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The Exponent Set of Symmetric Primitive (0, 1) Matrices with Zero Trace

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ABSTRACT

We prove that the exponent set of symmetric primitive (0, 1) matrices with zero trace (the adjacency matrices of the simple graphs) is $\{2,3,\ldots,2n-4\}\setminus S$, where S is the set of all odd numbers in $\{n-2,n-1,\ldots,2n-5\}$. We also obtain a characterization of the symmetric primitive matrices with zero trace whose exponents attain the upper bound $2n-4$.

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1. INTRODUCTION

An $n \times n$ (0,1) matrix A over the binary Boolean algebra $\{0,1\}$ is called a *primitive matrix* if there exists a positive integer k such that $A^k = J$. The least such k is called the *exponent* of A, denoted by $\gamma(A)$. The *associated digraph* of $A = (a_{ij})$, denoted by $G(A)$, is the digraph with vertex set $V(G(A)) = \{1, 2, \ldots, n\}$ such that there is an arc from i to j in $G(A)$ iff $a_{ij} = 1$. A digraph G is *primitive* if there exists an integer $k > 0$ such that for all ordered pairs of vertices $i, j \in V(G)$ (not necessarily distinct), there is a walk from i to j with length k. The least such k is called the *exponent* of the digraph G, denoted by $\gamma(G)$. Clearly, a matrix A is primitive iff its associated digraph $G(A)$ is primitive, and in this case we have $\gamma(A) = \gamma(G(A))$. Other definitions and notation not in this article can be found in [1].

Let $L(D) = \{\gamma_1, \gamma_2, \ldots, \gamma_\lambda\}$ denote the set of distinct lengths of cycles of the digraph D. Let i, j be vertices of the digraph D. The *exponent* from i to j, denoted by $\gamma(i,j)$, is the least integer γ such that there exists a walk of length m from i to j for all $m \geqslant \gamma$.

The properties of a primitive digraph and its exponent given in the following three propositions are well known.

PROPOSITION 1. *A digraph D is primitive iff D satisfies the following two conditions*:

(i) *D is strongly connected*;
(ii) gcd $\{\gamma_1, \ldots, \gamma_\lambda\} = 1$, *where* $L(D) = \{\gamma_1, \ldots, \gamma_\lambda\}$.

When A is a symmetric matrix, $G(A)$ can be regarded as an undirected graph. Since $G(A)$ must contain a 2-cycle in this case, it follows that:

PROPOSITION 2. *An undirected graph G is primitive iff G is connected and has odd cycles.*

By the definition, it is obvious that:

PROPOSITION 3. $\gamma(D) = \max_{i,j \in V(D)} \gamma(i,j)$.

We will find the following result very useful.

PROPOSITION 4. *Let G be a primitive simple graph, and let i and j be any pair of vertices in $V(G)$. If there are two walks P_1, P_2 from i to j with lengths k_1 and k_2 respectively, where k_1 and k_2 have different parity, then* $\gamma(i,j) \leqslant \max\{k_1, k_2\} - 1$.

Proof. Note that every vertex belongs to a cycle of length 2, so we can get walks of all lengths $k_m + 2t$, $m = 1$ or 2 and $t \geqslant 0$, from i to j by using 2-cycles. ■

2. THE MAIN RESULT

In the study of primitive matrices, we are interested in the problem of estimating the exponent $\gamma(A)$ and characterizing the exponent set. In 1950, H. Wielandt [7] stated the exact general upper bound $\gamma(A) \leqslant (n-1)^2 + 1$ for $n \times n$ primitive matrices. Let

$$E_n = \{m \in \mathbb{Z}^+ \mid m = \gamma(A) \text{ for some } n \times n \text{ primitive matrix } A\}.$$

Then

$$E_n \subseteq \{1, 2, \ldots, (n-1)^2 + 1\}.$$

The problem of determining the exponent set E_n is completely solved in [4] and [8].

One can also ask for the exponent set of some particular classes of primitive matrices. In [3] the exact upper bound of the exponent set of $n \times n$ doubly stochastic primitive matrices was determined. Then [2] and [5] considered the exponent set of $n \times n$ nearly reducible primitive matrices, and got a new estimate, the exact upper and lower bounds of the exponent set. Besides the primitive tournament matrices, the only complete description is in [6] for the exponent set $\tilde{E}_n$ of $n \times n$ symmetric primitive matrices:

$$\tilde{E}_n = \{1, 2, \ldots, 2n-2\} \setminus D, \qquad (\alpha)$$

where D is the set of odd numbers in $\{n, n+1, \ldots, 2n-2\}$.

In this paper, we consider a particular class of primitive matrices—symmetric primitive matrices with zero trace, i.e., the adjacency matrices of the simple primitive graphs. Denote its exponent set by $\hat{E}_n$. That is,

$$\hat{E}_n = \{m \in \mathbb{Z}^+ \mid m = \gamma(A) \text{ for some } n \times n \text{ symmetric primitive matrix } A \text{ with zero diagonal}\}.$$

Clearly, $\hat{E}_n \subseteq \tilde{E}_n$. In the following sections, we prove

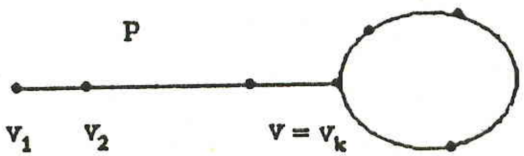


FIG. 1.

THEOREM 1. *$\hat{E}_n = \{2,3,\ldots,2n-4\}\setminus S$, where S is the set of all odd numbers in $\{n-2, n-1,\ldots,2n-5\}$.*

Let $P = v_1 \ldots v_k$ be a path of length $k-1$, and let C be a cycle of length l, with $v \in C$. The connected graph L is called $(v_1, v_k; k, l)$-lollipop, or a (v_1, v_k) lollipop for short, obtained by identifying the vertices v_k and v; see Figure 1. C is denoted by $C(L)$, and P by $P(L)$.

Let G be a primitive simple graph. By [6, Theorem 2.2], we have

$$\gamma(G) \leqslant 2n-4. \qquad (\beta)$$

We next characterize those G attaining this upper bound.

THEOREM 2. *$\gamma(G) = 2n-4$ iff $G \cong G_0$ where G_0 is a $(v_1, v_{n-2}; n-2, 3)$-lollipop.*

Proof. Sufficiency: It is easy to see that any walk of odd length from v_1 to v_r in G_0 must pass through all vertices of G. Thus any walk of odd length from v_1 to v_1 has length at least $2n-3$. Hence $\gamma(G_0) \geqslant 2n-4$ by definition. So $\gamma(G_0) = 2n-4$ by (β).

Necessity: If $\gamma(G) = 2n-4$, then there is at least one odd cycle C with length $r \geqslant 3$ in G, since G is primitive. On the other hand, by Proposition 3, there are two vertices (say v_1 and v_k) in G with $\gamma(v_1, v_k) = 2n-4$. Let P be a shortest path from v_1 to v_k in G, and l its length. If $P \cap C \neq \varnothing$, then there exist $v_i, v_e \in P \cap C$ (perhaps $v_i = v_e$), where v_i (v_e) is the first (last) vertex on C along P. Let v_i, v_e divide C into two parts C', C'', and let P_1 be the part of P from v_1 to v_i, and P_2 be the part of P from v_e to v_k. Thus the lengths of walks $P_1 \cup C' \cup P_2$ and $P_1 \cup C'' \cup P_2$ have different parity and are not greater than n, so $\gamma(v_1, v_k) < n$ by Proposition 4. This is a contradiction. Hence $P \cap C = \varnothing$. Let P_3 be a shortest path from P to C. Clearly, the length of P_3 is at most $n-r-l$. Also, the lengths of the walks P and $P \cup 2P_3 \cup C$ have different parity, and are not greater than $2n-r-l$. So $\gamma(v_1, v_k) \leqslant 2n-r-l-1$. Thus $l = 0$ and $r = 3$, since $\gamma(v_1, v_k) = 2n-4$. So the length of P_3 must be $n-3$. Therefore G is a $(v_1, v_{2n-2}; 2n-2, 3)$-lollipop. ■

3. EXISTENCE OF GRAPHS WITH GIVEN EXPONENT

In order to determine $\hat{E}_n$, we first establish that certain numbers belong to $\hat{E}_n$ and then in next section find the gaps of $\hat{E}_n$.

LEMMA 1. $\{2,4,\ldots,2n-4\} \subseteq \hat{E}_n$.

Proof. Consider the graph G_1 in Figure 2, where $k \in \{0,1,\ldots,n-3\}$. In the following, we will prove $\gamma(G_1) = 2(k+1)$.

Let $V_1 = \{v_1, v_2, \ldots, v_{n-k}\}$, $V_2 = \{v_{n-k+1}, v_{n-k+2}, \ldots, v_n\}$.

Case 1. If $v_i, v_j \in V_1$, by Proposition 4 we have $\gamma(v_i, v_j) \leqslant 2$, since there exist paths from v_i to v_j with length 2 and 3.

Case 2. If $v_i, v_j \in V_2$, let $i \leqslant j = n-k+\gamma$. Note that the lengths of the walks $v_i v_{i+1} \cdots v_j$ and $v_i v_{i-1} \cdots v_{n-k} v_1 v_{n-k-1} v_{n-k} v_{n-k+1} \cdots v_j$ have different parity. So, by Proposition 4, we have $\gamma(v_i, v_j) \leqslant 2(k+1)$. In particular, $\gamma(v_n, v_n) = 2(k+1)$.

Case 3. If $v_i \in V_1$, $v_j \in V_2$, let $j = n-k+\gamma$. If $v_i \neq v_1$, let $W_1 = v_i v_1 v_{n-k} v_{n-k+1} \cdots v_j$ and $W_2 = v_i v_{i+1} v_1 v_{n-k} v_{n-k+1} \cdots v_j$ or $v_i v_{i-1} v_1 v_{n-k} v_{n-k+1} \cdots v_j$; if $v_i = v_1$, let $W_1 = v_1 v_{n-k} v_{n-k+1} \cdots v_j$ and $W_2 = v_1 v_{n-k-1} v_{n-k} v_{n-k+1} \cdots v_j$. Then the lengths of W_1 and W_2 have different parity. So by Proposition 4 we have $\gamma(v_i, v_j) \leqslant \gamma + 2 \leqslant k+2 < 2(k+1)$.

Finally, from cases 1, 2 and 3, and we have $\gamma(G_1) = 2(k+1)$, $k = 0, 1, \ldots, n-2$. Therefore $\{2,4,\ldots,2n-4\} \subseteq \hat{E}_n$. ■

LEMMA 2.

(i) *If $n = 2k+1$, then $\{3,5,\ldots,2k-3 = n-4\} \subseteq \hat{E}_n$.*
(ii) *If $n = 2k$, then $\{3,5,\ldots,2k-3 = n-3\} \subseteq \hat{E}_n$.*

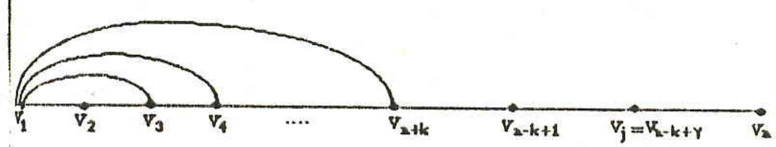
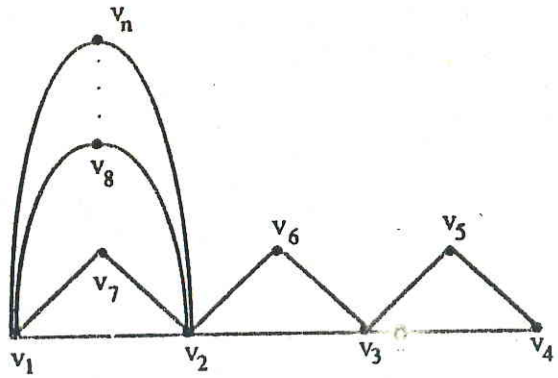


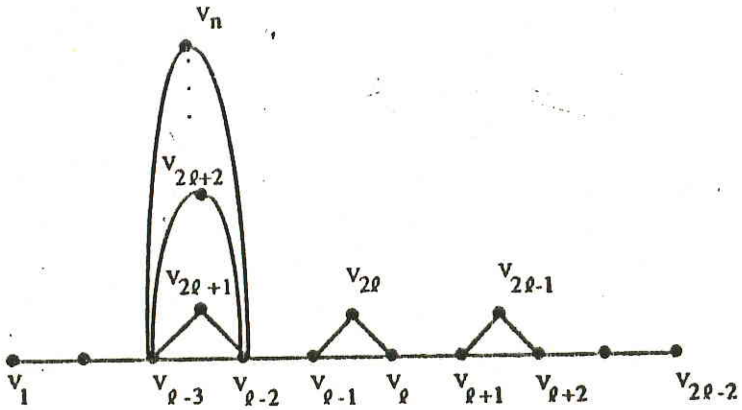
FIG. 2.

Proof. (i): If $n = 2k+1$, we consider G_2 and G_3 in Figure 3, where $l = 4,5,\ldots,(n-1)/2$. Using Proposition 4 as in proving Lemma 1, we have $\gamma(G_2) = 3$ and $\gamma(G_3) = 2l-3$, $l = 4,5,\ldots,(n-1)/2$. Hence $\{3,5,\ldots,2k-3 = n-4\} \subseteq \hat{E}_n$.

(ii): If $n = 2k$, we consider G_4 and G_5 in Figure 4, where $l = 4,5,\ldots,(n-1)/2$. Using Proposition 4 as in proving Lemma 1, we have $\gamma(G_4) = 3$ and $\gamma(G_5) = 2l-1$, $l = 4,5,\ldots,\frac{1}{2}n-1$. Hence $\{3,5,\ldots,2k-3\} \subseteq \hat{E}_n$. ∎

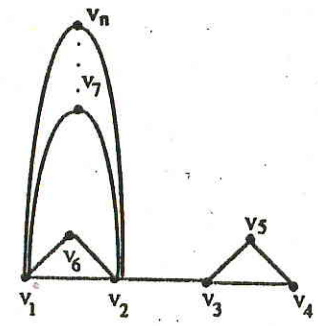


G_2

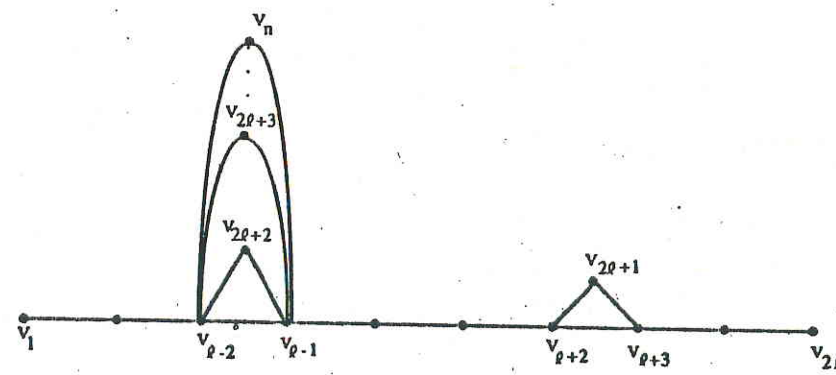


G_3

FIG. 3.



G_4



G_5

FIG. 4.

4. GAPS OF $\hat{E}_n$

Let L be a (v_1, v_k, k, l)-lollipop. Then $P(L) \cup C(L) \cup P(L)$ is a walk from v_1 to v_1 in L. Its length, $2k+l-2$, is called the *length* of L. Clearly, the length of L is odd iff l is odd.

PROPOSITION 5. *If there is a u-u walk of length k in a graph G, where k is odd, then there is a* (u,v)*-lollipop in G of odd length at most k, for some v.*

Proof. Note that there is an odd cycle in the u-u walk, since k is odd. ∎

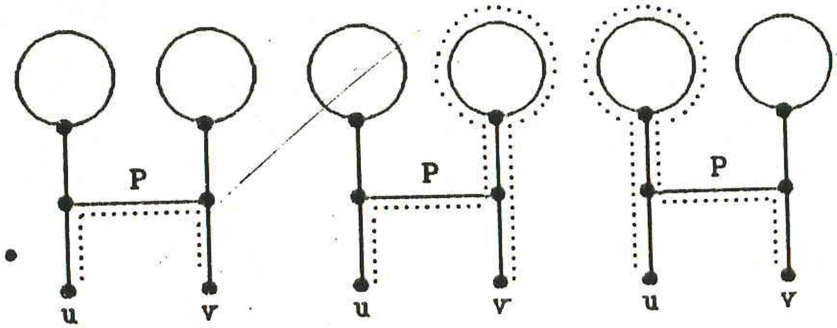


FIG. 5.

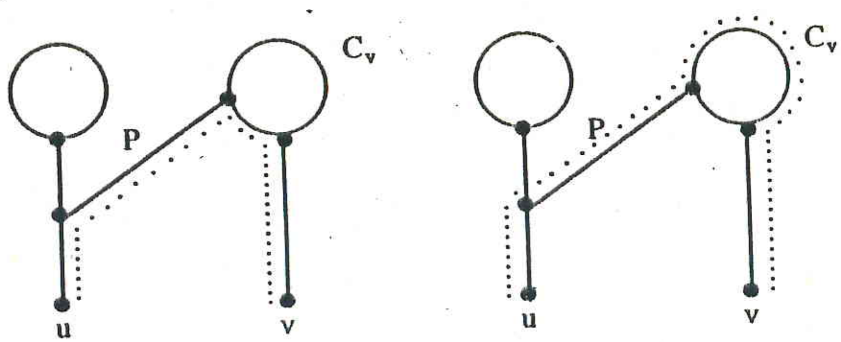


FIG. 6.

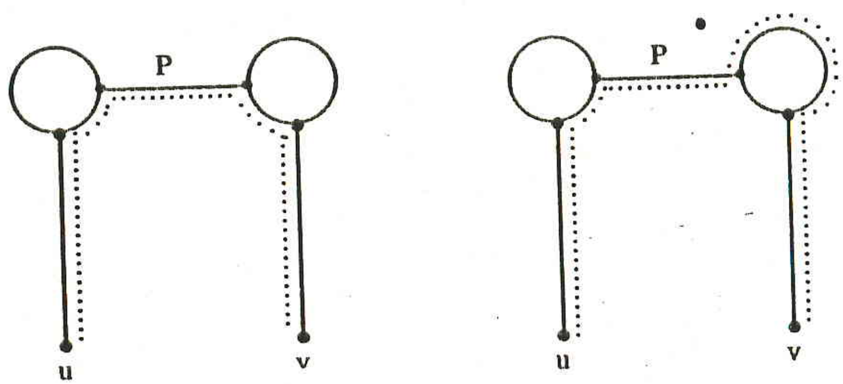


FIG. 7.

LEMMA 3. *If G is a primitive simple graph on n vertices, then $\gamma(G)$ is not* $\operatorname{odd}\{n-2, n-1\}$. *That is,* $\operatorname{odd}\{n-2, n-1\} \notin \hat{E}_n$.

Proof. Suppose there is a simple primitive graph G with $\gamma(G) = \operatorname{odd}\{n-2, n-1\}$. Then for any vertex $u \in V(G)$, there is a u-u walk of length $\operatorname{odd}\{n-2, n-1\}$. By Proposition 5, for any vertex $u \in V(G)$, there exists a (u, u_1)-lollipop L_u with odd length at most $\operatorname{odd}\{n-2, n-1\}$. So, in order to prove this lemma it is enough to prove that if u and v have lollipops L_u and L_v of odd lengths at most $\operatorname{odd}\{n-2, n-1\}$, then there is an even u-v walk of

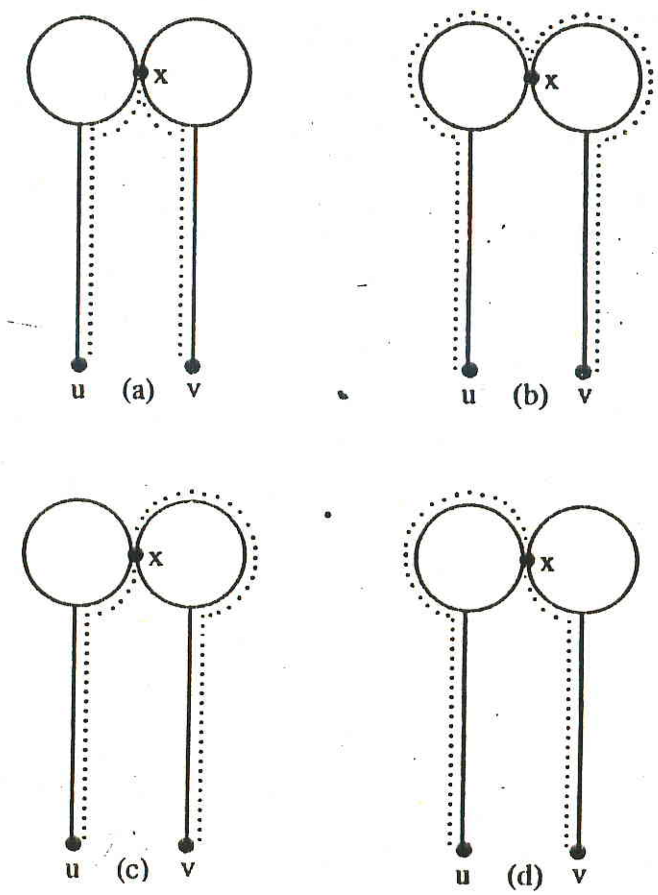


FIG. 8.

length at most $\text{odd}(n-2, n-1)-1$. Note that any u-v path will do as long as it avoids at least one vertex of G.

Let C_u, P_u denote $C(L_u), P(L_u)$ respectively. Similarly define C_v and P_v. In the following we divide the proof into four cases.

Case 1. $L_u \cap L_v = \varnothing$. Then, since G is connected, there exists a path P from L_u to L_v which is internally disjoint from L_u and L_v.

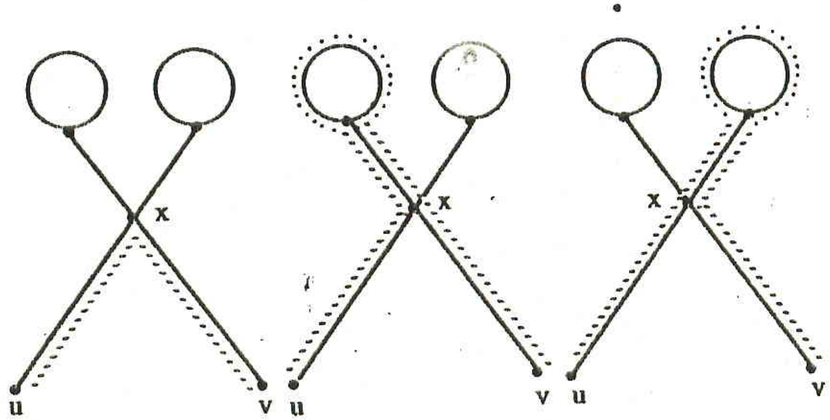
Case 1.1. *P joins P_u to P_v.* Then one of the walks in Figure 5, which all avoid at least one vertex of G, is even and short enough.

Case 1.2. *P joins P_u to C_v (or vice versa).* Then one of the paths in Figure 6, which each avoid a vertex of G, is even.

Case 1.3. *P joins C_u to C_v.* Then one of the paths in Figure 7, which each avoid a vertex of G, is even.

Case 2. $C_u \cap C_v \neq \varnothing$. Let x be a vertex in $C_u \cap C_v$. In the following, our diagrams may now contain vertices which appear to be different but are actually the same vertex, due to overlapping of the two lollipops. Consider the four walks (usually not paths) in Figure 8. Either (a) and (b) are both even or (c) and (d) are both even. The lengths of the walks in (a) and (b) add to (the length of L_u)+(the length of L_v) $\leqslant 2\,(\text{odd}(n-2, n-1))$. Hence one of (a) and (b) has length at most $\text{odd}(n-2, n-1)$. The same is true for (c) and (d). So one of the four walks has even length at most $n-2$.

Case 3. $P_u \cap P_v \neq \varnothing$. Let x denote the vertex in $P_u \cap P_v$ which is the closest to v, where the distance from v is measured along the path P_v. Consider the three walks in Figure 9. If (a) has even length, then we are



(a) (b) (c)

FIG. 9.

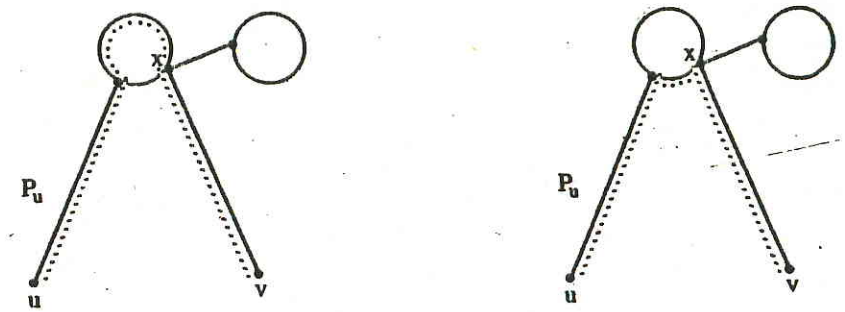


FIG. 10.

done, since its length is strictly less than the average of the lengths of L_u and L_v. Otherwise, the lengths of (b) and (c) are both even, so we are done as in case 2.

Case 4. $C_u \cap C_v = \varnothing$, $P_u \cap P_v = \varnothing$, $C_u \cap P_v \neq \varnothing$ (*or* $C_v \cap P_u \neq \varnothing$). Let x be the vertex in $C_u \cap P_v$ which is closest to v along P_v (similar to x in case 3). Consider the two paths in Figure 10. One of them is even, and both avoid a vertex in G (in particular, the vertex at the intersection of P_v and C_u). So we are done in this case. ■

Theorem 1 follows from (α), (β), Lemmas 1, 2, and 3, and the fact that $1 \notin \hat{E}_n$.

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