])

$$|V(c)| \geqslant \begin{cases} 2r + 2s & \text{if } m = 1 \\ 2r + 2s + m - 3 & \text{if } m = \text{odd} > 1 \\ 2r + 2s + m - 2 & \text{if } m = \text{even} \end{cases} \tag{1}$$

In the following, the proof splits into three cases depending on the value of m.

(1) $m \leqslant 3$, thus $r \geqslant k$, $s \geqslant h$. Hence $|V(C)| \geqslant 2(h + k)$ by Eq. (1).

(2) $m \geqslant 6$. By Lemma 6, without loss of generality, we may assume that P has property $d_p^-(v_1) + d_p^+(v_m) \leqslant m - 4$. Since $s \geqslant h - d_p^-(v_1)$, $r \geqslant k - d_p^+(v_m)$, we have $|V(P) \cup V(C)| \geqslant 2r + 2s + 2m - 3 \geqslant 2(h + k) + 5$ by Eq. (1). Hence R has a path of length at least $2(h + k) + 4$.

(3) $m = 4, 5$, thus $d_p^-(v_1) \leqslant 1$, $d_p^+(v_m) \leqslant 1$, hence $s \geqslant h - 1$, $r \geqslant k - 1$. And by Eq. (1) we have $|V(C)| \geqslant 2r + 2s + 2$. Therefore R has a cycle of length at least $2(h + k)$ except in the following case:

$$\begin{cases} s = h - 1 > 0 \\ r = k - 1 > 0 \\ |V(c)| = 2r + 2s + 2 \end{cases} \tag{2}$$

If Eq. (2) holds in R, then $h \geqslant 2$, $k \geqslant 2$ and $|V(C)| = 2(h + k) - 2 = l$. We may suppose that vertices $c_i \in C$, $v_j \in P$ belong to the same part of the bipartition if and only if i and j have the same parity.

Case 1. $m = 5$.

Relabeling the vertices of C if necessary, we may assume that

$$N_c^-(v_1) = \{c_{2k+2}, c_{2k+4}, \ldots, c_l\},$$
$$N_c^+(v_5) = \{c_6, c_8, \ldots, c_{2k+2}\}$$

by Eq. (2). Hence, besides C and P, we may assume that $C_1 = v_1v_2 \cdots v_5c_6 \cdots c_lv_1$ is also the longest cycle in R, $P_1 = c_1c_2c_3c_4c_5$ is also the longest path in R-C_1 and Eq. (2) holds, otherwise we are in a case considered before. So $N_{c_1}^-(c_1) = N_c^-(v_1)$ and $N_{c_1}^+(c_5) = N_c^+(v_5)$. Note that $v_4v_1, v_5v_2, c_4c_1, c_5c_2 \in A(R)$, hence v_1, v_5 (c_1, c_5) are both an initial and a terminal vertex of some longest path with the same vertex set in $R-C$ $(R-C_1)$. By Lemma 1, $N_R(v_1) \subseteq V(P) \cup V(C)$. Since $d_R(v_1) \geqslant h + k$ and the number of vertices adjacent to v_1 in $V(P) \cup V(C)$ is at most $h + k + 1$, at least one of $\{v_1c_2, v_1c_4\}$ belongs to $A(R)$. Similarly, in every pair of $\{c_2v_5, c_4v_5\}$, $\{c_1v_2, c_1v_4\}$, and $\{v_2c_5, v_4c_5\}$, at least one arc belongs to $A(R)$. On the other hand, since $v_1c_2 \cdots c_lc_1v_2v_3v_4v_1$, $v_1c_4 \cdots c_lc_1v_2v_3v_4v_1$, and $v_1c_2 \cdots c_lc_1v_4v_1$ are longer than C, it follows that $c_1v_2, v_1c_2 \notin A(R)$, thus $c_1v_4, v_1c_4 \in A(R)$. Since $v_1c_4 \cdots c_lc_1c_2v_5v_2v_3v_4v_1$ is longer than C, $c_2v_5 \notin A(R)$, and thus $c_4v_5 \in A(R)$. Finally, since $c_5 \cdots c_lc_1c_2c_3c_4v_5v_2v_3v_4c_5$ and $c_5 \cdots c_lc_1c_2c_3c_4v_5v_2c_5$ are longer than C, hence $v_2c_5, v_4c_5 \notin A(R)$, which contradicts that at least one of $\{v_2c_5, v_4c_5\}$ belongs to $A(R)$. Therefore, there doesn't exist a bipartite oriented graph with $m = 5$ that satisfies Eq. (2).

Case 2. $m = 4$.

As case 1, we may assume that

$$N_c^-(v_1) = \{c_{2k+2}, c_{2k+4}, \ldots, c_l\},$$
$$N(v_4) = \{c_5, c_7, \ldots, c_{2k+1}\}$$

by Eq. (2). Hence besides C and P we may assume that $C_2 = v_1v_2v_3v_4c_5 \cdots c_lv_1$ is also the longest cycle in R, $P_2 = c_1c_2c_3c_4$ is also the longest path in $R-C_2$, and Eq. (2) holds, otherwise we are in a case considered before. So $N^-_{c_2}(c_1) = N^-_c(v_1)$ and $N^+_{c_2}(c_4) = N^+_c(v_4)$. Note that $v_4v_1 \in A(R)$, hence each $v_i \in V(P)$ is both an initial and a terminal vertex of some longest path with the same vertex set in $R-C$. Hence we can prove, as in case 1, that in every pair of $\{v_1c_2, v_1c_4\}$ and $\{c_1v_2, c_1v_4\}$, at least one of arc belongs to $A(R)$ and at least one of arc between v_2 and $\{c_1, c_3\}$ belongs to $A(R)$. Since $c_1 \cdots c_lv_1v_2c_1$, $c_1v_2v_3v_4v_4v_1c_2 \cdots c_lc_1$, and $c_1v_2v_3v_4v_1c_4 \cdots c_lc_1$ contradict the choice of C, c_1 and v_2 are nonadjacent, and thus $c_1v_4 \in A(R)$. Since $c_1v_4v_1c_2 \cdots c_lc_1$ is longer than C, we deduce that $v_1c_2 \notin A(R)$ and $v_1c_4 \in A(R)$. Finally, since $c_3v_2v_3v_4v_1c_4 \cdots c_lc_1c_2c_3$ and $v_2c_3 \cdots c_lc_1v_4v_1v_2$ are longer than C, c_3 and v_2 are nonadjacent. Thus it follows that v_2 and $\{c_1, c_3\}$ are nonadjacent, which is a contradiction. Therefore, there does not exist a bipartite oriented graph with $m = 4$ that satisfies Eq. (2).

Up to now we have exhausted all possible cases, therefore the proof of Theorem 2 is completed. ∎

Corollary [3, Corollary 8.2]. Every diregular bipartite tournament is Hamiltonian.

Proof. Since a k-diregular bipartite tournament T has exactly $4k$ vertices, T hasn't any path of length $4k + 3$. Hence the conclusion of the corollary follows immediately from Theorem 2. ∎

For any integer $r > 0$, R_r is defined as follows: $V(R_r) = \{v_0, v_1, v_2, \ldots, v_{4r}\}$, $v_iv_j \in A(R_r) \leftrightarrow j - i \equiv 1 \pmod 4$. We take two disjoint copies of R_r first, and then v_0 of each copy is identified. The resulting graph is denoted by R_r'.

The results of Theorem 1 and 2 do not imply each other they depict the properties of a bipartite oriented graph in different ways and are best possible in view of R_r. Consider R_r, where $\delta = 2r$, $h = k = r$. R_r is neither Hamiltonian nor quasi-Hamiltonian, since $|V(R_r)| = 4r + 1$ is odd. Hence R_r has a path with length $2\delta = 4r$ by Theorem 1. On the other hand, R_r hasn't any path with length $2(h + k) + 3 = 4r + 3$, hence R_r has a cycle of length at least $2(h + k) = 4r$ by Theorem 2. It is easy to directly check that the path with length $4r$ and $4r$-cycle are the longest path and cycle in R_r, respectively.

We conclude by suggesting that the Jackson conjecture (see [3, Conjecture 1]) remains valid for a bipartite oriented graph.

Conjecture 1. Every 2-diconnected bipartite oriented graph with $\delta^+ \geqslant k$ contains either a Hamilton cycle or else a cycle of length at least $4k$.

Conjecture 2. Every bipartite oriented graph with $\delta \geqslant h$, $\delta^+ \geqslant k$ contains a cycle of length at least $2(h + k)$.

The following are some results in support of Conjecture 2.

Theorem [4, Theorem 2]. Let R be a bipartite oriented graph with $\delta^- \geqslant k$, $\delta^+ \geqslant k$ $(k \geqslant 3)$, and with at most $4k + 4$ vertices. Then R has a cycle of length at least $4k$.

Theorem [3, Theorem 8]. Let T be a diconnected bipartite tournament such that whenever u and v are vertices of T and $uv \notin A(T)$,

$$d_T^+(u) + d_T^-(v) \geqslant n\,.$$

Then T contains a cycle of length at least $2n$.

Both Conjectures would be, in a sense, best possible.

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